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Numerical methods for singularly perturbed differential equations

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بسم الله الرحمن الرحيم

Dedicace

I Dedicate this work to

My Parents

My Brother and My Sister

My Uncle in the eternal memory

All My Family

Liverpool

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Hilal Ardjani

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List of Abbreviations

ODE	:	Ordinary Differential Equation
BVP	:	Boundary Value Problem
SPP	:	Singular Perturbation Problem
SPDE	:	Singularly Perturbed Differential Equation
SPBVP	:	Singularly Perturbed Boundary Value Problem
FDM	:	Finite Difference Method
ERBFM	:	Energitic Robin Boundary Functions Method

Notations

L	:	differential operator
L^*	:	adjoint operator
$f(v)$	:	functional f applied to v
$C^l(\Omega)$	:	function space
$L_p(\Omega)$	:	function space, $1 \leq p \leq \infty$
$\|\cdot\|_{L_p}$ or $\|\cdot\|\|_{0,p}$	:	norm in $L_p(\Omega)$
$\|\cdot\|_{L_p(\Omega),d}$	:	discrete norm in $L_p(\Omega)$
ε	:	singular perturbation parameter
C	:	generic constant, independent of ε
$\mathcal{O}(\cdot), o(\cdot)$	:	Landau symbols
h	:	mesh parameter in space
L_h	:	difference operator
u, u_h, u_i	:	unknown(s)
u_0	:	reduced solution
D^+, D^-	:	forward and backward difference operators
D^0, D^+D^-	:	central difference operators

Introduction

Imagine a river? a river flowing strongly and smoothly. Liquid pollution pours into the water at a certain point. What shape does the pollution stain form on the surface of the river?

Two physical processes operate here: the pollution diffuses slowly through the water, but the dominant mechanism is the swift movement of the river, which rapidly convects the pollution downstream. Convection alone would carry the pollution along a one-dimensional curve on the surface; diffusion gradually spreads that curve, resulting in a long thin curved wedge shape.

When convection and diffusion are both present in a linear differential equation and convection dominates, we have a convection-diffusion problem.

The simplest mathematical model of a convection-diffusion problem is a two-point boundary value problem (BVP) of the form

$$-\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x) \quad \text{for } 0 < x < 1,$$

with $u(0) = u(1) = 0$, where ε is a small positive parameter and a, b and f are some given functions. Here the term u'' corresponds to diffusion and its coefficient $-\varepsilon$ is small. The term u' represents convection, while u and f play the rôle of a source and driving term respectively. (See [11] for a detailed explanation about the modelisation of diffusion and convection by second order and first-order derivatives respectively.)

The study of singularly perturbed differential equations (SPDE) is so important and it appears in several branches of engineering and applied mathematics.

Numerical analysis and asymptotic analysis are the two principal approaches for solving singular perturbed problems, numerical analysis tries to provide quantitative information about a particular problem, whereas the asymptotic analysis tries to gain insight into the qualitative behavior of a family of problems and only semi-quantitative information about any particular member of the family.

Numerical methods are intended for a broad class of problems and to minimize demands upon the problem solver.

Asymptotic methods treat comparatively restricted class of problems and require the problem solver to have some understanding of the behavior of the solution. Since the mid-1960s, singular perturbations have nourished, the subject is now commonly a part of graduate students training in applied mathematics and in many fields of engineering.

In this work we are interested in the study of the singularly perturbed two-point boundary value problems with presenting methods for their numerical solutions. The thesis is structured as follows

The first Chapter begins by positioning the problem with an exposition of the technique of matched asymptotic expansions.

In the second Chapter we present the Finite Difference methods (FDM), classical and upwind schemes are studied, and we propose a Matlab implementation which is tested in some examples for the purpose of comparison.

In the last Chapter, we discuss the Collocation method based on Energitic Robin boundary functions, and we compare the obtained results by Liu and Li in [4] with the results of the FDM using the upwind schemes on the same example.

Chapter 1

The Analytical Behaviour of Solutions

In this Chapter we shall discuss the linear singularly perturbed boundary value problem (SPBVP) for Ordinary Differential Equations.

1.1 Position of the Problem

Consider the linear SPBVP

$$Lu := -\varepsilon u'' + b(x)u' + c(x)u = f(x) \quad \text{for } x \in (d, e),$$

with the boundary conditions

$$\alpha_d u(d) - \beta_d u'(d) = \gamma_d,$$
$$\alpha_e u(e) - \beta_e u'(e) = \gamma_e.$$

Where $0 < \varepsilon << 1$ is a small parameter, the functions b, c and f are continuous. The constants α_d , α_e , β_d , β_e, γ_d and γ_e are given.

In general, we can obtain an equivalent problem with homogeneous boundary conditions $\gamma_d = \gamma_e = 0$ by choosing a smooth function ψ which satisfies the original boundary conditions and substracting it from u.

Example 1.1.1 *Given Dirichlet boundary conditions $u(d) = \gamma_d$ and $u(e) = \gamma_e$, we take*

$$\psi(x) = \gamma_d \frac{x-e}{d-e} + \gamma_e \frac{x-d}{e-d},$$

now set $u^*(x) = u(x) - \psi(x)$. Then u^* is the solution of a differential equation of the same type but with homogeneous boundary conditions.

We can also without loss of generality assume that $x \in [0, 1]$ by means of the linear transformation

$$x \mapsto \frac{x-d}{e-d}.$$

The analytical behaviour of the solution of a SPBVP depends on the nature of the boundary conditions, and when these conditions are Dirichlet it becomes the most complicated case from the numerical analyst's point of view, We consequently pay scant attention to other cases.

Then we investigate in next sections the singularly perturbed problem

$$Lu := -\varepsilon u'' + b(x)u' + c(x)u = f(x) \quad \text{for } x \in (0,1), \tag{1.1a}$$

$$u(0) = u(1) = 0, \quad \text{with } c(x) \geq 0 \quad \text{for } x \in [0,1], \tag{1.1b}$$

under the same conditions on ε, b, c and f stated earlier.

Remark 1.1.1 *The problem* (1.1) *is a typical convection diffusion problem, because in general we assume that b is not identically zero.*

Then let us state the following useful lemma.

Lemma 1.1.1 (Comparaison principle) *Suppose that v and w are functions in $C^2(0,1) \cap C[0,1]$ that satisfy*

$$Lv(x) \leq Lw(x), \quad \textit{for all } x \in (0,1),$$

and $v(0) \leq w(0)$, $v(1) \leq w(1)$. Then

$$v(x) \leq w(x) \quad \textit{for all } x \in [0,1]\,.$$

At this stage we have existence and the comparison principle ensures uniqueness of the solution u of (1.1), but we know nothing about its behaviour as ε tends to zero.

Remark 1.1.2 *The condition $c \geq 0$ cannot in general be eliminated, it's evident from the problem*

$$-\varepsilon u'' + \lambda u = 0 \quad \textit{on } (0,1), \quad u(0) = u(1) = 0,$$

which has multiple solutions when $\lambda < 0$.

For a first insight into the structure of u when ε is small, we study a simple example.

Example 1.1.2 *The boundary value problem*

$$\begin{cases} -\varepsilon u'' + u' = 1 \quad \textit{on } (0,1)\,, \\ u(0) = u(1) = 0, \end{cases}$$

has the solution

$$u(x) = x - \frac{\exp\left(-\dfrac{1-x}{\varepsilon}\right) - \exp\left(-\dfrac{1}{\varepsilon}\right)}{1 - \exp\left(-\dfrac{1}{\varepsilon}\right)}.$$

Hence for $a \in [0,1)$

$$\lim_{x\to a}\lim_{\varepsilon\to 0} u(x) = a = \lim_{\varepsilon\to 0}\lim_{x\to a} u(x),$$

but

$$1 = \lim_{x\to 1}\lim_{\varepsilon\to 0} u(x) \neq \lim_{\varepsilon\to 0}\lim_{x\to 1} u(x) = 0.$$

This inequality means that the problem is singularly perturbed and that the solution changes abruptly as x *approaches* 1*, and we say that there is a boundary layer at* $x = 1$ *(See Figure 1.1).*

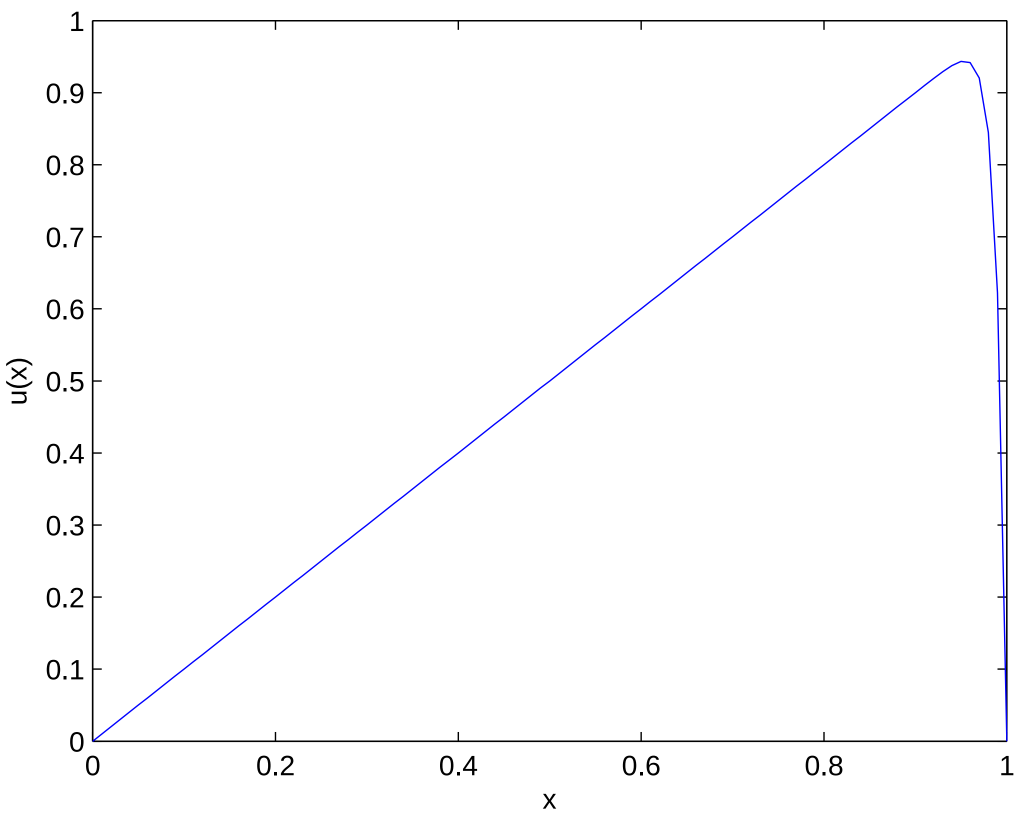

Figure 1.1: Solution of Example 1.1.2 with a boundary layer at $x = 1$ for $\varepsilon = 10^{-2}$.

Example 1.1.3 *Modifying the sign of* b *gives the problem*

$$\begin{cases} -\varepsilon u'' - u' = 1 \quad \textit{on } (0,1)\,, \\ u(0) = u(1) = 0. \end{cases}$$

The change of variable $x \mapsto 1-x$ transforms the problem to the problem in Example 1.1.2. Hence

$$u(x) = 1 - x - \frac{\exp\left(-\frac{x}{\varepsilon}\right) - \exp\left(-\frac{1}{\varepsilon}\right)}{1 - \exp\left(-\frac{1}{\varepsilon}\right)},$$

and the boundary layer develops in the neighbourhood of $x = 0$. (See Figure 1.2)

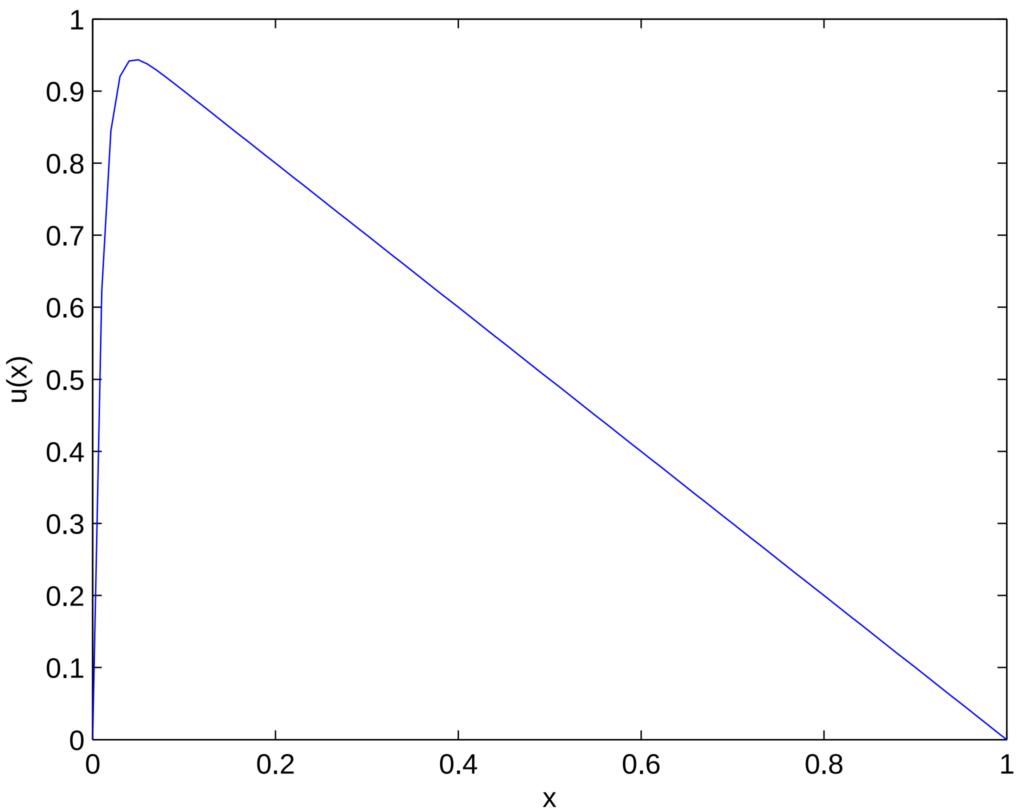

Figure 1.2: Solution of Example 1.1.3 with a boundary layer at $x = 0$ for $\varepsilon = 10^{-2}$.

Can we find a simple known function as an approximation for the solution u of (1.1)? Yes, by using The Method Of Matched Asymptotic Expansions which is a standard technique in the theory of singular perturbations.

1.2 Asymptotic Expansions

The constructed function by this technique is an asymptotic expansion of u, it illuminates its nature, and thus is valuable information. First let us recall a basic definition.

Definition 1.2.1 *The function u_{as} is an asymptotic expansion of order m of u, if there is a constant C such that*

$$|u(x) - u_{as}(x)| \leq C\varepsilon^{m+1} \quad \textit{for all } x \in [0,1] \textit{ and all } \varepsilon \textit{ sufficiently small.}$$

In the construction of u_{as} for (1.1), we assume that b, c and f are sufficiently smooth on $[0,1]$. Then the first step is to try to find a global (or regular or outer) expansion u_g, which is a good approximation of u away from any layer(s). We set

$$u_g(x) = \sum_{\nu=0}^{m} \varepsilon^{\nu} u_{\nu}(x), \tag{1.2}$$

where the $u_{\nu}(x)$ are yet to be determined (Here, as for regular perturbations, we try to expand the solution in a Taylor-type series). By formally setting $\varepsilon = 0$ in L we define

$$L_0 v := bv' + cv.$$

Substituting u_g into (1.1) and comparing powers of ε, we see that we need

$$L_0 u_0 = f,$$

$$L_0 u_{\nu} = u''_{\nu-1} \quad \text{for} \quad \nu = 1, \ldots, m.$$

Zeros of b are called turning points, if $b(x)$ has a zero in $[0,1]$ it will be difficult to define the coefficients u_v of the global expansion. We study only linear second order problems without turning points, and for more other cases we refer the reader to [9].

Suppose that $b(x) \neq 0$ for all $x \in [0,1]$, then in principle we can calculate $u_0, u_1, \ldots, u_m$ explicitly, provided that we have an additional condition for each unknown to ensure its uniqueness. We should use one of the boundary conditions (1.1b) to define u_0, so the crucial question is: which one should we discard? Guided by Example 1.1.2, we state the following cancellation law, which tells us which boundary condition to drop.

- If $b > 0$, then the boundary layer is located at $x = 1$, and we cancel the boundary condition at $x = 1$.
- If $b < 0$ then the boundary layer is located at $x = 0$ and the boundary condition at $x = 0$ is dropped.

Remark 1.2.1 *The transformation $x \mapsto 1 - x$ reduces the case $b < 0$ to $b > 0$; thus it suffices to study the case $b > 0$ in detail.*

The coefficients in the global expansion u_g are defined by

$$L_0 u_0 = f, \quad u_0(0) = 0, , \tag{1.3a}$$

$$L_0 u_\nu = u''_{\nu-1}, \quad u_\nu(0) = 0 \quad \text{for } \nu = 1, \ldots, m. \tag{1.3b}$$

We call (1.3a) the reduced problem and u_0 is the reduced solution.

The condition $u_0(0) = 0$ comes from (1.1b), while the conditions $u_\nu(0) = 0$ for $\nu \geq 1$ ensure that $u_g(0) = u(0)$.

The aim of the method is to construct an approximation of u for all $x \in [0,1]$. But u_g is not such an approximation, as it fails to satisfy the boundary condition at $x = 1$.

Therefore we add a local correction to u_g near $x = 1$. we notice that the difference $w = u - u_g$ satisfies

$$Lw = \varepsilon^{m+1} u''_m,$$

$$w(0) = 0, \quad w(1) = -\sum_{\nu=0}^{m} \varepsilon^\nu u_\nu(1).$$

Write $L = \varepsilon L_1 + L_0$. Recalling that a local correction is needed near $x = 1$, where u has a boundary layer, we introduce a change of scale by using the local variable

$$\xi = \frac{1-x}{\delta}, \qquad \text{where } \delta > 0 \text{ is small.}$$

We choose δ so that L_0 and εL_1 have formally the same order, with respect to ε, after the independent variable is transformed from x to ξ. That is, since $b \neq 0$, one sets

$$\varepsilon \delta^{-2} \approx \delta^{-1},$$

this leads to the choice $\varepsilon = \delta$.

In terms of the new variable ξ, we use Taylor expansions to write

$$b(1-\varepsilon\xi) = \sum_{\nu=0}^{\infty} b_\nu \varepsilon^\nu \xi^\nu, \qquad \text{with } b_0 = b(1),$$
$$c(1-\varepsilon\xi) = \sum_{\nu=0}^{\infty} c_\nu \varepsilon^\nu \xi^\nu, \qquad \text{with } c_0 = c(1).$$

Consequently, for any sufficiently differentiable function g, we can express L in terms of ξ as

$$\varepsilon L_1 g + L_0 g = \frac{1}{\varepsilon} \sum_{\nu=0}^{\infty} \varepsilon^\nu L_\nu^* g,$$

with

$$L_0^* := -\frac{d^2}{d\xi^2} - b_0 \frac{d}{d\xi},$$
$$L_1^* := \quad -b_1 \xi \frac{d}{d\xi} + c_0,$$

etc.

Now we introduce the local expansion

$$v_{loc}(\xi) = \sum_{\mu=0}^{m+1} \varepsilon^\mu v_\mu(\xi), \tag{1.4}$$

In order that v_{loc} approximates $w = u - u_g$, the local corrections v_μ should satisfy the boundary layer equations

$$L_0^* v_0 = 0, \tag{1.5a}$$

$$L_0^* v_\mu = -\sum_{\kappa=1}^{\mu} L_\kappa^* v_{\mu-\kappa}, \qquad \text{for } \mu = 1, \ldots, m+1. \tag{1.5b}$$

To correct the boundary conditions at $x = 1$, we need $v_\kappa(0) = u_\kappa(1)$. for $\kappa = 0, 1, \ldots, m$. As the differential equations (1.5) are of second order, so two boundary conditions are needed. The second condition must guarantee the local character of the local correction, one requires that $\lim_{\xi \to \infty} v_\mu(\xi) = 0$. With these two boundary conditions the problem (1.5) has a unique solution, because the characteristic equation corresponding to L_0^* is

$$-\lambda^2 - b(1)\lambda = 0,$$

which has exactly one negative root. For example, the first-order correction v_0 is

$$v_0(\xi) = -u_0(1)e^{-b(1)\xi}.$$

Remark 1.2.2 *A critical question in this method is whether or not the equations* (1.5) *for the local correction possess a number of decaying solutions that is equal to the number of boundary conditions that are not satisfied by the global approximation. If one cancels the wrong boundary condition when defining the reduced problem, this can lead to boundary layer equations without decaying solutions and the method then fails.*

Theorem 1.2.1 *If the coefficients and the right-hand side of the boundary value problem* (1.1) *are sufficiently smooth and* $b(x) > \beta > 0$ *on* $[0, 1]$, *then its solution* u *has a matched asymptotic expansion of the form*

$$u_{as}(x) = \sum_{\nu=0}^{m} \varepsilon^\nu u_\nu(x) + \sum_{\mu=0}^{m} \varepsilon^\mu v_\mu \left(\frac{1-x}{\varepsilon} \right), \tag{1.6}$$

such that for any sufficiently small fixed constant ε_0 *one has*

$$|u(x) - u_{as}(x)| \le C\varepsilon^{m+1} \quad \text{for } x \in [0,1] \text{ and } \varepsilon \le \varepsilon_0.$$

Here C *is independent of* x *and* ε*.*

Proof. We first consider

$$u^*_{as}(x) := \sum_{\nu=0}^{m} \varepsilon^\nu u_\nu(x) + \sum_{\mu=0}^{m+1} \varepsilon^\mu v_\mu \left(\frac{1-x}{\varepsilon}\right).$$

Based on our construction, we have

$$L\left(u - u^*_{as}\right) = O\left(\varepsilon^{m+1}\right),$$
$$\left(u - u^*_{as}\right)(0) = O\left(\varepsilon^{\kappa}\right), \quad \left(u - u^*_{as}\right)(1) = O\left(\varepsilon^{m+1}\right),$$

where $\kappa > 0$ is arbitrary. Now apply the comparison principle of Lemma 1.1.1 with the barrier function $w(x) = C\varepsilon^{m+1}(1+x)$ this choice of w exploits the property $b \ge b_0 > 0$. This leads to

$$\left|\left(u - u^*_{as}\right)(x)\right| \le |w(x)| \le C\varepsilon^{m+1} \quad \text{for all } x \in [0,1].$$

But

$$|u_{as}(x) - u^*_{as}(x)| = \left|\varepsilon^{m+1} v_{m+1}((1-x)/\varepsilon)\right| \le C\varepsilon^{m+1},$$

so a triangle inequality completes the argument. ∎

A formal differentiation of (1.6) leads to the following conjecture:

If b, c and f are sufficiently smooth and $b > 0$ (so turning points are excluded), the solution u of the boundary value problem (1.1) satisfies

$$\left|u^{(i)}(x)\right| \leq C\left[1+\varepsilon^{-i}\exp\left(-b(1)\frac{1-x}{\varepsilon}\right)\right].$$

1.3 Stability Estimates

If the coefficients of the boundary value problem (1.1) are sufficiently smooth and no turning points are present, then the asymptotic expansion procedure will describe precisely the behaviour of the solution u as $\varepsilon \to 0$.

If however any of these hypotheses are violated then this approach may fail, so we now present an alternative source of information about u and its derivatives.

The comparison principle provides a simple proof of the typical stability inequality

$$\|v\|_\infty \leq C\|Lv\|_\infty, \quad \text{for all } v \text{ with } v(0)=v(1)=0, \tag{1.7a}$$

under the assumption that $b(x) \geq b_0 > 0$, where

$$\|z\|_\infty := \max_{x\in[0,1]} |z(x)|,$$

Indeed, $w(x) = (1+x)C\|Lv\|_\infty$ is a barrier function for v.

Note that the stability constant C in (1.7a) is independent of ε. When applied to the exact solution u, (1.7a) yields

$$\|u\|_\infty \leq C\|f\|_\infty. \tag{1.7b}$$

We have here a typical result: a stability inequality implies an a priori estimate for the exact solution. The inequality (1.7b) tells us that u is bounded, uniformly with respect to ε, in the maximum norm.

The numerical analysis of discretization methods requires information about the derivatives of u. We therefore present a lemma of an a priori estimate for u' and a stability result stronger than (1.7a).

Lemma 1.3.1 *Assume that $b(x) > \beta > 0$ and b, c , f are sufficiently smooth. Then for $i = 1, 2, \ldots, q$ the solution u of* (1.1) *satisfies*

$$\left|u^{(i)}(x)\right| \leq C\left[1 + \varepsilon^{-i}\exp\left(-\beta\frac{1-x}{\varepsilon}\right)\right] \qquad \textit{for } 0 \leq x \leq 1,$$

where the maximal order q depends on the smoothness of the data.

Proof. Set $h = f - cu$. Using an integrating factor we integrate $-\varepsilon u'' + bu' = h$ twice, obtaining

$$u(x) = u_p(x) + K_1 + K_2 \int_x^1 \exp\left(-\varepsilon^{-1}(B(1) - B(t)\right)dt\ ,$$

where

$$u_p(x) := -\int_x^1 z(t)dt, \quad z(x) := \int_x^1 \varepsilon^{-1}h(t)\exp\left(-\varepsilon^{-1}(B(t) - B(x))\right)dt,$$
$$B(x) := \int_0^x b(t)dt,$$

here the constants of integration (K_1 and K_2) may depend on ε.

The boundary condition $u(1) = 0$ implies that $K_1 = 0$. We see that $u'(1) = -K_2$. Now $u(0) = 0$ gives

$$K_2\int_0^1 \exp\left[-\varepsilon^{-1}(B(1) - B(t))\right]dt = -u_p(0).$$

The bound (1.7b) implies that

$$|z(x)| \leq C\varepsilon^{-1}\int_x^1 \exp\left(-\varepsilon^{-1}(B(t) - B(x))\right)dt.$$

Applying the inequality

$$\exp\left(-\varepsilon^{-1}(B(t) - B(x)\right) \leq \exp\left(-b_0\varepsilon^{-1}(t-x)\right) \quad \text{for} \quad x \leq t\ ,$$

we obtain

$$|z(x)| \leq C\varepsilon^{-1}\int_x^1 \exp\left(-b_0\varepsilon^{-1}(t-x)\right)dt \leq C.$$

Hence $|u_p(0)| \leq C$. Set $\bar{b} = \max_{x\in[0,1]} b(x)$ Then

$$\int_0^1 \exp\left(-\varepsilon^{-1}(B(1) - B(t))dt\right) \geq \int_0^1 \exp\left(-\bar{b}\varepsilon^{-1}(1-t)\right)dt \geq C\varepsilon.$$

It now follows that

$$|K_2| \leq C\varepsilon^{-1}.$$

Finally

$$u'(x) = z(x) - K_2 \exp\left(-\varepsilon^{-1}(B(1) - B(x))\right),$$

implies that

$$|u'(x)| \leq C\left(1 + \varepsilon^{-1}\exp\left(-b_0\frac{1-x}{\varepsilon}\right)\right).$$

The proof for $i > 1$ follows by induction and repeated differentiation of (1.1). ■

From Lemma 1.3.1 we have immediately

Corollary 1.3.1

$$\int_0^1 |u'(x)|\,dx \leq C.$$

Hence we state the following result

Theorem 1.3.1 *Let us assume that*

1. $c, f \in L_1(0,1)$,

2. $b \in L_\infty[0,1], \quad b(x) \geq b_0 > 0.$

Then (1.1) *is strongly uniformly stable for* $0 < \varepsilon \le \varepsilon_0$ *, that is,*

$$\|u\|_\infty + \varepsilon \|u'\|_\infty \le C\|Lu\|_{L_1*}. \tag{1.8}$$

In Theorem 1.3.1 the term $\varepsilon \|u'\|_\infty$ can be replaced by the L_1 norm of u', as we now show.

Theorem 1.3.2 *Under the assumptions of Theorem 1.3.1 , the operator L satisfies the stability estimate*

$$\|u\|_\infty + \|u'\|_{L_1} \le C\|Lu\|_{L_1}. \tag{1.9}$$

Proof. See [8]. ■

Chapter 2

Finite Difference Methods

In this chapter we will present and analyse the finite difference methods for singularly perturbed differential equations.

2.1 Classical Convergence Theory for Central Differencing

In order to introduce the basic terminology of finite difference methods we first present the fundamental ideas and notations used for classical (non-singularly perturbed) two-point boundary value problems. Let us consider the linear two-point boundary value problem

$$Lu := -u'' + b(x)u' + c(x)u = f(x), \quad u(0) = u(1) = 0, \tag{2.1}$$

under the assumptions that b, c and f are smooth and $c(x) \geq 0$.

Finite difference methods will be studied on an equidistant grid with mesh size $h = 1/N$; that is, we set

$$x_i = ih \quad \text{for } i = 0, 1, \ldots, N, \quad \text{with } x_0 = 0 \text{ and } x_N = 1.$$

A finite difference method is a discretization of the differential equation using the grid points x_i, where the unknowns u_i (for $i = 0, \ldots, N$) are approximations of the values $u(x_i)$. It is natural to approximate $u'(x)$ by the central difference

$$\left(D^0 u\right)(x) := [u(x+h) - u(x-h)]/(2h).$$

Composing the forward and backward differences

$$\left(D^+ u\right)(x) := [u(x+h) - u(x)]/h \text{ and } \left(D^- u\right)(x) := [u(x) - u(x-h)]/h,$$

yields the following central approximation for $u''(x)$

$$\left(D^+ D^- u\right)(x) := [u(x+h) - 2u(x) + u(x-h)]/h^2.$$

The order of accuracy of every finite difference approximation depends on the smoothness of u. For instance, Taylor's formula yields

$$u(x \pm h) = u(x) \pm hu'(x) + h^2 \frac{u''(x)}{2} \pm h^3 \frac{u'''(x)}{6} + R_4,$$

with

$$R_4 = \int_x^{x \pm h} [u'''(\xi) - u'''(x)] \frac{(x \pm h - \xi)^2}{2} d\xi.$$

Hence

$$\left|\left(D^+ D^- u\right)(x) - u''(x)\right| \leq K h^2 \quad \text{ if } u \in C^4, \tag{2.2}$$

and we say that $D^+ D^-$ is second-order accurate, which is sometimes written as $O(h^2)$ accurate. Using the notation

$$g_i = g\left(x_i\right), \quad \text{ where } g \text{ can be } b, c \text{ or } f.$$

The classical central difference scheme for the boundary value problem (2.1) is

$$-D^+D^-u_i + b_iD^0u_i + c_iu_i = f_i \quad \text{for } i = 1, \ldots, N-1, \tag{2.3a}$$

$$u_0 = u_N = 0. \tag{2.3b}$$

This is a tridiagonal system of linear equations:

$$r_iu_{i-1} + s_iu_i + t_iu_{i+1} = f_i \quad \text{for } i = 1, \ldots, N-1, \quad \text{with } u_0 = u_N = 0, \tag{2.4}$$

where

$$r_i = -\frac{1}{h^2} - \frac{1}{2h}b_i, \quad s_i = c_i + \frac{2}{h^2}, \quad t_i = -\frac{1}{h^2} + \frac{1}{2h}b_i. \tag{2.5}$$

Two questions must now be tackled: what properties does the discrete problem (2.3) enjoy? What can we say about the errors $|u(x_i) - u_i|$?

Classical convergence theory for finite difference methods is based on the complementary concepts of consistency and stability. First, formally write (2.3) (or any difference scheme) as

$$L_hu_h = f_h, \tag{2.6}$$

where L_h is a matrix, and we have

$$\begin{cases} u_h := (u_h(x_0), u_h(x_1), \ldots, u_h(x_N))^T := (u_0, u_1, \ldots, u_N)^T, \\ f_h := (f(x_0), f(x_1), \ldots, f(x_N))^T. \end{cases}$$

Functions defined on the grid such as u_h and f_h, are called grid functions. The restriction of a function $v \in C[0,1]$ to a grid function is denoted by R_hv , viz.,

$$R_hv = (v(x_0), v(x_1), \ldots, v(x_N)).$$

We sometimes omit R_h when the meaning is clear. The discrete maximum norm on the space of grid functions is

$$\|v_h\|_{\infty,d} := \max_i |v_h(x_i)|.$$

Definition 2.1.1 *Consider a difference scheme of the form $L_h u_h = R_h(Lu)$, where we incorporate the boundary conditions into the scheme by taking the first and last rows of L_h to be identical to the first and last rows respectively of the identity matrix, with $(R_h Lu)_0 = u_0$ and $(R_h Lu)_N = u_N$. This scheme is consistent of order k in the discrete maximum norm if*

$$\|L_h R_h u - R_h Lu\|_{\infty,d} \leq K h^k,$$

where the positive constants K and k are independent of h.

As in (2.2) we can apply Taylor's formula to prove

Lemma 2.1.1 *Under the assumption $u \in C^4[0,1]$, the central difference scheme* (2.3) *is consistent of order two.*

Applying the discrete operator L_h to the error at the interior grid points yields

$$L_h(R_h u - u_h) = L_h R_h u - f_h = L_h R_h u - R_h Lu. \tag{2.7}$$

In order to estimate $R_h u - u_h$ from (2.7) and the consistency order, it is natural to introduce the concept of stability.

Definition 2.1.2 *A discrete problem $L_h u_h = f_h$ is stable in the discrete maximum norm, if there exists a constant K (the stability constant) that is independent of h, such that*

$$\|u_h\|_{\infty,d} \leq K \|L_h u_h\|_{\infty,d}, \tag{2.8}$$

for all mesh functions u_h.

Our last ingredient is the following

Definition 2.1.3 *A difference method for* (2.1) *is convergent (of order k) in the discrete maximum norm if there exist positive constants K and k that are independent of h for which*

$$\|u_h - R_h u\|_{\infty,d} \leq K h^k.$$

The main result of classical convergence theory for finite difference methods now follows immediately

$$\text{Consistency} \; + \; \text{Stability} \implies \text{Convergence}.$$

The investigation of the order of consistency is usually based on Taylor's formula and is straightforward. But to prove stability one needs some new tools.

The material that follows uses the natural ordering of vectors, viz., $x \leq y$ if and only if $x_i \leq y_i$ for all i. Sometimes we simply write $z \geq 1$ when we mean that $z_i \geq 1$ for all i.

Definition 2.1.4 *For each matrix $A = (a_{ij})$, the inequality $A \geq 0$ means that $a_{ij} \geq 0$ for all i and j.*

Definition 2.1.5 *A matrix A is called inverse-monotone if A^{-1} exists and $A^{-1} \geq 0$.*

Lemma 2.1.2 (Discrete comparison principle) *Let A be inverse-monotone. Then $Av \leq Aw$ implies that $v \leq w$.*

Proof. Multiply $A(v - w) = b \leq 0$ by A^{-1} and use $A^{-1} \geq 0$. ∎

The class of M-matrices is an important subset of inverse-monotone matrices class.

Definition 2.1.6 *A matrix A is an M-matrix if its entries a_{ij} satisfy $a_{ij} \leq 0$ for $i \neq j$ and its inverse A^{-1} exists with $A^{-1} \geq 0$.*

The diagonal entries of an M-matrix satisfy $a_{ij} > 0$.

While the condition $a_{ij} \leq 0$ is easy to check, it may be difficult to verify directly the inequality $A^{-1} \geq 0$. Fortunately, several equivalent but more tractable characterizations of M-matrices are known. The following result is frequently used in the context of discretization methods.

Theorem 2.1.1 *Let the matrix A satisfy $a_{ij} \leq 0$ for $i \neq j$. Then A is an M-matrix if and only if there exists a vector $e > 0$ such that $Ae > 0$. Furthermore, we have*

$$\left\|A^{-1}\right\|_{\infty,d} \leq \frac{\|e\|_{\infty,d}}{\min_k (Ae)_k}. \tag{2.9}$$

In Theorem 2.1.1 the vector e is called a majorizing element for the matrix A.

This theorem allows us to verify that the coefficient matrix of a given discretization is an M-matrix while simultaneously estimating the stability constant from (2.9) provided that we are able to find a majorizing element.

The following recipe for construction of this element is often successful:

- Find a function $e > 0$ such that $Le(x) > 0$ for $x \in (0,1)$, this is a majorizing element for the differential operator L.

- Restrict e to a grid function e_h.

In general, if the first step in this method is feasible then the method will work (at least for sufficiently small h) provided the discretization is consistent to some positive order.

For homogeneous boundary conditions one usually eliminates the variables u_0 and u_N before applying Theorem 2.1.1.

Example 2.1.1 *Consider the special case where $b(x) \equiv 0$ in the differential operator L of (2.1). Choose $e(x) := x(1-x)/2$. Then*

$$Le(x) = 1 + c(x)e(x) \geq 1.$$

On setting $e(x) := R_h e$ *one obtains*

$$L_h e_h \geq (1, \ldots, 1)^T,$$

since D^+D^- *discretizes quadratic functions exactly at the interior grid points. Now inequality* (2.9) *provides a stability constant of* $1/8$.

In the general case of (2.1), the construction of a majorizing element is slightly more complicated. Define $e(x)$ to be the solution of the boundary value problem

$$-w'' + b(x)w' = 1, \quad w(0) = w(1) = 0.$$

Then $e(x) > 0$ for $x \in (0, 1)$ and $e(x)$ is bounded. The inequality $c(x) \geq 0$ and the consistency of the discretization imply that at the interior grid points one has

$$L_h e_h = R_h Le + (L_h e_h - R_h Le) \geq 1/2,$$

for all sufficiently small h, because $R_h L_e = 1$. This gives

Lemma 2.1.3 *For all sufficiently small* h*, the central difference scheme for the boundary value problem* (2.1) *is stable in the discrete maximum norm; moreover, the corresponding coefficient matrix is then an* M*-matrix.*

One can clearly combine Lemmas 2.1.1 (consistency) and 2.1.3 (stability) to obtain a second-order convergence result.

2.2 Upwind Schemes

We study now difference schemes for the SPBVP

$$Lu := -\varepsilon u'' + b(x)u' + c(x)u = f(x) \quad \text{on } (0,1), \quad u(0) = u(1) = 0. \tag{2.10}$$

when turning points are excluded, i.e., when $b(x) \neq 0$ for all $x \in [0,1]$. We also assume that $c \geq 0$ on $[0,1]$ and that the functions b, c and f are smooth. Recall that for $b > 0$ there is an exponential boundary layer at $x = 1$, and for $b < 0$ the boundary layer is at $x = 0$. The conditions "$b < 0$" and "$b > 0$" are equivalent: the change of variable $x \mapsto 1 - x$ transforms the problem from one formulation to the other.

Suppose that $\varepsilon > 0$ is small. If u exhibits a boundary layer, this adversely affects both consistency and stability. If instead the boundary conditions are such that u has no layer, then the consistency error improves but stability may still be a problem.

To begin, the central difference scheme is applied to the example

$$-\varepsilon u'' + u' = 0 \text{ on } (0,1), \quad u(0) = 0, u(1) = 1.$$

A transformation $u(x) = x + v(x)$ would give homogeneous boundary conditions, but one can use the scheme directly with inhomogeneous conditions. The discrete problem is

$$-\varepsilon D^+D^-u_i + D^0u_i = 0, \quad u_0 = 0, u_N = 1.$$

It is easy to solve this exactly:

$$u_i = \frac{r^i - 1}{r^N - 1}, \quad \text{with} \quad r = \frac{2\varepsilon + h}{2\varepsilon - h}.$$

If $h \gg 2\varepsilon$, then $r \approx -1$ so this computed solution oscillates badly and is not close to the true solution

$$u(x) = \frac{e^{-(1-x)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}.$$

If we assume that $h < 2\varepsilon$, then the central difference scheme works, but from the practical point of view this assumption is unsatisfactory when for instance $\varepsilon = 10^{-5}$. A fortiori, in two or three dimensions such a mesh restriction would lead to unacceptably large numbers of mesh points, as for small ε the dimension of the algebraic system generated would be too large for computer solution.

Returning to the general problem (2.10) write the central difference scheme in the form of (2.5), viz.,

$$r_i = -\frac{\varepsilon}{h^2} - \frac{1}{2h}b_i, \quad s_i = c_i + \frac{2\varepsilon}{h^2}, \quad t_i = -\frac{\varepsilon}{h^2} + \frac{1}{2h}b_i. \tag{2.11}$$

This gives an M-matrix and hence stability if we assume that

$$h \leq h_0(\varepsilon) = \frac{2\varepsilon}{\|b\|_\infty},$$

which generalizes the observation of the example above. Note that $h_0(\varepsilon) \to 0$ if $\varepsilon \to 0$. This conclusion is not confined to the central difference scheme: Classical numerical methods on equidistant grids yield satisfactory numerical solutions for singularly perturbed boundary value problems only if one uses an unacceptably large number of grid points. In this sense, classical methods fail.

An alternative heuristic explanation for the failure of central differencing in the above example is that when $\varepsilon \ll h$ the scheme is essentially $D^0 u_i = 0$, which implies in particular that $u_{N-2} \approx u_N = 1$, so u_{N-2} is a poor approximation to $u\left(x_{N-2}\right) \approx 0$.

This argument also shows that we would do well to avoid any difference approximation of $u'\left(x_{N_1}\right)$ that uses u_N. The simplest candidate meeting this requirement is the approximation

$$u'(x_i) \approx \frac{u_i - u_{i-1}}{h}. \tag{2.12}$$

An inspection of the signs of the matrix entries of the earlier discrete problem, with the aim of modifying the difference scheme in order to generate an M-matrix, also motivates (2.12).

Thus for the general case where the sign of b may be positive or negative, consider the scheme

$$-\varepsilon D^+ D^- u_i + b_i D^{\aleph} u_i + c_i u_i = f_i \quad \text{for } i = 1, \ldots, N-1, \tag{2.13a}$$

$$u_0 = u_N = 0, \tag{2.13b}$$

with

$$D^{\aleph} = \begin{cases} D^+ & \text{if} \quad b < 0, \\ D^- & \text{if} \quad b > 0. \end{cases} \tag{2.12c}$$

This is the simple upwind scheme. (We saw in the Introduction that convection dominates the problem and assigns a direction to the flow; upwind means that the finite difference approximation of the convection term is taken on the upstream side of each mesh point).

We now begin our analysis of the upwind scheme. Write L_h for the matrix of the scheme after eliminating u_0 and u_N. In the form (2.4), the coefficients of the discrete problem are

$$r_i = -\frac{\varepsilon}{h^2} - \frac{1}{h}\max\{0, b_i\}, \quad s_i = c_i + \frac{2\varepsilon}{h^2} + \frac{1}{h}|b_i|$$
$$t_i = -\frac{\varepsilon}{h^2} + \frac{1}{h}\min\{0, b_i\}.$$

Now the off-diagonal matrix entries are non-positive, irrespective of the relative sizes of h and ε.

Lemma 2.2.1 *Assume that $b(x) \neq 0$ for all $x \in [0,1]$. Then the coefficient matrix L_h for the upwind scheme* (2.13) *is an M-matrix and the upwind scheme is uniformly stable with respect to the perturbation parameter:*

$$\|u_h\|_{\infty,d} \leq C \|L_h u_h\|_{\infty,d}.$$

Proof. For definiteness assume that $b(x) \geq \beta > 0$. We construct a suitable majorizing vector. Choose $e(x) := x$, so $Le(x) \geq \beta$. A direct computation yields $L_h e_h \geq \beta$. By Theorem 2.1.1 the matrix is an M-matrix and one gets the desired stability bound with stability constant $C = 1/\beta$. ■

Theorem 2.2.1 *Assume that $b > \beta > 0$ and $c \geq 0$. Then there exists a positive constant B^*, which depends only on β, such that the error of the simple upwind scheme* (2.13) *at the inner grid points $\{x_i : i = 1, \ldots, N-1\}$ satisfies*

$$|u(x_i) - u_i| \leq \begin{cases} Ch\left[1 + \varepsilon^{-1} \exp\left(-\beta^* \left(1 - x_i\right)/\varepsilon\right)\right] & \text{if } h \leq \varepsilon, \\ Ch + C \exp\left(-\beta^* \left(1 - x_{i+1}\right)/\varepsilon\right) & \text{if } h \geq \varepsilon. \end{cases}$$

Proof. *We estimate the consistency error using Taylor's formula. At the grid point x_i, we obtain*

$$|\tau_i| := |L_h u(x_i) - f(x_i)| \leq C \int_{x_{i-1}}^{x_{i+1}} \left(\varepsilon \left|u^{(3)}(t)\right| + \left|u^{(2)}(t)\right|\right) dt. \tag{2.13}$$

The crude bound $\left|u^{(k)}\right| \leq C\varepsilon^{-k}$ combined with the stability result of Lemma 2.2.1 yields only $|u(x_i) - u_i| \leq Ch/\varepsilon^2$, so a more precise bound on $\left|u^{(k)}\right|$ is needed. Invoking Lemma

1.3.1 yields the inequality

$$
\begin{aligned}
|\tau_i| &\le Ch + C\varepsilon^{-2}\int_{x_{i-1}}^{x_{i+1}} \exp(-\beta(1-t)/\varepsilon)dt \\
&\le Ch + C\varepsilon^{-1}\sinh\left(\frac{\beta h}{\varepsilon}\right)\exp\left(-\frac{\beta\,(1-x_i)}{\varepsilon}\right).
\end{aligned}
$$

Consider first the case when $h \le \varepsilon$. Then $\beta h/\varepsilon$ is bounded. Now $\sinh t \le Ct$, when t is bounded, so

$$|\tau_i| \le Ch\left[1 + \varepsilon^{-2}\exp\left(-\frac{\beta\,(1-x_i)}{\varepsilon}\right)\right].$$

At first sight, this inequality seems unable to deliver the desired power of ε (viz., ε^{-1} instead of ε^{-2}) when Lemma 2.1.3 is applied. But if one considers the boundary value problem

$$-\varepsilon w'' + bw' + cw = C\varepsilon^{-1}\exp\left(-\frac{\beta(1-x)}{\varepsilon}\right), \quad w(0) = w(1) = 0,$$

then using the barrier function

$$w^*(x) = C\exp\left(-\frac{\beta^*(1-x)}{\varepsilon}\right),$$

where $\beta^* > \beta$, the comparison principle of Lemma 1.1.1 yields the estimate

$$|w(x)| \le C\exp\left(-\frac{\beta^*(1-x)}{\varepsilon}\right),$$

where w has gained a power of ε compared with Lw ! The same calculation at the discrete level, using the discrete comparison principle of Lemma 2.1.2, completes the proof of the theorem when $h \le \varepsilon$.

In the more difficult case $h \geq \varepsilon$, we decompose the solution as

$$u(x) = -u_0(1) \exp\left(-\frac{b(1)(1-x)}{\varepsilon}\right) + z(x).$$

By imitating the proof of Lemma 1.3.1 one finds that

$$\left|z^{(i)}(x)\right| \leq C\left[1 + \varepsilon^{1-i} \exp\left(-\frac{b(1)(1-x)}{\varepsilon}\right)\right].$$

Set

$$v(x) = -u_0(1) \exp\left(-\frac{b(1)(1-x)}{\varepsilon}\right),$$

and define v_h and z_h by

$$L_h v_h = Lv \quad \text{and} \quad L_h z_h = Lz.$$

where v_h and z_h agree with v and z, respectively, at x_0 and x_N . Then

$$|u(x_i) - u_i| = |v(x_i) + z(x_i) - (v_i + z_i)| \leq |v(x_i) - v_i| + |z(x_i) - z_i|.$$

For the consistency error associated with z, similarly to before one gets

$$|\tau_i(z)| \leq Ch + C \sinh\left(\frac{\beta h}{\varepsilon}\right) \exp\left(-\frac{\beta(1-x_i)}{\varepsilon}\right).$$

As now $h \geq \varepsilon$, we use the inequality $\sinh t \leq Ce^t$. Hence

$$|\tau_i(z)| \leq Ch + C \exp\left(-\frac{\beta(1-x_{i+1})}{\varepsilon}\right).$$

The consistency error due to v must still be bounded. The definition of v gives

$$|Lv(x)| \leq C\varepsilon^{-1}|v(x)|.$$

Thus

$$\left|(L_h v_h)_i\right| = |Lv(x_i)| \leq C\varepsilon^{-1} \exp\left(-\frac{\beta(1-x_i)}{\varepsilon}\right).$$

Appealing again to the discrete comparison principle, one obtains

$$|v(x_i) - v_i| \leq |v(x_i)| + |v_i| \leq C \exp\left(-\frac{\beta(1-x_i)}{\varepsilon}\right).$$

Combining the various estimates proves the result for the case $h \geq \varepsilon$. ■

Theorem 2.2.1 shows that outside the boundary layer (i.e., in the interval $[0, 1-\delta]$ for any fixed $\delta > 0$) simple upwinding gives first-order convergence with a convergence constant independent of ε. But inside the layer the theorem does not prove convergence, and indeed the story here is disappointing: take the example

$$-\varepsilon u'' - u' = 0, \quad u(0) = 0, u(1) = 1,$$

which has a boundary layer at $x = 0$. Then the simple upwind scheme yields

$$u_i = \frac{1 - r^i}{1 - r^N}, \quad \text{with} \quad r = \frac{\varepsilon}{\varepsilon + h}.$$

Thus for $h = \varepsilon$ one gets

$$u_1 = \frac{1/2}{1 - (1/2)^N} \quad \text{but} \quad u(x_1) = \frac{1 - e^{-1}}{1 - e^{-1/\varepsilon}}.$$

Several options are available for the construction of upwind schemes that achieve higher-order convergence outside the layer. (Here "upwind" means that the first-order derivative in the differential equation is approximated by a non-centred difference approximation.)

First, taking $b > 0$ for convenience, the simple upwind scheme (2.13) can be rearranged as

$$-\left(\varepsilon + \frac{b_i h}{2}\right) D^+ D^- u_i + b_i D^0 u_i + c_i u_i = f_i, \quad u_0 = u_N = 0. \tag{2.14}$$

This resembles the central difference scheme, but the diffusion coefficient has been modified from ε to $\varepsilon + b_i h/2$. That is, simple upwinding applied to (2.10) is the same as central differencing applied to a modified version of (2.10). For $\varepsilon > b_i h/2$ the dominant diffusion is $O(\varepsilon)$, but in the more interesting case $\varepsilon < b_i h/2$ it becomes $O(b_i h/2)$. The scheme (2.14) is said to have artificial diffusion or artificial viscosity. It is the simplest example of a general strategy: add artificial diffusion to the given differential equation to stabilize a standard discretization method.

It turns out that too much artificial viscosity will "smear" the computed solution (that is, the computed layers are too wide).

Artificial diffusion can be introduced directly by means of a fitting factor δ, as in the following fitted upwind scheme, which generalizes (2.14):

$$-\varepsilon\sigma\left(q\left(x_i\right)\right) D^+ D^- u_i + b_i D^0 u_i + c_i u_i = f_i \quad \text{for } i = 1, \ldots, N-1, \tag{2.15a}$$

$$u_0 = u_N = 0, \tag{2.15b}$$

$$\text{with} \quad q(x) := \frac{b(x)h}{2\varepsilon}. \tag{2.15c}$$

If $\delta(q) = 1 + q$, this becomes the simple upwind scheme (2.14).

Which choices of δ will generate good upwind schemes? As part of the answer to this question, we generalize Lemma 2.2.1 to the following stability result.

Lemma 2.2.2 *Assume that $b(x) > \beta > 0$, $c \geq 0$, and $\delta(q) > q$. Then the coefficient* *matrix of the fitted upwind scheme* (2.15) *is an M-matrix and the method is stable in the discrete maximum norm, uniformly in ε.*

Proof. See [9]. ■

2.3 Matlab Implemententaion

In order to validate the theoretical results, in this section we propose MATLAB codes to solve the linear two-point boundary value problem (2.10) without turning points for two different schemes. The codes are based on the schemes (2.11) and (2.13).

The main code of central difference scheme

```
Ia=0;Ib=1;n = 100 ; h = 1/n ; xh = Ia : h: Ib ;
eps=0.001;%epsilon value
%eps=eps+h.*b(xh(2:end-1))./2
f= @(x)-1;%f(x)
b=@(x)1;%b(x)
c=@(x)0;%c(x)
ybegin=0;yend=0;
A1 = diag(ones(n - 2, 1) ,1)*(-1) ;
A2 = diag(ones(n - 2, 1) ,-1)*(-1) ;
A3 = diag(ones(n-1, 1))*(2) ;
Ah1 = eps*(A1 + A2 + A3);
idA2=diag(zeros(n-1,1));
sdA2=diag(b(xh(2:end-2)).*ones(n-2,1)',1);
ddA2=diag(-b(xh(3:end-1)).*ones(n-2,1)',-1);
Ah2=(h./2)*(idA2+sdA2+ddA2);
```

```
idA3=diag(c(xh(2:end-1)).*ones(n-1,1)');
Ah3=h^2*(idA3);
A = Ah1 + Ah2+ Ah3 ;
fx=f(xh(2:end-1)).*ones(n-1,1)';
fh=h^2*fx+[(eps+b(xh(1))*h/2)*ybegin zeros(1,n-3) (eps-b(xh(
    end))*h/2)*yend];
yh=inv(A)*fh';
yhh=[ybegin, yh',yend];
%%%%%%%%%%%%%%%%%%%%
syms y(x)
ode = -eps*diff(y,x,2)+b.*diff(y,x)+c.*y == f;
cond1 = y(0) ==ybegin;
cond2 = y(1) == yend;
conds = [cond1 cond2];
ySol(x) = dsolve(ode,conds);
x = 0:h:1;
y = ySol(x);
u = zeros(1, n+1);
for i=1:n+1,
u(i)= ySol(x(i));
end
r=abs(yhh-u);
errm=max(r)
plot(xh,yhh,'-',x,u,x,r)
xlabel('x')
legend('Numerical solution','Exact Solution','Error')
```

```
figure;
plot(x,r)
ylabel('Error')
xlabel('x')
ylim auto
```

The main code of upwind scheme

```
Ia=0;Ib=1;n = 100 ; h = 1/n ; xh = Ia : h: Ib ;
eps=0.001;%epsilon value
f= @(x)-1;%f(x)
b=@(x)1;%b(x)
c=@(x)0;%c(x)
ybegin=0;yend=0;
A1 = diag(ones(n - 2, 1) ,1)*(-1);
A2 = diag(ones(n - 2, 1) ,-1)*(-1);
A3 = diag(ones(n-1, 1))*(2);
Ah1 = eps*(A1 + A2 + A3);
idA2=diag(abs(b(xh(2:end-1))).*ones(n-1,1)');
ddA2=diag(-max(b(xh(3:end-1)),0).*ones(n-2,1)',-1);
sdA2=diag(min(0,b(xh(2:end-2))).*ones(n-2,1)',1);
Ah2=(h).*(idA2+ddA2+sdA2);
idA3=diag(c(xh(2:end-1)).*ones(n-1,1)');
Ah3=h^2*(idA3);
A = Ah1 + Ah2 + Ah3;
fx=f(xh(2:end-1)).*ones(n-1,1)';
```

```
fh=h^2*fx+[(eps+max(0,b(xh(1)))*h)*ybegin zeros(1,n-3) (eps-
   min(0,b(xh(end)))).*yend];
yh=inv(A)*fh'
yhh=[ybegin, yh',yend];
%%%%%%%%%%%%%%%%%%%%
syms y(x)
ode =- eps.*diff(y,x,2)+b.*diff(y,x)+c.*y == f;
cond1 = y(0) == ybegin;
cond2 = y(1) == yend;
conds = [cond1 cond2];
ySol(x) = dsolve(ode,conds);
x = 0:h:1;
y = ySol(x);
u = zeros(1, n+1);
for i=1:n+1,
u(i)= ySol(x(i));
end
r=abs(yhh-u);
errm=max(r)
plot(xh,yhh,'-',x,u,x,r)
xlabel('x')
legend('Numerical solution','Exact Solution','Error')
figure;
plot(x,r)
ylabel('Error')
xlabel('x')
```

```
ylim auto
```

Example 2.3.1 *Let us consider the same SPBVP discussed in Section* (2.2)*,i.e.,*

$$-\varepsilon u'' + u' = 0 \ on \ (0,1), \quad u(0) = 0, u(1) = 1,$$

a transformation $u(x) = x + v(x)$ would give homogeneous boundary conditions, then applying the code of central difference scheme for $\varepsilon = 10^{-2}$ and $n = 10^3$ we get

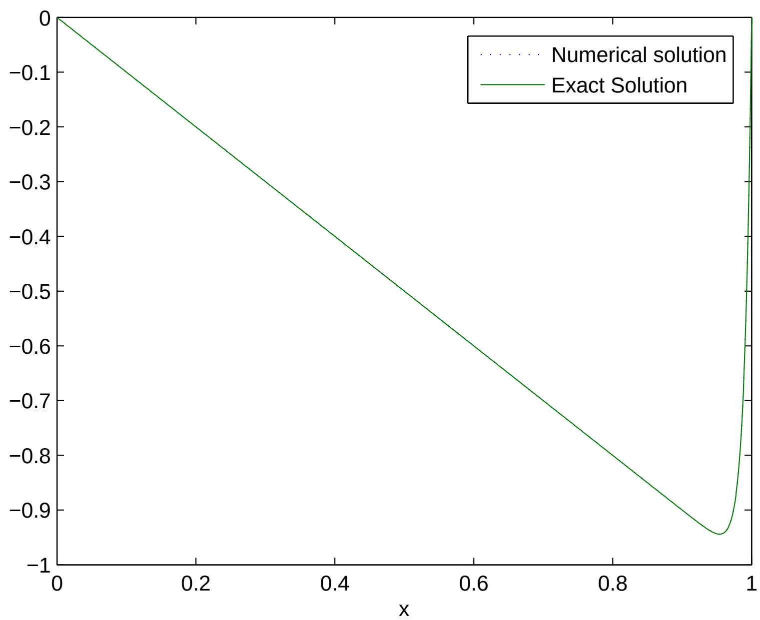

Figure 2.1: Numerical solution of Example 2.3.1 using central difference scheme for $\varepsilon = 10^{-2}$ and $n = 10^3$.

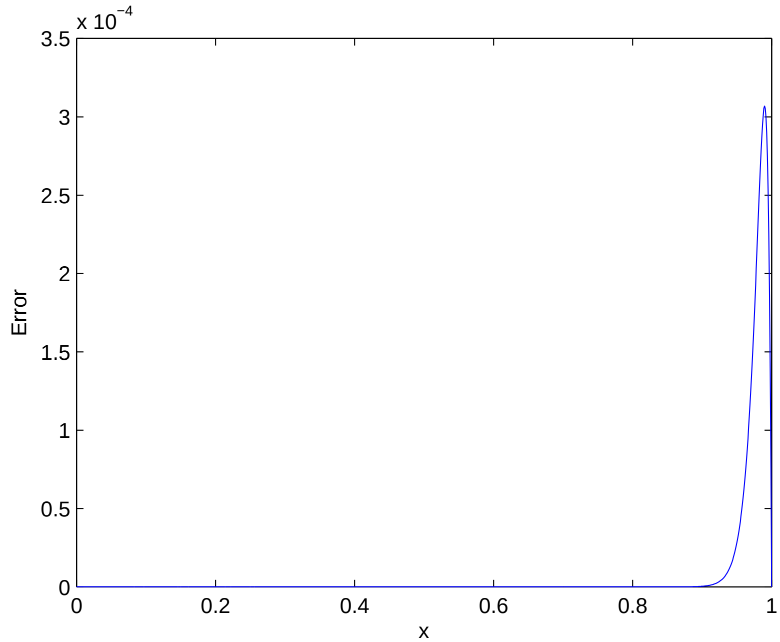

Figure 2.2: Error on the computed solution of Example 2.3.1 using central difference scheme for $\varepsilon = 10^{-2}$ and $n = 10^3$.

We find that the central difference scheme works with 3.069×10^{-4} *as a maximum error (See Figure 2.2), but in the case* $h > 2\varepsilon$ *the method fails, we can validate the result by taking* $h = 10^{-2}$ *and* $\varepsilon = 10^{-3}$ *(See Figure 2.3).*

On the other hand, the numerical behaviour of the upwind scheme is much better than the central one in the case $h > 2\varepsilon$, *it suffices to apply the code by taking* $h = 10^{-2}$ *and* $\varepsilon = 10^{-3}$ *to get more satisfactory results (See Figure 2.4).*

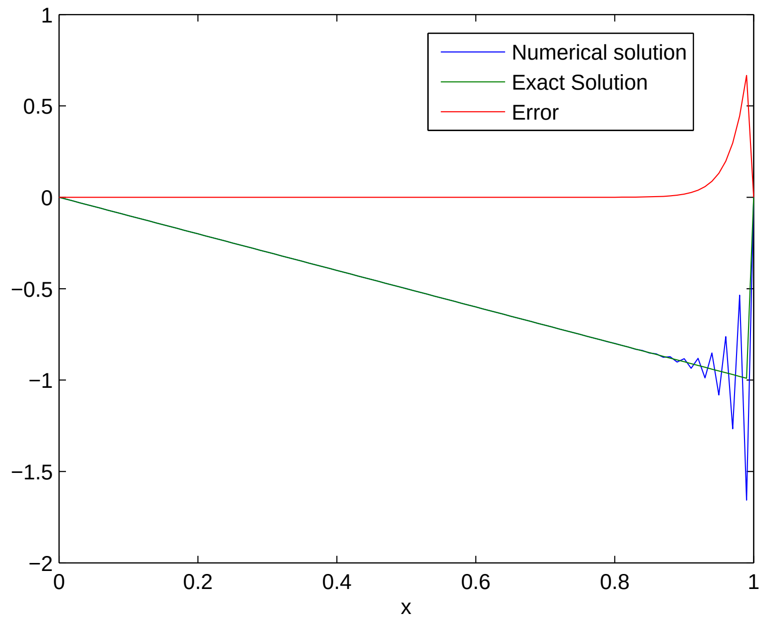

Figure 2.3: Comparing the numerical solutions with the exact solution of Example 2.3.1 using central difference scheme for $\varepsilon = 10^{-3}$ and $n = 10^2$ and showing error.

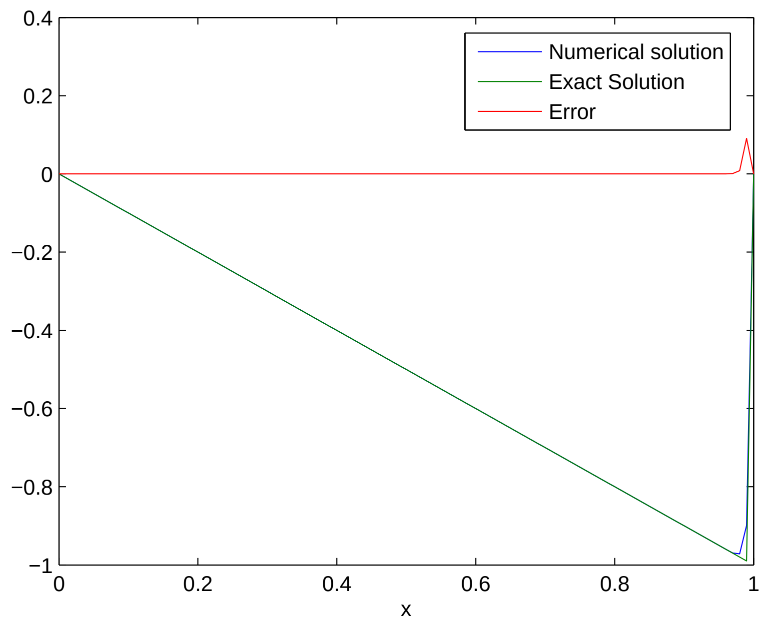

Figure 2.4: Comparing the numerical solutions with the exact solution of Example 2.3.1 using upwind scheme for $\varepsilon = 10^{-3}$ and $n = 10^2$ and showing error.

Example 2.3.2 *Consider the following singular perturbation problem (SPP)*

$$\varepsilon u''(x) + u'(x) - u(x) = 0, \quad u(0) = 1, \quad u(1) = 1. \tag{2.16}$$

The exact solution is given by

$$u(x) = \frac{1}{e^{p_2} - e^{p_1}} \left[\left(e^{p_2} - 1\right) e^{p_1 x} + \left(1 - e^{p_1}\right) e^{p_2 x} \right],$$

where

$$p_1 = \frac{-1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon}, \quad p_2 = \frac{-1 - \sqrt{1 + 4\varepsilon}}{2\varepsilon}.$$

A transformation $u(x) = 1 + v(x)$ would give homogeneous boundary conditions, then applying the code of upwind scheme for $\varepsilon = 10^{-2}$ and $n = 200$ we get 4.82×10^{-2} as a maximum error (See Figure 2.5).

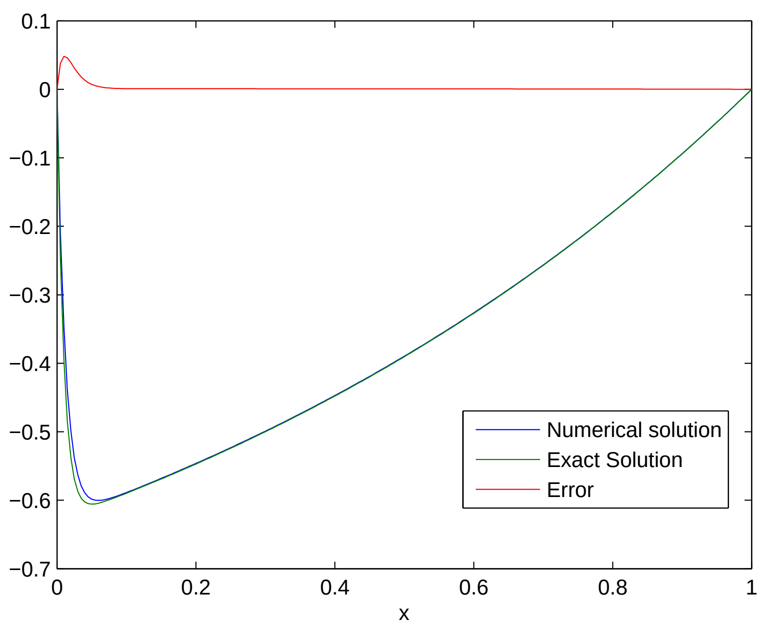

Figure 2.5: Comparing the numerical solutions with the exact solution of Example 2.3.1 using upwind scheme for $\varepsilon = 10^{-2}$ and $n = 200$ and showing error.

Chapter 3

The Collocation Method

In this chapter we will present the energetic Robin boundary functions method for solving the singularly perturbed ordinary differential equations (ODE) under the Robin boundary conditions.

3.1 Introduction and Homogenization

Let the following second order boundary value problem under the Robin type boundary conditions

$$\varepsilon u''(x) + p(x)u'(x) + q(x)u(x) = H(x), \quad 0 < x < 1, \tag{3.1a}$$

$$a_1 u(0) + b_1 u'(0) = c_1, \quad a_2 u(1) + b_2 u'(1) = c_2. \tag{3.1b}$$

where a_1 , b_1 satisfy $a_1^2 + b_1^2 > 0$, and a_2 , b_2 satisfy $a_2^2 + b_2^2 > 0$, c_1 , c_2 are given constants, and $[0, 1]$ is an interval of our problem. We suppose that $p(x)$, $q(x)$ and $H(x) \in C[0, 1]$ However, in many applications the independent variable t may be in an interval $[a, b]$, of which after taking the variable transform $x = (t - a)/(b - a)$ we have the problem in the interval $x \in [0, 1]$, again, and the ODE and the Robin boundary conditions should be adjusted accordingly. When $\varepsilon = 1$ we have the usual ODE, while for $0 < \varepsilon << 1$ we have a singularly perturbed ODE.

In the construction of the energy method, the first step is the homogenization technique, such that for the new variable

$$y(x) = u(x) - B_0(x), \tag{3.2}$$

the Robin boundary conditions are homogeneous. If $c_1^2 + c_2^2 = 0$ we can skip the following processes and go to the next section directly.

We divide the derivations of the homogenization function $B_0(x)$ into two parts.

Part One. if $a_1 = 0$, hence $b_1 \neq 0$, then we can derive

$$B_0(x) = a_0 x + b_0 x^{\nu}, \quad \nu \geqslant 2, \tag{3.3a}$$

$$a_0 = \frac{c_1}{b_1}, \tag{3.3b}$$

$$b_0 = \frac{b_1 c_2 - a_2 c_1 - b_2 c_1}{b_1 a_2 + b_1 b_2 \nu}. \tag{3.3c}$$

There are many values of ν such that $a_2 + b_2\nu \neq 0$ (hence, $b_1 a_2 + b_1 b_2 \nu \neq 0$) and one can choose it easily.

Part Two. if $a_1 \neq 0$, we can derive

$$B_0(x) = a_0 + b_0 x^{\nu}, \quad \nu \geqslant 2, \tag{3.4a}$$

$$a_0 = \frac{c_1}{a_1}, \tag{3.4b}$$

$$b_0 = \frac{a_1 c_2 - a_2 c_1}{a_1 a_2 + a_1 b_2 \nu}. \tag{3.4c}$$

There are many values of ν such that $a_2 + b_2\nu \neq 0$ (hence, $a_1 a_2 + a_1 b_2 \nu \neq 0$) and one can choose it easily.

The above function $B_0(x)$ includes a parameter ν. Let

$$B_0(x) = a_0 + b_0 x, \tag{3.5}$$

and through some derivations we can obtain

$$a_0 = \frac{c_1 b_2 + c_1 a_2 - c_2 b_1}{a_1 b_2 + a_1 a_2 - a_2 b_1}, \tag{3.6a}$$

$$b_0 = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 + a_1 a_2 - a_2 b_1}. \tag{3.6b}$$

In the case with $a_1 b_2 + a_1 a_2 - a_2 b_1 = 0$, we must employ the above (3.3) and (3.4) to set up the function $B_0(x)$.

Through the variable transformation (3.2), we obtain a new BVP with the homogeneous Robin boundary conditions

$$\varepsilon y''(x) + p(x)y'(x) + q(x)y(x) = F(x), \quad 0 < x < 1, \tag{3.7a}$$

$$a_1 y(0) + b_1 y'(0) = 0, \quad a_2 y(1) + b_2 y'(1) = 0. \tag{3.7b}$$

Such that

$$F(x) = H(x) - \varepsilon B_0''(x) - B_0'(x)p(x) - B_0(x)q(x).$$

3.2 Energetic Robin Boundary Functions Method

By multiplying both sides of (3.7a) by $y(x)$, integrating it from $x = 0$ to $x = 1$, one can derive

$$\int_0^1 \left[\varepsilon y''(x)y(x) + p(x)y'(x)y(x) + q(x)y^2(x)\right] \mathrm{d}x = \int_0^1 F(x)y(x)\mathrm{d}x. \tag{3.8}$$

If there exists an exact solution $y(x)$ of (3.7a) and (3.8), it must satisfy the above equation. The resulting equation is an energy equation and we will use it as a mathematical tool to solve $y(x)$.

The next step is searching the Robin boundary functions which automatically satisfy (3.8). In terms of polynomials we can derive

$$B_j(x) = 1 - \frac{a_1}{b_1}x + \frac{a_1b_2 + a_1a_2 - a_2b_1}{b_1a_2 + (j+1)b_1b_2}x^{j+1}, \quad j \geqslant 1 \text{ if } b_1 \neq 0, \tag{3.9a}$$

$$B_j(x) = x - \frac{a_2 + b_2}{a_2 + (j+1)b_2}x^{j+1}, \quad j \geqslant 1 \text{ if } b_1 = 0. \tag{3.9b}$$

For the homogeneous Robin boundary conditions in (3.8) we may encounter the case that there exists a positive integer j_0 such that $a_2 + (j_0 + 1)\, b_2 = 0$, for example, when $a_2 = 4$, $b_2 = -1$, $j_0 = 3$. With this situation we can skip this j_0 in (3.9a) and (3.9b) and they are modified to

$$B_j(x) = 1 - \frac{a_1}{b_1}x + \frac{a_1b_2 + a_1a_2 - a_2b_1}{b_1a_2 + (j+1)b_1b_2}x^{j+1}, \quad j \geqslant 1, j \neq j_0, \text{ if } b_1 \neq 0, \tag{3.10a}$$

$$B_j(x) = x - \frac{a_2 + b_2}{a_2 + (j+1)b_2}x^{j+1}, \quad j \geqslant 1, j \neq j_0, \text{ if } b_1 = 0. \tag{3.10b}$$

They are at least second-order polynomial functions which satisfy the following homogeneous Robin boundary conditions

$$a_1B_j(0) + b_1B_j'(0) = 0, a_2B_j(1) + b_2B_j'(1) = 0, \quad j \geqslant 1. \tag{3.11}$$

For a BVP if the boundary conditions make the coefficient preceding x_{j+1} be zero, then (3.9a) and (3.9b) are not applicable. For this case we can enrich the boundary functions by including other type functions.

From (3.9a) and (3.11) it is obvious that when $B_j(x)$ is a Robin boundary function, $\beta B_j(x), \beta \in \mathbb{R}$ is also a Robin boundary function, and when $B_j(x)$ and $B_k(x)$ are Robin boundary functions, $B_j(x) + B_k(x)$ is also a Robin boundary function. The Robin boundary functions are closed under scalar multiplication and addition. Therefore, the set of

$$\{B_j(x)\}, \quad j \geqslant 1, \tag{3.12}$$

and the zero element constitute a linear space of the Robin boundary functions, denoted by $\mathcal{B}$.

Let us now state the following result.

Theorem 3.2.1 *In the linear space $\mathcal{B}$ there exist Robin boundary functions*

$$E_j(x) = \gamma_j B_j(x), \quad j \geqslant 1, j \text{ not summed.} \tag{3.13}$$

Where

$$e_2 = \int_0^1 \left[\varepsilon B_j''(x)B_j(x) + p(x)B_j'(x)B_j(x) + q(x)B_j^2(x)\right] \mathrm{d}x, \tag{3.14a}$$

$$e_1 = \int_0^1 B_j(x)F(x)\mathrm{d}x, \tag{3.14b}$$

$$\gamma_j = \frac{e_1}{e_2}, \tag{3.14c}$$

are such that $E_j(x)$ satisfies the following energy integral equation

$$\int_0^1 \left[\varepsilon E_j''(x)E_j(x) + p(x)E_j'(x)E_j(x) + q(x)E_j^2(x)\right] \mathrm{d}x = \int_0^1 F(x)E_j(x)\mathrm{d}x. \tag{3.15}$$

Proof. See [11]. ■

The Robin boundary function $E_j(x)$ in (3.13) endowed with the multiplier γ_j in (3.14c) not only satisfies the homogeneous Robin boundary conditions but also preserves the energy in (3.15). The multiplier γ_j is determined by using the energy identity (3.15). Hence, $E_j(x)$ is an energetic Robin boundary function, and correspondingly the numerical method based on $E_j(x)$ is an energetic Robin boundary functions method (ERBFM).

3.2.1 Deriving the linear system by collocation method

The numerical procedure for solving $y(x)$ is given in the following form: to find the expansion coefficients c_j in

$$y(x) = \sum_{j=1}^{n} c_j s_j E_j(x), \quad \left[u(x) = B_0(x) + \sum_{j=1}^{n} c_j s_j E_j(x) \right], \tag{3.16}$$

where $E_j(x)$ acts as the basis in the numerical solution of $y(x)$. It can be seen that $y(x)$ in (3.16) automatically satisfies (3.7b).

Because the boundary conditions are automatically satisfied by (3.16), we only need to guarantee that the governing equation (3.7a) is satisfied. First we set $s_j = 1$. Inside the interval $(0, 1)$ we can collocate n_q points $x_i = i/(n_q + 1), i = 1, \ldots, n_q$, to satisfy (3.7a) by inserting (3.16) for $y(x)$, so that we have a linear system

$$\mathbf{A}\mathbf{c} = \mathbf{F}, \tag{3.17}$$

which can be used to determine the expansion coefficients $c := c_j$, whose number is n. In the above, the components of A and F are given, respectively, by

$$a_{ij} = \varepsilon E_j''(x_i) + p(x_i) E_j'(x_i) + q(x_i) E_j(x_i),$$

and

$$F_i = F(x_i).$$

The dimension of A is $n_q \times n$, and (3.17) is an over-determined system with $n_q > n$.

In general, the norms of the columns of the coefficient matrix A are not equal. If one asks the norms of the columns of the coefficient matrix of A to be equal, the multiple-scale s_j is determined by (3.8)

$$s_j = \frac{R_0}{\|\mathbf{a}_j\|}, \tag{3.18}$$

where a_j denotes the jth column of A in (3.17) and R_0 is a parameter. Hence, we have $||a_j|| = R_0$, $j = 1, \ldots, n$.

3.2.2 Normalized exponential trial functions

In the strong-form formulation of differential equations it is known that the selection of trial functions is very important, for which we suppose that the set of trial functions is complete, linearly independent, and satisfying the boundary conditions exactly. In general, the used polynomial basis is hard to match the singularity behaviour for the SPBVP. We will see a different set of trial functions which are used in [11] to treat the second-order singularly perturbed problems

$$\varphi_j(x) = \frac{\mathrm{e}^{jx} - 1}{\mathrm{e}^{j} - 1}, \quad \varphi_j(0) = 0, \quad \varphi_j(1) = 1, \tag{3.19a}$$

$$\varphi_0(x) = x, \varphi_0(0) = 0, \quad \varphi_0(1) = 1. \tag{3.19b}$$

To avoid the divergence of e^{jx}, we have introduced a normalized factor $e^j - 1$ in the denominator. Therefore, $\varphi_j(x)$ is a normalized exponential trial function. In order to let $\varphi_j(x)$ satisfy the homogeneous Robin boundary conditions we can derive

$$B_j(x) = 1 - \frac{x^2}{a_2 + 2b_2}\left[a_2 - \frac{a_1 a_2\left(\mathrm{e}^j - 1\right)}{j b_1} - \frac{a_1 b_2 \mathrm{e}^j}{b_1}\right] - \frac{a_1\left(\mathrm{e}^j - 1\right)}{j b_1}\varphi_j(x), \quad j \in \mathbb{Z} \text{ if } b_1 \neq 0, \tag{3.20}$$

$$B_j(x) = x^2 - \frac{(a_2 + 2b_2)\left(\mathrm{e}^j - 1\right)}{a_2\left(\mathrm{e}^j - 1\right) + j b_2 \mathrm{e}^j}\varphi_j(x), \quad j \in \mathbb{Z} \text{ if } b_1 = 0. \tag{3.21}$$

The special case $B_0(x)$ can be obtained by applying the L'Hospital rule to the above equations.

Then, by applying Theorem 3.2.1 to the above $B_j(x)$, we can derive the trial functions $E_j = \gamma_j B_j(x)$. We suppose that the solution $y(x)$ can be expanded by

$$y(x) = \sum_{j=-m_1}^{m_2} a_j s_j E_j(x), \quad \left[u(x) = B_0(x) + \sum_{j=-m_1}^{m_2} a_j s_j E_j(x) \right], \tag{3.22}$$

where $n = m_1 + m_2 + 1$ and the unknown coefficients a_j have to be determined.

3.3 Numerical Examples

In order to assess the performance of the newly developed ERBFM, We investigate the following examples of Lui and li [4].

Example 3.3.1 *We solve the Example 2.3.2, i.e.,*

$$\varepsilon u''(x) + u'(x) - u(x) = 0, \quad u(0) = 1, \quad u(1) = 1. \tag{3.23}$$

The exact solution is given by

$$u(x) = \frac{1}{\mathrm{e}^{p_2} - \mathrm{e}^{p_1}} \left[(\mathrm{e}^{p_2} - 1)\, \mathrm{e}^{p_1 x} + (1 - \mathrm{e}^{p_1})\, \mathrm{e}^{p_2 x} \right],$$

where

$$p_1 = \frac{-1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon}, \quad p_2 = \frac{-1 - \sqrt{1 + 4\varepsilon}}{2\varepsilon}.$$

We expand the solution $u(x)$ by (3.22). *Under the parameters $\varepsilon = 0.01$, $m_1 = 100$, $m_2 = 1$, $n_q = 200$, and $R_0 = 0.1$, we can find that the solution $u(x)$ is very close to the exact one with the maximum error being 8.09×10^{-8} as shown in Figure 3.1. Obviously, the maximum error is much smaller than that calculated by the FDM using the upwind* *scheme in the Example 2.3.2 for the same parameters and even though for a smaller stepsize.*

The method gives also much better results than calculated ones by Varner and Choudhury [12], and by Reddy and Chakravarthy [7], who used $h = 0.001$ *as a stepsize.*

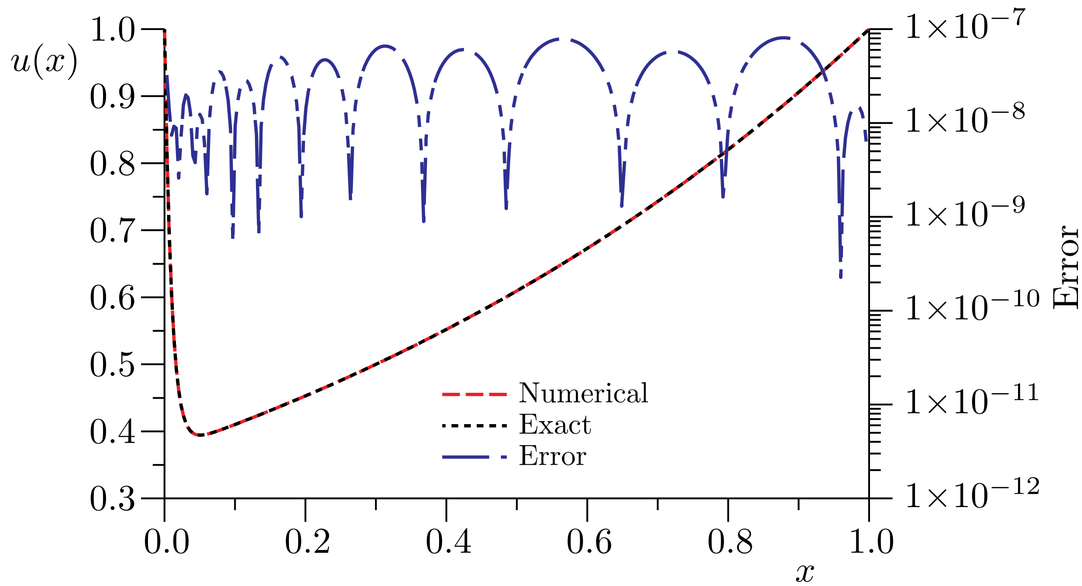

Figure 3.1: Comparing the numerical solutions of Example 3.3.1 obtained by the ERBFM with the exact solutions and showing the errors.

Example 3.3.2 *We revisit Example 3.3.1 again, let us consider the Robin type boundary conditions*

$$\varepsilon u''(x) + u'(x) - u(x) = 0,$$
$$u(0) + u'(0) = 1 + \frac{1}{e^{p_2} - e^{p_1}} \left[p_1 \left(e^{p_2} - 1 \right) + p_2 \left(1 - e^{p_1} \right) \right], \quad u(1) = 1,$$

where p_1 *and* p_2 *were defined in the previous example.*

Under the parameters $\varepsilon = 0.01$, $m_1 = 100$, $m_2 = 1$, $n_q = 300$, $\nu = 2$, *and* $R_0 = 1$, *we can find that* 5.53×10^{-7} *as a maximum error (See Figure 3.2), The accuracy is slightly worse than that in Example 3.3.1.*

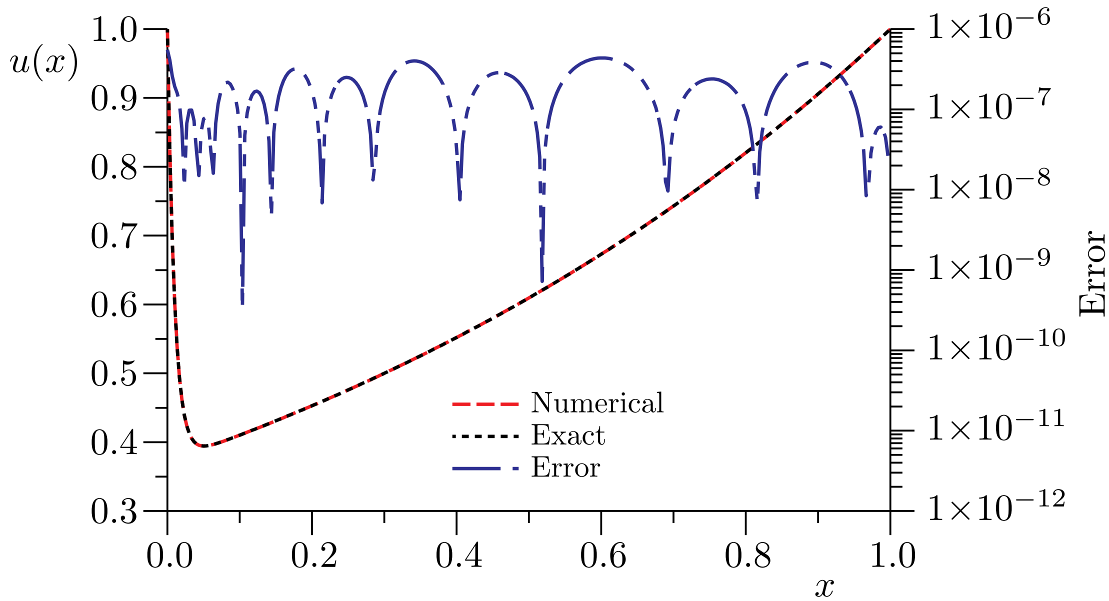

Figure 3.2: Comparing the numerical solutions of Example 3.3.2 obtained by the ERBFM with the exact solutions and showing the errors.

Conclusion

In this present work, we introduced two numerical methods to solve the singularly perturbed BVP, we saw that the collocation method based on energitic Robin boundary conditions is highly accurate and stable than the finite difference methods with classical and upwind schemes.

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Abstract

This work aims to study some numerical methods for the singularly perturbed boundary value problems, the method of finite difference has been applied with upwind schemes to reduce errors and Matlab implementation was established, we also apply the collocation method based on energitic Robin boundary functions with giving a numerical comparison on the same example between the used methods.

Key words. Singular perturbations, boundary layer, finite differences, collocation method.

ملخص

يهدف هذا العمل إلى دراسة بعض الطرق العددية لحل المسائل الحدية ذات الاضطرابات غير المنتظمة، وقد تمّ تطبيق طريقة الفروق المنتهية بمخطّطات عددية معدّلة من أجل تحسين الخطأ الناتج مع برمجتها في الماتلاب، كما تمّ تطبيق طريقة التجميع التي تعتمد على توابع الطاقة الحدية لروبين مع إجراء مقارنة عددية بين الطريقتين على نفس المثال.

الكلمات المفتاحية. الاضطرابات غير المنتظمة، الطبقة الحدودية، الفروق المنتهية، طريقة التجميع.

Résumé

Ce travail vise à étudier quelques méthodes numériques pour les problèmes aux limites singulièrement perturbés, la méthode de la différence finie a été appliquée avec des schémas décentré amont pour réduire les erreurs et l'implémentation sous Matlab a été établie, Nous avons également appliqué la méthode de collocation basée sur les fonctions énergitiques aux limites de Robin avec une comparaison numérique entre les deux méthodes sur le même exemple.

Mots clés. Perturbations singulières, couche limite, différences finies, méthode de collocation.