# Computation of Normal and Critical Depths in Parabolic Cross Sections

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**Abstract:** With a suitable non-dimensionalization, the available charts for normal depth computation in parabolic sections are reduced to a single dimensionless curve when either *Manning*'s or *Chezy*'s resistance equation is used. Moreover, simple approximate solutions are presented to compute normal depth with a quite adequate degree of accuracy. Upon rearranging the critical-flow criterion equation, relevant explicit relationships are derived for this state of flow. The procedure of normal and critical depths calculation is clearly described through some examples.

## 1. INTRODUCTION

Normal depth plays a significant role in the design of open channels and in the analysis of the non-uniform flow as well. Searching for earlier literature, one can find some methods of uniform flow computation. For the most part, the well known *Manning* and *Chezy* resistance equations are extensively used. Due to their implicit form, graphical methods have been presented in the past for uniform flow computation in the common rectangular, trapezoidal, triangular and circular cross sections [1-3]. For these, explicit solutions for normal depth have been proposed afterwards [4, 5]. The most relevant recent study is certainly that of *Swamee* and *Rathie* [6], in which exact analytical equations for normal depth have been reported for rectangular, trapezoidal and circular cross sections. For round-bottomed triangular, round-cornered rectangular and parabolic cross sections, exact or approximate solutions are not yet available. For these sections, graphical methods have been proposed by *Babaeyan-Koopaei* [7], using the *Manning*'s resistance equation. The form of the considered parabola is defined by:

$$Y = aX^2 \tag{1}$$

in which $a$ is the shape factor of the parabolic channel. By the use of Eq. (1), the top width $T$, water area $A$ and wetted perimeter $P$ are given in terms of the maximum channel depth $h$ and maximum permissible side slope $1/Z$ . These are expressed respectively as:

$$T = 4Zh \tag{2}$$

$$A = \frac{8}{3}Zh^2 \tag{3}$$

$$P = 2Z^2h\left[Z^{-1}\sqrt{1+Z^{-2}} + \ln\left(Z^{-1} + \sqrt{1+Z^{-2}}\right)\right] \tag{4}$$

Furthermore, from *Manning* equation, the following identities are deduced:

$$\frac{nQ}{T^{8/3}\sqrt{S}} = \frac{AR^{2/3}}{T^{8/3}} = K \tag{5}$$

where $n$ is the *Manning* roughness coefficient, $Q$ is the discharge, $S$ is the channel slope and $R$ is the hydraulic radius. Using Eqs. (2), (3) and (4), the parameter $K$ can be expressed as :

$$K = 3.23044(y/T)^{8/3}Z\left[\sqrt{1+Z^{-2}} + Z\ln\left(Z^{-1} + \sqrt{1+Z^{-2}}\right)\right]^{-2/3} \tag{6}$$

With the aid of Eqs. (5) and (6) dimensionless curves showing the variation of $y/T = f(K,Z)$ have been plotted. These permit the determination of the relative normal depth $y_n/T$ , provided $K$ and $Z$ are given.

This paper aims to present a new graphical solution for normal depth in parabolic sections by the only use of a single curve. This was obtained with the commonly *Manning* and *Chezy* resistance equations in which a new parameter is introduced. Furthermore, approximate solutions for normal depth are proposed and new relations for the critical flow characteristics are presented.

## 2. GEOMETRICAL CONSIDERATIONS

The parabola defined by Eq. (1) is represented in Fig. (**1**) for $X \geq 0$ , where $X$ is the longitudinal coordinate. Three points are particularly considered namely: $P(T_m/2, y_m)$ which is well defined by the geometrical elements $T_m$ and $y_m$ of the parabolic channel, $N(T_n/2, y_n)$ which is connected to the uniform flow characterized by the top width $T_n$ and the normal depth $y_n$ , and $E(T_o/2, y_o)$ which translate the fact that the top width $T_o$ is equal to the depth $y_o$ . Otherwise, the corresponding water area $A_o$ is inscribed in a square of length side $y_o = T_o$ such that the aspect ratio $\zeta_o = y_o/T_o = 1$ . This special case is one and only for a given parabola. For a slender parabola the point $E$ is located below the point $P$ corresponding

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to $y_o < y_m$, whereas $y_o > y_m$ for a widened or a much more opened parabola.

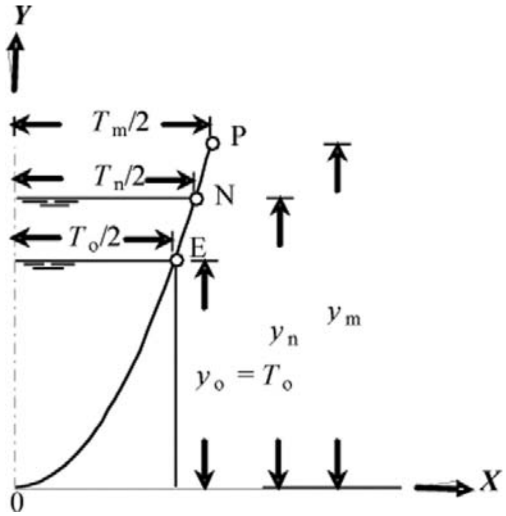


**Fig. (1).** Definition sketch of a parabolic channel flow.

For $P(T_m/2, y_m)$, Eq. (1) gives $y_m = a(T_m/2)^2$ or :

$$a = 4y_m / T_m^2 \tag{7}$$

Inserting Eq. (7) into Eq. (1), results in :

$$Y = \frac{4}{B}X^2 \tag{8}$$

in which :

$$B = T_m^2 / y_m \tag{9}$$

The parameter $B$ is thus a linear dimension and it is well defined by the geometric elements $T_m$ and $y_m$ of the parabolic channel.

For $N(T_n/2, y_n)$ and $E(T_o/2, T_o)$, Eq. (8) gives respectively:

$$B = T_n^2 / y_n \tag{10}$$

$$B = T_o \tag{11}$$

Equation (11) indicates that the parameter $B$ is also equal to the length side of the square defined in Fig. (**1**), such that $y_o = T_o = B$.

## 3. CHARACTERISTICS OF UNIFORM FLOW IN PARABOLIC SECTIONS

Assume $\zeta_n$ as the aspect ratio of the normal water area $A_n$, defined by:

$$\zeta_n = y_n / T_n \tag{12}$$

With the aid of Eqs. (10) and (12), $\zeta_n$ can be written as :

$$\zeta_n = T_n / B \tag{13}$$

Eliminating $T_n$ between Eqs. (10) and (12), $\zeta_n$ can be also written as :

$$\zeta_n = \sqrt{y_n / B} \tag{14}$$

The normal depth $y_n$ is thus:

$$y_n = B\zeta_n^2 \tag{15}$$

For a parabolic channel, the normal water area $A_n$ is given by:

$$A_n = \frac{2}{3}T_n y_n \tag{16}$$

which can be written, with the aid of Eqs. (13) and (15), as :

$$A_n = \frac{2}{3}B^2\zeta_n^3 \tag{17}$$

On the other hand, the normal wetted perimeter $P_n$ is given by:

$$P_n = \frac{T_n^2}{8y_n}\left[\sigma\sqrt{1+\sigma^2} + \ln\left(\sigma + \sqrt{1+\sigma^2}\right)\right] \tag{18}$$

in which :

$$\sigma = 4y_n / T_n \tag{19}$$

or :

$$\sigma = 4\zeta_n \tag{20}$$

Inserting Eqs. (10) and (20) into Eq. (18), yields :

$$P_n = \frac{B}{8}\left[4\zeta_n\sqrt{1+16\zeta_n^2} + \ln\left(4\zeta_n + \sqrt{1+16\zeta_n^2}\right)\right] \tag{21}$$

With the aid of Eqs. (17) and (21), the normal hydraulic radius $R_n$, defined as the ratio of the normal water area $A_n$ to the normal wetted perimeter $P_n$, is thus :

$$R_n = \frac{16}{3}\frac{B\zeta_n^3}{4\zeta_n\sqrt{1+16\zeta_n^2} + \ln\left(4\zeta_n + \sqrt{1+16\zeta_n^2}\right)} \tag{22}$$

## 4. NORMAL DEPTH COMPUTATION

### 4.1. Manning Equation

*Manning*'s equation is given by:

$$Q = \frac{1}{n}A_n R_n^{2/3}\sqrt{S} \tag{23}$$

which can be written, using Eqs. (17) and (22), as follows :

$$Q = \frac{\alpha}{n}B^{8/3}\sqrt{S}\,\psi(\zeta_n) \tag{24}$$

where $\alpha = \text{constant} = 4\times(2/3)^{5/3} \cong 2.035$ and $\psi(\zeta_n)$ is defined by :

$$\psi(\zeta_n)=\zeta_n^5\left[4\zeta_n\sqrt{1+16\zeta_n^2}+\ln\left(4\zeta_n+\sqrt{1+16\zeta_n^2}\right)\right]^{-2/3} \quad (25)$$

In terms of non-dimensional parameters, Eq. (24) is changed to :

$$M=\alpha\psi(\zeta_n) \quad (26)$$

in which :

$$M=\frac{nQ}{B^{8/3}\sqrt{S}} \quad (27)$$

Notice that $\zeta_n(M=0.29)=1$, corresponding to $y_n=T_n=B$. The implicit relationship (26) is plotted in Fig. (**2**) showing the relation between $\zeta_n$ and $\sqrt{M}$ .

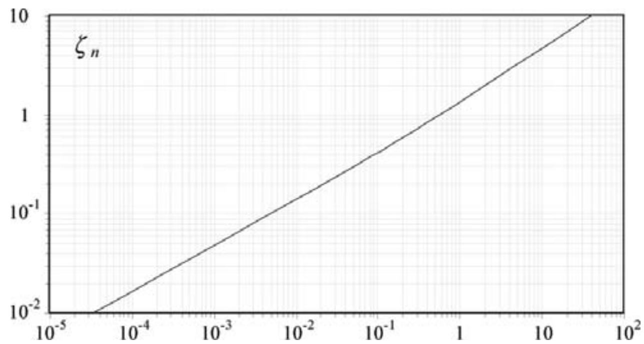


**Fig. (2).** Dimensionless curve for normal depth computation in parabolic cross sections, using *Manning* resistance equation.

The normal depth $y_n$ can be then evaluated with respect to the following four steps:

i. Knowing the geometric elements $T_m$ and $y_m$ of the considered parabolic channel, $B$ is then computed using Eq. (9).

ii. With $Q$, $B$, $S$ and $n$, Eq. (27) gives the value of $M$ and, hence, $\sqrt{M}$ .

iii. Using $\sqrt{M}$ , one can read from Fig. (**2**) the aspect ratio $\zeta_n$ .

iv. Knowing $\zeta_n$ and $B$, the normal depth $y_n$ follows then immediately from Eq. (15).

### *4.1.1. Approximate Solutions*

The aspect ratio $\zeta_n(M)$ of the implicit relation (26) can be reasonably written as the following power law:

$$\zeta_n=\vartheta M^{\delta} \quad (28)$$

The parameters $\vartheta$ and $\delta$ have been evaluated and reported in Table **1**, depending on the limitations of *M*. Also indicated is the maximum deviation between Eqs. (26) and (28). As it can be seen, Eq. (28) is quiet satisfactory for practical applications.

### *4.1.2. Example*

A parabolic channel with the geometric elements $T_m=5m$ and $y_m=2.5m$ , a slope $S=0.001$ and $n=0.025s/m^{1/3}$, carries a discharge $Q=5m^3/s$ . Compute the normal depth $y_n$ and the normal top width $T_n$ using *Manning*'s resistance equation.

i. According to Eq. (9), the parameter *B* is :

$B=T_m^2/y_m=5^2/2.5=10m$ . With respect to Eq. (8), the equation of the parabola is thus :

$$Y=(4/B)X^2=0.4X^2$$

ii. Applying Eq. (27), the parameter *M* is then :

$M=\dfrac{nQ}{B^{8/3}\sqrt{S}}=0.025\times5/(10^{8/3}\times\sqrt{0.001})\cong0.00851$, corresponding to $\sqrt{M}=0.0923$

i. With $\sqrt{M}=0.0923$ , one can read from Fig. (**2**) $\zeta_n\cong0.41$ . This can be obtained using Eq. (28), along with Table **1**, whence :

$$\zeta_n=1.35M^{0.25}=1.35\times0.00851^{0.25}=0.41$$

ii. From Eq. (15), the normal depth $y_n$ is thus :

$$y_n=B\zeta_n^2=10\times0.41^2\cong1.681m$$

iii. Eq. (13) gives the top width $T_n$ as :

$$T_n=B\zeta_n=10\times0.41=4.10m$$

## 4.2. Chezy Equation

*Chézy*'s resistance equation is given by:

$$Q=CA_n\sqrt{R_nS} \quad (29)$$

in which *C* is the *Chezy*'s roughness coefficient. With the aid of Eqs. (17) and (22), Eq. (29) can be written as :

**Table 1. Values of $\vartheta$ and $\delta$ for Computation of the Aspect Ratio $\zeta_n$ by Eq. (28)**

| M | $\zeta_n$ | $\vartheta$ | $\delta$ | Maximum deviation (%) |
|---|---|---|---|---|
| $M\le0.00114$ | $\zeta_n\le0.25$ | 1.197 | 0.232 | 0.7 |
| $0.00114<M\le0.29$ | $0.25<\zeta_n\le1$ | 1.350 | 0.250 | 1 |
| $0.29<M\le1484.6$ | $1<\zeta_n\le10$ | 1.388 | 0.270 | 0.6 |

$$Q = \beta C B^{5/2} \sqrt{S} \varphi(\zeta_n) \tag{30}$$

where $\beta$ = constant $= 8/(3\sqrt{3}) \cong 1.539$ and $\varphi(\zeta_n)$ is defined by :

$$\varphi(\zeta_n) = \zeta_n^{9/2} \left[ 4\zeta_n \sqrt{1+16\zeta_n^2} + \ln\left( 4\zeta_n + \sqrt{1+16\zeta_n^2} \right) \right]^{-1/2} \tag{31}$$

In terms of non-dimensional parameters, Eq. (30) can be written as follows :

$$C^* = \beta \varphi(\zeta_n) \tag{32}$$

in which :

$$C^* = \frac{Q}{C B^{5/2} \sqrt{S}} \tag{33}$$

Notice that $\zeta_n (C^* = 0.357) = 1$. The implicit relationship (32) is plotted in Fig. (**3**) showing the variation in $\zeta_n$ with $\sqrt{C^*}$ . The normal depth can be then graphically computed, following the same steps than those indicated in paragraph 4.1, once $\sqrt{C^*}$ is determined.

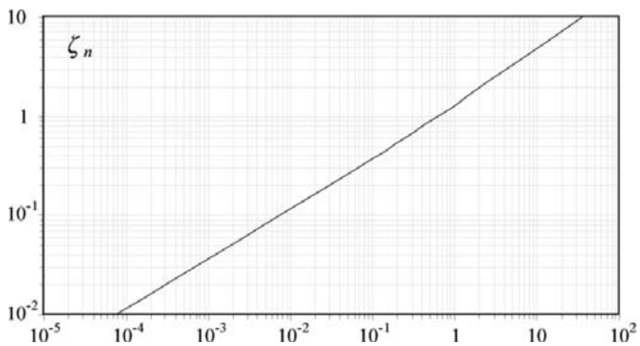


**Fig. (3).** Dimensionless curve for normal depth computation in parabolic cross sections, using *Chezy*'s resistance equation.

### *4.2.1. Approximate Solutions*

The aspect ratio $\zeta_n(C^*)$ of the implicit relationship (32) can be directly computed by the use of the following power law:

$$\zeta_n = \theta C^{*\gamma} \tag{34}$$

The numerical values of $\theta$ and $\gamma$ are reported in Table **2** with respect to the limitations of $C^*$. The maximum deviation between Eqs. (32) and (34) is also indicated. As it can be seen, the effect of errors involved in estimating the aspect ratio $\zeta_n$ for the determination of the normal depth $y_n$ by Eq. (34) is small. Consequently, one may consider Eq. (34) as a satisfactory approximate solution.

### *4.2.2. Example*

A parabolic channel with the geometric elements $T_m = 8m$ and $y_m = 4m$ , a slope $S = 0.002$ and $C = 85m^{0.5}/s$, carries a discharge $Q = 8m^3/s$. Compute the normal depth $y_n$ and the normal top width $T_n$ using *Chezy*'s resistance equation.

i. According to Eq. (9), the parameter $B$ is :

$$B = T_m^2 / y_m = 8^2/4 = 16m$$

ii. Using Eq. (33), $C^*$ is then :

$$C^* = \frac{Q}{C B^{5/2}\sqrt{S}} = 8/(85 \times 16^{5/2} \times \sqrt{0.002}) \cong 0.002055 ,$$ corresponding to $\sqrt{C^*} \cong 0.0453$

iii. With $\sqrt{C^*} = 0.0453$, one can read from Fig. (**3**) $\zeta_n \cong 0.25$ , which can be computed otherwise by the use of Eq. (34) along with Table **2**, whence :

$$\zeta_n = 1.310 C^{*0.251} = 1.310 \times 0.002055^{0.267} \cong 0.251$$

iv. From Eq. (15), the normal depth $y_n$ is then :

$$y_n = B\zeta_n^2 = 16 \times 0.251^2 = 1.008 \cong 1m$$

v. Eq. (13) gives the normal top width $T_n$ as :

$$T_n = B\zeta_n = 16 \times 0.251 = 4.016 \cong 4m$$

## 5. CRITICAL FLOW

### 5.1. Critical Flow Characteristics in Parabolic Sections

The well known criterion for critical flow states that:

$$\frac{Q^2 T_c}{g A_c^3} = 1 \tag{35}$$

where the subscript "*c*" denotes the condition of the critical state of flow and *g* is the acceleration due to gravity. Fur-

**Table 2. Values of $\theta$ and $\gamma$ for Computation of the Aspect Ratio $\zeta_n$ by Eq. (34)**

| $C^*$ | $\zeta_n$ | $\theta$ | $\gamma$ | Maximum deviation (%) |
|---|---|---|---|---|
| $C^* \le 0.00198$ | $\zeta_n \le 0.25$ | 1.187 | 0.251 | 0.7 |
| $0.00198 < C^* \le 0.357$ | $0.25 < \zeta_n \le 1$ | 1.310 | 0.267 | 1 |
| $0.357 < C^* \le 1215.3$ | $1 < \zeta_n \le 10$ | 1.329 | 0.284 | 0.6 |

thermore, from Eq. (10), the critical top width $T_c$ is then :

$$T_c = \sqrt{By_c} \tag{36}$$

The water area $A_c$ is given by :

$$A_c = \frac{2}{3} T_c y_c$$

which can be rewritten with the aid of Eq. (36) as :

$$A_c = \frac{2}{3} y_c^{3/2} \sqrt{B} \tag{37}$$

Inserting Eqs. (36) and (37) into Eq. (35), results in :

$$\frac{Q^2 \sqrt{By_c}}{g y_c^{9/2} B^{3/2}} = (2/3)^3$$

whence :

$$y_c = \sqrt[4]{\frac{27Q^2}{8gB}} \tag{38}$$

Knowing the discharge *Q* and the parameter *B*, Eq. (38) permits then a direct determination of the critical depth $y_c$, bearing in mind that *B* is well defined by the geometric elements $T_m$ and $y_m$ of the parabolic channel according to Eq. (9). Furthermore, eliminating $y_c$ between Eqs. (36) and (38), the critical top width $T_c$ is explicitly expressed as :

$$T_c = \left( \frac{27B^3Q^2}{8g} \right)^{1/8} \tag{39}$$

On the other hand, the aspect ratio $\zeta_c$ is given by Eq. (14) for $y_n = y_c$, whence :

$$\zeta_c = \sqrt{\frac{y_c}{B}} \tag{40}$$

Eliminating $y_c$ between Eqs. (38) and (40), yields :

$$\zeta_c = \left( \frac{27Q^2}{8gB^5} \right)^{1/8} \tag{41}$$

As it can be seen, the aspect ratio $\zeta_c$ is then well defined by the discharge *Q* and the parameter *B*. Consequently, the water area $A_c$ and the hydraulic radius $R_c$ can be deduced from Eqs. (17) and (22) respectively, when substituting $\zeta_n$ by $\zeta_c$. Furthermore, the critical slope $S_c$ is thus explicitly evaluated using *Manning*'s or *Chezy*'s equation, provided *n* or *C* is given.

### 5.2. Example

A parabolic channel with the geometric elements $T_m = 4m$ and $y_m = 2m$, a *Chezy*'s roughness coefficient $C = 85m^{0.5}/s$, carries a discharge $Q = 8m^3/s$. Compute the critical depth $y_c$ and the critical slope $S_c$.

i. According to Eq. (9), the parameter *B* is :

$$B = T_m^2 / y_m = 4^2/2 = 8m$$

ii. With *Q* and *B*, Eq. (38) gives :

$$y_c = \sqrt[4]{\frac{27Q^2}{8gB}} = \left( \frac{27 \times 8^2}{8 \times 9.81 \times 8} \right)^{1/4} = \left( \frac{27}{9.81} \right)^{1/4} \cong 1.288m$$

iii. The aspect ratio $\zeta_c$ can be computed using Eq. (40) or Eq. (41), whence :

$$\zeta_c = \sqrt{\frac{y_c}{B}} = \sqrt{1.288/8} \cong 0.40125$$

iv. According to Eqs. (17) and (22), the water area $A_c$ and the hydraulic radius $R_c$ are respectively

$$A_c = \frac{2}{3} B^2 \zeta_c^3 = \frac{2}{3} \times 8^2 \times 0.40125^3 \cong 2.756m^2$$

$$R_c = \frac{16}{3} \frac{B\zeta_c^3}{4\zeta_c \sqrt{1+16\zeta_c^2} + \ln(4\zeta_c + \sqrt{1+16\zeta_c^2})} =$$

$$\frac{16}{3} \times \frac{8 \times 0.401^3}{4 \times 0.401 \times \sqrt{1+16 \times 0.401^2} + \ln(4 \times 0.401 + \sqrt{1+16 \times 0.401^2}} \cong 0.642m$$

From the given data and using Eq. (29) under critical flow condition, the critical slope is :

$$S_c = \frac{Q^2}{C^2 A_c^2 R_c} = \frac{8^2}{85^2 \times 2.756^2 \times 0.642} \cong 0.00181$$

v. This step aims to verify Eq. (35) with the computed data, once the top width $T_c$ is determined. According to Eq. (13), $T_c$ is :

$$T_c = B\zeta_c = 8 \times 0.40125 = 3.21m$$

Inserting the values of *Q*, $A_c$ and $T_c$ in Eq. (35), results in:

$$\frac{Q^2 T_c}{g A_c^3} = \frac{8^2 \times 3.21}{9.81 \times 2.756^3} = 1.00041 \cong 1$$

## 6. CONCLUSION

A single dimensionless curve for normal depth computation in parabolic sections is obtained by an appropriate non-dimensionalization of both *Manning*'s and *Chezy*'s resistance equation. This is possible with the aid of the new parameter *B* which is well defined by the geometric elements $T_m$ and $y_m$ of the channel according to Eq. (9). When *Manning*'s equation is used, the non-dimensional parameter *M* [Eq. (27)] permits a graphical determination of the aspect ratio

$\zeta_n$ by reading Fig. (**2**). The same procedure is to be reproduced on Fig. (**3**) with *Chezy*'s equation, once the non-dimensional parameter $C^*$ is determined from Eq. (33). In spite of the fact that Figs. (**2** and **3**) can be used without incurring any tangible error, approximate solutions for a direct computation of $\zeta_n$ are proposed and presented under a simple power law form [Eqs. (28) and (34)]. These are circumstantially reported in Tables **1** and **2** from which one can observe the sufficient degree of their accuracy depending, however, on the limitations of $M$ and $C^*$. The parameter $B$ plays also a significant role in critical flow computation, leading to explicit and simple formulae. In addition to critical slope, the flow characteristics such as critical depth, critical top width and critical aspect ratio can be directly computed using Eqs. (38), (39) and (41) respectively. Some examples are taken to explain the procedure of calculation.

## NOTATION

$A_c$ = critical water area

$A_n$ = normal water area

$B$ = geometric element equal to $T_m^2 / y_m$

$C$ = *Chezy*'s roughness coefficient

$C^*$ = dimensionless parameter equal to $C^* = Q/(CB^{5/2}\sqrt{S})$

$g$ = acceleration due to gravity

$M$ = dimensionless parameter equal to $M = nQ/(B^{8/3}\sqrt{S})$

$n$ = *Manning*'s roughness coefficient

$P_n$ = normal wetted perimeter

$Q$ = discharge

$R_n$ = normal hydraulic radius equal to $A_n / P_n$

$S$ = channel bed slope

$S_c$ = critical slope

$T_c$ = top width at the critical depth

$T_m$ = top width of a parabolic channel

$T_n$ = top width at the normal depth

$X$ = longitudinal coordinate

$y_c$ = critical depth

$y_m$ = height of a parabolic channel

$y_n$ = normal depth

$\zeta_n$ = aspect ratio of normal water area

$\zeta_c$ = aspect ratio of critical area

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