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Applied Mathematics

SUBJECT OF THE THESIS:

Existence and asymptotic behavior of solutions for some hyperbolic equations with time delay

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Abstract

The thesis aims to provide the reader with how to use the most popular method for studying the existence and uniqueness of the solution and the general energy decay of some wave problems with strong delay and distributed delay, similar to the Kirchhoff system and the Lamé system. The first chapter deals with introducing some basic notions in bounded and unbounded operators and some main theorems in functional analysis. In second chapter, we proved the well-posedness and an exponential decay result under a suitable assumptions on the weight of the damping and the weight of the delay for a wave equation with a strong damping and a strong constant (respectively, distributed) delay. Finally "third and forth chapters", we proved the global existence of Kirchhoff's coupled system and general decay for the coupled system of Kirchhoff and Lamé with a distributed term delay.

Key words : Distributed delay term, Global existence, General Decay, Lyapunov functional, Strong delay, Viscoelastic term, Wave equations.

Résumé

Le but de cette thèse est de fournir au lecteur comment utiliser la méthode la plus populaire pour étudier l'existence et l'unicité de la solution et la décroissance générale d'énergie de certains problèmes d'ondes à fort retard et retard distribué, similaire au système de Kirchhoff et de Lamé. Dans le premier chapitre, nous avons introduit quelques notions de base sur les opérateurs bornés et non bornés et quelques théorèmes principaux en analyse fonctionnelle. Dans le deuxième chapitre, nous avons prouvé l'existence et l'unicité avec un résultat de décroissance exponentielle sous des hypothèses appropriées sur le poids de l'amortissement et le poids du retard pour une équation d'onde avec un fort amortissement et un fort retard constant (respectivement, distribué). Finalement "troisième et quatrième chapitres", nous avons prouvé l'existence globale du système couplé de Kirchhoff et la décroissance générale pour le système couplé de Kirchhoff et de Lamé avec un terme retard distribué.

Mots clés : Existence globale, Fonctionnelle de Lyapunov, Décomposition générale, Équations des ondes, Retard fort, Terme de retard distribué, Terme viscoélastique.

ملخص

الهدف من هذه الأطروحة هو تزويد القارئ بكيفية إستخدام الطريقة الأكثر شيوعًا لدراسة وجود ووحدانية الحل والانحلال العام لبعض مشاكل الموجة مع تأخير شديد وتأخر موزع، على غرار نظام كيرشوف ونظام لامي. في الفصل الأول، قدمنا بعض المفاهيم الأساسية في المؤثرات المحدودة وغير المحدودة وبعض النظريات الرئيسية في التحليل الدالي. في الفصل الثاني، أثبتنا وجود ووحدانية الحل نتيجة الإستقرار الأسي في ظل وجود فرضيات مناسبة على وزن التخميد ووزن التأخير لمعادلة موجية مع تخميد قوي و تأخير قوي ثابت (على التوالي ، موزع). في الفصلين الثالث والرابع ، أثبتنا الوجود العام والانحلال العام للنظام المزدوج لكيرشوف ولامي مع تأخير موزع.

الكلمات المفتاحية: الوجود العام، دالة ليبونوف، التحلل العام، معادلات الموجة، التأخير القوي، التأخير الموزع، المرونة اللزجة.

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Dedication

I dedicate this work to:
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My dear husband DEBBAR Mouhammed Ali, who has encouraged me all the way and taken with me charge of this study
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I dedicate as well :

To the whole of my dear family .
To all the teachers of Department od Mathematics.
To all my friends.

Doudi Nadjet

Contents

Notations

$\mathbb{N}$	:	The set of natural numbers.
$\mathbb{R}$	:	The set of real numbers.
$\mathbb{R}_+$	:	The set of positive real numbers.
$\mathbb{C}$	:	The set of complex numbers.
$(X, \|.\|_X)$	:	Banach space.
H	:	Hibert space.
$\|.\|_X$	:	Norme in X.
$\langle .,. \rangle_H$	:	The inner product in H.
$\text{Re}\,\langle .,. \rangle_H$	:	Real of the inner product in H.
$\mathcal{L}(X,Y)$	:	Space of bounded linear operators from X into Y.
$\mathcal{L}(X)$	:	Space of bounded linear operators from X into X.
A	:	Linear operator.
$\mathcal{D}(A)$	:	The domaine of A.
$\text{Ran}(A)$	:	The range of A.
$\text{Ker}(A)$	:	The kernel of A.
$\text{G}(A)$	:	The graph of A.
$\rho(A)$	:	The resolvent set of A.
$\text{R}(\lambda, A)$	:	The resolvent set of A at point λ.
$\sigma(A)$	:	The spectrum of A.
$\text{Im}(A)$	:	Image of A.
$(T(t))_{t>0}$	:	Semigroup of operators.
A_λ	:	The Yosida approximation of A
J_λ	:	The resolvent set of A.
I	:	Identity application.

Ω	:	A bounded and regular domain of $\mathbb{R}^n$.
$\partial\Omega$	:	Boundary of Ω.
$span\{u_1, u_2, ..., u_n\}$	:	Space spanned by $u_1, u_2, ..., u_n$.
Δ	:	The Laplacien operator.
$\dfrac{\partial u}{\partial t}, u_t$	:	The partial derivative of u with respect to t.

Introduction

Time delays arise in many applications since, in most instances, physical, chemical, biological, thermal, and economic phenomena naturally not only depend on the present state but also on some past occurrences, see [68], [62], [26], [38], and [18] and examples therein. Recently, the control of partial differential equations with time delay effects has become an active and attractive area of research. In fact, in many cases it was shown that delay can be a source of instability and even an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used.

The aim of this thesis is to study the stability of solutions, the energy decay and the existence of solution to some wave problems with delay term or distributed delay term. In fact, we prove that under some assumptions on the parameters in the systems and on the size of the initial data, the existence of solutions can be proved by using Hille-Yosida theorem or Faedo-Galerkin method and the general decay of solutions using the appropriate Lyapunov functionals, by considering the following three problems:

- A wave equation with strong damping and strong delay.
- Coupled kirchhoff system with a distributed delay term.
- Coupled Lamé system with viscoelastic dampings and distributed delay.

More precisely, we will consider at first a wave equation with a strong damping and a strong constant (respectively, distributed) delay. As mentioned earlier, this delay can be a source of instability. For instance, Datko in [19] showed that the time delay in the velocity term can destabilize the system

$$\begin{cases} u_{tt}(x,t) = u_{xx} - 2au_t\left(x, t-\tau\right), & \text{in } (0,1)\times(0,+\infty),\\ u(0,t) = u(1,t) = 0, & t\in(-\tau,+\infty),\\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } (0,1). \end{cases} \tag{1}$$

Datko et al. [21] obtained the same result by replacing the internal delay in equation (1) by a time delay in the boundary feedback control. However, the system without delay is uniformly asymptotically stable [16]. In [20], Datko presented two examples of hyperbolic partial differential equations which are destabilized by small time delays in the boundary feedback controls.

In the n-dimensional case, it is well-known that the problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a_0 u_t(x,t) + a u_t(x,t-\tau) = 0, & \text{in} \quad \Omega \times (0,+\infty), \\ u(x,t) = 0, & \text{on} \quad \Gamma_0 \times (0,+\infty), \\ \frac{\partial u}{\partial v}(x,t) = 0, & \text{on} \quad \Gamma_1 \times (0,+\infty), \end{cases} \tag{2}$$

is exponentially stable in the absence of delay ($a = 0, a_0 > 0$). See [76] and [42]. In the presence of delay ($a > 0$), Nicaise and Pignotti [59] examined system equation (2) and proved, under the assumption that the weight of the feedback is larger than the weight of the delay ($a < a_0$), that the energy is exponentially stable. However, in the opposite case, they could produce a sequence of delays for which the corresponding solution is instable. The same results were obtained for the case of boundary delay. See also [2] for the treatment of this problem in more general abstract form and [55], [60] and [61] for analogous results in the case of time-varying delay. When the delay term in equation (2) is replaced by the distributed delay

$$\int_{\tau_1}^{\tau_2} a(s) u_t(x,t-s)\,ds,$$

exponential stability results have been obtained in [58] under the condition

$$\int_{\tau_1}^{\tau_2} a(s)\,ds < a_0.$$

For coupled systems in thermoelasticity Racke [66] considered the following system

$$\begin{cases} u_{tt}(x,t) - a u_{xx}(x,t-\tau) + b\theta_x(x,t) = 0, & \text{in}\,(0,L) \times (0,\infty), \\ \theta_t(x,t) - d\theta_{xx}(x,t) + b u_{tx}(x,t) = 0, & \text{in}\,(0,L) \times (0,\infty), \end{cases} \tag{3}$$

where u is the transversal displacement and θ is the difference of temperature of a beam of length L, and proved that the internal time delay leads to an ill-posedness system. However, the system without delay is exponentially stable [57, 33]. Mustafa and Kafini [56] dropped the time delay in the harmonic term of the elastic equation of system (3) and added a distributed delay term of the form $\int_{\tau_1}^{\tau_2} \mu(s)\theta_{xx}(x,t-s)\,ds$ in the heat equation and established an exponential decay result under the assumption

$$\int_{\tau_1}^{\tau_2} \mu(s)\,ds < d.$$

The time delay in Timoshenko type systems of the form

$$\begin{cases} \rho_1 u_{tt} - K(u_x + \psi)_x = 0, & \text{in } (0, L) \times (0, \infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(u_x + \psi) + \mu_1 \psi_t(x, t) + \mu_2 \psi_t(x, t - \tau) = 0, & \text{in } (0, L) \times (0, \infty), \end{cases}$$

was proposed by Said Houari and Laskri [69], where u is the transversal displacement and ψ is the rotational angle of filament. They obtained an exponential decay result under the assumption $\mu_2 < \mu_1$. Kirane *et al.* [36] extended this result to time varying delay term. Kafini *et al.* [34] examined a coupling Timoskenko-thermoelasticity type III system with time delay and established exponential and polynomial stability results depending on the wave propagation speeds. Recently Apalara [6] considered the following Timoshenko type system with thermoelasticity of second sound, in the presence of distributed delay

$$\begin{cases} \rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi)_x + \mu\varphi_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\, \varphi_t(x, t - s)\, ds = 0, & \text{in } (0, 1) \times (0, \infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, & \text{in } (0, 1) \times (0, \infty), \\ \rho_3 \theta_t + q_x + \delta\psi_{tx} = 0, & \text{in } (0, 1) \times (0, \infty), \\ \tau q_t + \beta q + \theta_x = 0, & \text{in } (0, 1) \times (0, \infty), \end{cases} \tag{4}$$

and proved an exponential decay result under the assumption

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds < \mu.$$

System (4) in the absence of frictional damping ($\mu = 0$) and with the presence of constant delay instead of distributed delay, was investigated by Apalara and Messaoudi [5], and an exponential stability was established under a smallness condition on the delay.

In this thesis, we consider the following problem with strong damping and strong delay

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) - \mu_1 \Delta u_t(x,t) - \mu_2 \Delta u_t(x, t - \tau) = 0, & \text{in } \Omega \times (0, \infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0, \infty), \\ u_t(x, t - \tau) = f_0(x, t - \tau), & t \in (0, \tau), \\ u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{5}$$

where Ω is a bounded and regular domain of $\mathbb{R}^n$, $\tau > 0$ represents the time delay, μ_1, μ_2 are real numbers such that $|\mu_2| < \mu_1$ and u_0, u_1, f_0 are given data. Our equation can be regarded as a Kelvin-Voight linear model for a viscoelastic material in the presence of a delay response. In such a model, we have a spring and two dashpots in parallel whose total stress is given by the following stress-strain relation

$$\sigma(x,t)=\varepsilon(x,t)+\mu_1\frac{d\varepsilon(x,t)}{dt}+\mu_2\frac{d\varepsilon(x,t-\tau)}{dt}$$

where ε is the strain and τ is the delay time, and $\mu_1>0$, μ_2 are real parameters. For small deformations, the strain is proportional to the deformation gradient $\varepsilon=\kappa\nabla u$. Hence, by substituting in the motion equation

$$u_{tt}(x,t)=\text{div }\sigma(x,t),$$

we obtain our equation. We refer the reader to [64] for application of the model when $\mu_2=0$ in biological tissues. To the best of our knowledge, this problem has not been discussed before. In the second section, the constant delay term in (5) is replaced by the distributed delay term of the form $-\int_{\tau_1}^{\tau_2}\mu_2(s)\Delta u_t(x,t-s)ds$, where $\mu_2:[\tau_1,\tau_2]\to\mathbb{R}$ is a bounded function and $\tau_1<\tau_2$ are two positive constants. We establish well-posedness and exponential decay results under suitable conditions on the weights of the constant (respectively, distributed) delay and the weight of the damping terms.

After that, we are interested in the following nonlinear viscoelastic Kirchhoff system with a distributed delay and general coupling terms

$$\begin{cases} |u_t|^l u_{tt}-M(\|\nabla u\|^2)\Delta u-\Delta u_{tt}+\displaystyle\int_0^t g_1(t-s)\Delta u(s)ds \\ \quad -k_1\Delta u_t-\displaystyle\int_{\tau_1}^{\tau_2}\mu_1(\varrho)\Delta u_t(x,t-\varrho)d\varrho+\alpha v=0, \\ |v_t|^l v_{tt}-M(\|\nabla v\|^2)\Delta v-\Delta v_{tt}+\displaystyle\int_0^t g_2(t-s)\Delta v(s)ds \\ \quad -k_2\Delta v_t-\displaystyle\int_{\tau_1}^{\tau_2}\mu_2(\varrho)\Delta v_t(x,t-\varrho)d\varrho+\alpha u=0, \end{cases} \tag{6}$$

where

$$(x,\varrho,t)\in\Omega\times(\tau_1,\tau_2)\times(0,\infty).$$

With the initial data and boundary conditions

$$\begin{cases} (u(x,0),v(x,0))=(u_0(x),v_0(x)), & \text{in } \Omega, \\ (u_t(x,0),v_t(x,0))=(u_1(x),v_1(x)), & \text{in } \Omega, \\ (u_t(x,-t),v_t(x,-t))=(f_0(x,t),g_0(x,t)), & \text{in } \Omega\times(0,\tau_2), \\ u(x,t)=v(x,t)=0, & \text{on } \partial\Omega\times(0,\infty), \end{cases} \tag{7}$$

where Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, l,α is positive constants. The second integral represents the distributed delay and $\mu_1,\mu_2:[\tau_1,\tau_2]\to\mathbb{R}$ are a bounded functions, where τ_1,τ_2 are two real numbers satisfying $0\le\tau_1<\tau_2$, and g_1,g_2 are the relaxation functions. M is a smooth function defined by

$$M \; : \; \mathbb{R}_+ \to \mathbb{R}_+ \tag{8}$$

$$r \mapsto M(r) = a + br^{\gamma} \tag{9}$$

with $a, b > 0$. In 1876 Kirchhoff proposed an equation named after him, which is a generalization of the D'Alembert equation, as it belongs to the wave equation models, which describe the transverse vibration of a chain fixed at its end. The problem we are studying is a description of the axially moving viscoelastic that consist of two heterogeneous wires (such as electrical wires) that will have an effect on their movements, especially acceleration, and many authors have addressed these issues, including ([49],[50]), where the proof of the global existence and the stability of solutions were established. The study of wave equations with the delay in the last period forms a fertile and active region (see [17], [30], [31], [32], [46], [67], [73] and [74]). As the delay appears in many different physical, chemical, biological, and engineering fields, where there is a time difference that can affect the stability of the system. Recently, Mezouar and Boulaaras [47] have studied the viscoelastic non-degenerate Kirchhoff equation with varying delay term in the internal feedback, and in [48] the authors considered the generalized coupled non-degenerate Kirchhoff system with a time varying delay term, they proved the global existence of solutions and they showed the exponential stability result.

Finally, we proceeded to the study of another type of equation, we are concerned with studying the polynomial decay rate of the following Lamé system in $\Omega \times \mathbb{R}_+$

$$\begin{cases} u_{tt} - \Delta_e u + \displaystyle\int_0^t g_1(t-s)\,\Delta u(s)\,ds - k_1 \Delta u_t - \int_{\tau_1}^{\tau_2} \mu_1(\varrho)\Delta u_t(x, t-\varrho)d\varrho = f_1(u, v), \\ v_{tt} - \Delta_e v + \displaystyle\int_0^t g_2(t-s)\,\Delta v(s)\,ds - k_2 \Delta v_t - \int_{\tau_1}^{\tau_2} \mu_2(\varrho)\Delta v_t(x, t-\varrho)d\varrho = f_2(u, v). \end{cases} \tag{10}$$

Equations (10) are associated with the following boundary and initial conditions

$$\begin{cases} u(x,t) = v(x,t) = 0, \text{ on } \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), x \in \Omega, \\ (u_t(x,-t), v_t(x,-t)) = (f_0(x,t), g_0(x,t)), \text{ in } \Omega \times (0, \tau_2), \end{cases} \tag{11}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n = 1, 2, 3)$, with smooth boundary $\partial\Omega$. The elasticity differential operator Δ_e is given by

$$\Delta_e u = \mu \Delta u + (\mu + \lambda) \nabla (\text{div } u),$$

and the Lamé constants μ and λ are satisfying the following conditions

$$\mu > 0, \mu + \lambda > 0.$$

The parameters k_1, k_2, τ_1 and τ_2 are positive constants with $\tau_1 < \tau_2$. The functions μ_1, $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ are bounded. The functions $f_1(u, v)$ and $f_2(u, v)$ which represent the source terms will be specified later. After several authors have studied the problems of coupled systems and hyperbolic systems, their stability is associated with velocities and are proven under conditions imposed on the subgroup [48]. The researchers also studied behavior of the energy in a limited field with non-linear damping and external force, and a varying delay of time to find solutions to the Lamé system [8, 11].

Recently, problems that contain viscoelasticity have been addressed, and many results have been found regarding the global existence and stability of solutions (see [8, 11]), under conditions on the relaxation function, whether exponential or polynomial decay. In addition, in [13], Boulaaras obtained the stability result of the global solution to the Lamé system with the flexible viscous by adding logarithmic nonlinearity, even though the kernel is not necessarily decreasing in contrast to what he studied [8].

Introducing a distributed delay term makes our problem different from those considered so far in the literature. The importance of this term appears in many works and this is due to the fact that many phenomena depends on their past. Also, it is influence on the asymptotic behavior of the solution for the different types of problems such that Timoshenko system ([28], [35], [4],[6]), transmission problem [43], wave equation [59], thermoelastic system ([51],[56]).

At the end of this introduction, this thesis is organized as follows:

First chapter

The first is introductory chapter which include some basic notions in bounded and unbounded operators and some main theorems in functional analysis.

Second chapter

This chapter is the subject of publication in Journal of Mathematical physics [44]. The contents of this chapter is organized as follows. In section 2, we present the semigroup setting of the wave problem (5) with constant delay and establish the existence of a unique solution. In section 3, we show that the energy associated to the solution of problem (5) decays exponentially. In section 4, we give a short proof of the well-posedness of the problem with distributed delay and establish the exponential decay of the solution.

Third chapter

In this chapter, we extend the result obtained by Mezouar and Boulaaras in [48] for a coupled system (6). We have added the term of distributed delay and established the global existence and stability of the solution under conditions on the kernel. This work has been published in the Journal of Revista de la Real Academia de Ciencias Exactas RACSAM [23].

The outline of this chapter is the following: In the second section, some hypotheses related to the problem are given and we state our main result. Then in section 3, we prove the global existence of weak solutions. Finally, in the last section, we give the stability result.

Fourth chapter

In this chapter, we extend the general decay result obtained by Baowei Feng in [29] to the case of distributed term delay, namely, we will make sure that the result is achieved if the distributed delay term exists. This result has been published in Journal of Function Spaces (Hindawi) [24].

Chapter 1

Preliminary

In this chapter, we introduce some basic notions in bounded and unbounded operators and some main theorems in functional analysis. As the semi-groups theory, Lax-Milgram theorem and Faedo-Glerkin method. We are not trying to give a complete development, but rather review the basic definitions and theorems, mostly without proof. We refer to [10], [65], [14], [15], [9], [41], [25], [40], [75], [22].

1.1 Bounded and unbounded linear operators in Banach spaces

Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be two Banach spaces over $\mathbb{C}$ [65], [9].

Definition 1. *A linear operator $A : X \to Y$ is a transformation which maps linearly X in Y, that is*

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y), \forall x, y \in X \text{ and } \alpha, \beta \in \mathbb{C}.$$

Definition 2. *A linear operator $A : X \to Y$ is* ***bounded*** *if there exists $C \geq 0$, such that*

$$\|Au\|_Y \leq C \|u\|_X \,, \ \forall u \in X.$$

That is, we say that A is bounded if $\|Au\|_Y$ remains bounded when $u \in \{x \in X, \|x\|_X \leq 1\}$. Otherwise, A is said to be ***unbounded****.*

Then, we cet

$$\|A\|_{\mathcal{L}(X,Y)} = \sup_{x \in X, x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X}, \ \forall x \in X,$$

where $\mathcal{L}(X,Y)$ is the set of all bounded linear operators from X into Y. Moreover, the set of all bounded linear operators from X into X is denoted by $\mathcal{L}(X)$.

The preceding inequality holds with $C = \|A\|$ (the optimal constant).

Definition 3. *A linear operator A from X into Y is a pair $(A, \mathcal{D}(A))$, where $\mathcal{D}(A)$ is a linear manifold of X and A is a linear map from $\mathcal{D}(A) \subset X$ to Y. $\mathcal{D}(A)$ is called the domain of A.*

One says that the operator $(A, \mathcal{D}(A))$ is densely defined if $\mathcal{D}(A)$ is dense in X.

For such an operator, its range $Ran(A)$ is defined by

$$Ran(A) := A\left(\mathcal{D}(A)\right) = \left\{f \in Y | \, f = Ag \text{ for some } g \in \mathcal{D}(A)\right\}.$$

In addition, one defines the kernel $Ker(A)$ of A by

$$Ker(A) := \left\{f \in \mathcal{D}(A) | \, Af = 0\right\}.$$

Definition 4. *The graph $G(A)$ of A is the set $\{(x, Ax) \;\; x \in \mathcal{D}(A)\}$. Since A is linear, $G(A)$ is a subspace of $X \times Y$. If the graph of A is closed in $X \times Y$, then A is said to be closed in X. When there is no ambiguity concerning the space X, we say that A is closed.*

Remark 1.

1. *A is closed if and only if $\{x_n\}$ in $\mathcal{D}(A)$, $\{x_n\} \to x$, $Ax_n \to y$, imply $x \in \mathcal{D}(A)$ and $Ax = y$.*

2. *If $\mathcal{D}(A)$ is closed and A is continuous, then A is closed.*

Definition 5. *(Resolvent set and spectrum) Let X be a Banach space and let the closed linear operator $A : \mathcal{D}(A) \subset X \to X$, the set:*

$$\rho(A) = \{\lambda \in \mathcal{C} : \lambda I - A : \mathcal{D}(A) \to X \text{ is bijective}\}$$

is called the resolvent set of A and its complement $\sigma(A) = \mathbb{C} \backslash \rho(A)$ the spectrum of A. For $\lambda \in \rho(A)$ the reverse

$$R(\lambda, A) = (\lambda I - A)^{-1}$$

is a bounded operator in X called the resolvent of A at point λ.

1.2 Strongly continuous semigroups

In this section, we state some definitions and properties of a continuous semi-group of linear and bounded operators [65], [9], [25].

Definition 6. *A family* $(T(t))_{t>0}$ *of bounded linear operators on a Banach space* X *is called a strongly continuous (one-parameter) semigroup (or* C_0*-semigroup) if it satisfies the functional equation*

$$\begin{cases} T(t+s) = T(t).T(s), \text{ for all } t,s > 0, \\ T(0) = I \end{cases} \tag{1.2.1}$$

and is strongly continuous in the following sense. For every $x \in X$ *the orbit maps*

$$\xi_x : t \to \xi_x(t) := T(t)x \tag{1.2.2}$$

are continuous from $\mathbb{R}_+$ *into* X *for every* $x \in X$.

The continuity of the orbit maps (1.2.2) at each $t \geq 0$ and for each $x \in X$ is already implied by much weaker properties.

Proposition 1. *For a semigroup* $(T(t))_{t\geq 0}$ *on a Banach space* X*, the following assertions are equivalent:*

1. $(T(t))_{t\geq 0}$ *is strongly continuous.*
2. $\lim_{t\to 0} T(t)x = x$ *for all* $x \in X$.
3. *There exist* $\delta > 0$, $M \geq 1$, *and a dense subset* $D \subset X$ *such that*
   - (a) $\|T(t)\| \leq M$ *for all* $t \in [0, \delta]$,
   - (b) $\lim_{t\to 0} T(t)x = x$ *for all* $x \in D$.

Proposition 2. *For every strongly continuous semigroup* $(T(t))_{t\geq 0}$, *there exist constants* $w \in \mathbb{R}$ *and* $M \geq 1$ *such that*

$$\|T(t)\| \leq Me^{wt}$$

for all $t \geq 0$.

In particular, if $w = 0$ *then the corresponding semigroup is uniformly bounded. Moreover, if* $M = 1$ *then* $(T(t))_{t\geq 0}$ *is said to be a* C_0*-semigroup of contraction.*

Definition 7. *[65] (Infinitesimal generator of the semigroup) The linear operator* A *defined by*

$$\mathcal{D}(A) = \left\{ x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{t}|_{t=0} \text{ for } x \in \mathcal{D}(A)$$

is the infinitesimal generator of the semigroup $T(t)$, $\mathcal{D}(A)$ *is the domain of* A.

Definition 8. *An unbounded linear operator $(A, \mathcal{D}(A))$ on X, is said to be dissipative if*

$$Re \langle Au, u \rangle_X \leq 0, \ \forall u \in \mathcal{D}(A).$$

Definition 9. *An unbounded linear operator $(A, \mathcal{D}(A))$ on X, is said to be maximal dissipative (m-dissipative) if*

- *A is a dissipative operator.*
- *$\exists \lambda_0 > 0$ such that $Im(\lambda_0 I - A) = X$.*

Theorem 1. *[14] (Hille-Yosida) A linear (unbounded) operator A is the infinitesimal generator of a C_0-semigroup of contractions $(T(t))_{t\geq 0}$ if and only if*

- *A is closed and $\overline{\mathcal{D}(A)} = X$.*
- *The resolvent set $\rho(A)$ of A contains $\mathbb{R}_+$ and for every $\lambda > 0$*

$$\left\| (\lambda I - A)^{-1} \right\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}.$$

Theorem 2. *[14] (Lumer-Phillips) Let A be a linear operator with dense domain $\mathcal{D}(A)$ in X. A is the infinitesimal generator of a C_0-semigroup of contractions if and only if it is a m-dissipative operator.*

1.3 The Hille-Yosida theorem and maximal monotone operators

In this section, we explain how we may associate a C_0-semigroup to the evolution equation as a mere consequence of the linearity of the equation and of the existence and uniqueness result. Throughout this section H denotes a Hilbert space equipped with the scalar product $\langle ., . \rangle_H$ and the corresponding norm $\|.\|_H$.

Definition 10. *(Maximal Monotone Operators) An unbounded linear operator $A : \mathcal{D}(A) \subset H \to H$ is said to be monotone if it satisfies*

$$\langle Av, v \rangle_H \geq 0, \quad \forall v \in \mathcal{D}(A).$$

It is called maximal monotone if, in addition, $Im(I + A) = H$, i.e.,

$$\forall f \in H, \exists u \in \mathcal{D}(A) \text{ such that } u + Au = f.$$

Proposition 3. *[15] Let A be a maximal monotone operator. Then*

1. *$\mathcal{D}(A)$ is dense in H,*

2. *A is a closed operator,*

3. *For every $\lambda > 0$, $(I + \lambda A)$ is bijective from $\mathcal{D}(A)$ onto H, $(I + \lambda A)^{-1}$ is a bounded operator, and $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(H)} \leq 1$.*

Definition 11. *Let A be a maximal monotone operator. For every $\lambda > 0$, set*

$$J_\lambda = (I + \lambda A)^{-1} \text{ and } A_\lambda = \frac{1}{\lambda}(I - J_\lambda);$$

J_λ is called the resolvent of A, and A_λ is the Yosida approximation (or regularization) of A. Keep in mind that $\|J_\lambda\|_{\mathcal{L}(H)} \leq 1$.

Theorem 3. *[15] (Hille–Yosida) Let A be a maximal monotone operator. Then, given any $u_0 \in \mathcal{D}(A)$ there exists a unique function*

$$u \in C^1\left([0, \infty); H\right) \cap C\left([0, \infty); \mathcal{D}(A)\right)$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + Au = 0, & \text{on } [0, +\infty), \\ u(0) = u_0. \end{cases} \tag{1.3.1}$$

Moreover

$$|u(t)| \leq |u_0| \text{ and } \frac{du}{dt} = |Au(t)| \leq |Au_0|, \ \forall t \geq 0.$$

Remark 2. *The main interest of theorem 3 lies in the fact that we reduce the study of an "evolution problem" to the study of the "stationary equation" $u + Au = f$ (assuming we already know that A is monotone, which is easy to check in practice) [15].*

Theorem 4. *[15] (Hille-Yosida) Let $(A, \mathcal{D}(A))$ be an unbounded linear operator on H. Assume that A is the infinitesimal generator of a C_0-semigroup of contractions $(T(t))_{t\geq 0}$.*

1. *For $u_0 \in \mathcal{D}(A)$, the problem (1.3.1) admits a unique strong solution*

$$u(t) = T(t)u_0 \in C^0\left(\mathbb{R}_+; \mathcal{D}(A)\right) \cap C^1\left(\mathbb{R}_+; H\right).$$

2. *For $u_0 \in H$, the problem (1.3.1) admits a unique weak solution*

$$u(t) \in C^0\left(\mathbb{R}_+; H\right).$$

1.4 Semigroups for delay equations

In this section, we present a systematic semigroup approach to linear partial differential equations with delay, we will see that the appropriate setting for linear delay differential equations is that of an abstract Cauchy problem in an appropriate Banach space.

Everybody who has had a basic Ordinary Differential Equations course knows that the fundamental solution to equation (1.3.1) is given by the exponential function $t \to e^{tA}$. More precisely, for every $x \in \mathbb{C}$ the function $u(t) := e^{tA}x$ is the unique solution of equation (1.3.1) with initial value x.

Let us now modify equation (1.3.1) slightly by considering

$$u'(t) = Au(t-\tau),\ \ t \geq 0,$$

where $\tau > 0$. Do we still find an exponential function solving this equation of course, the function $t \to e^{tA}$ does not work anymore and there is no other matrix $B \in \mathcal{L}(\mathbb{C})$ such that $t \to e^{tB}$ is a fundamental solution. Nevertheless, the answer is still "yes" provided we look at it in the right setting.

To do so we have to change our finite-dimensional viewpoint into an infinite-dimensional one. Take a Banach space X and consider a function $u : [-\tau, \infty) \to X$. For each $t \geq 0$, we call the function

$$u_t : \sigma \in [-\tau, 0] \mapsto u(t+\sigma) \in X,$$

history segment with respect to $t \geq 0$. The history function of u is then the function

$$h_u : t \mapsto u_t$$

on $\mathbb{R}_+$. A delay differential equation is of the form

$$u'(t) = \frac{d}{dt}u(t) = \varphi(u(t), u_t)\ \ t \geq 0, \tag{1.4.1}$$

where $\varphi(.,.)$ is an X-valued mapping. The explanation for this terminology follows.

In many concrete situations, the derivative $u'(t)$ actually depends on $u(t)$ and on $u(t-\tau)$ for some fixed $\tau > 0$ (often with τ normalized to $\tau = 1$) and one has to study differential equations of the form

$$u(t) = \Psi(u(t), u_t(-\tau)), \tag{1.4.2}$$

for some function Ψ from $X \times X$ into X. Thus, interpreting t as time values of u have an effect on u' with a certain delay τ. If we now define φ as

$$\varphi(u(t), u_t(-\tau)) := \Psi(u(t), u_t(-\tau)),$$

we arrive at equation (1.4.1).

We will show in this section that this point of view allows us to read delay equations as vector-valued ordinary differential equations.

Hypothesis 1. *(The standing hypotheses) Assume that*

(H_1) *X is a Banach space;*

$(H2)$ *$B : \mathcal{D}(B) \subseteq X \to X$ is a closed, densely defined, linear operator;*

$(H3)$ *Z is a Banach space such that $\mathcal{D}(B) \overset{d}{\hookrightarrow} Z \overset{d}{\hookrightarrow} X$;*

$(H4)$ *$1 \le p < \infty$, $f \in L^p([-1, 0], Z)$ and $x \in X$;*

$(H5)$ *$\Phi : W^{1,p}([-1, 0], Z) \to X$ is a bounded linear operator, called the delay operator; and*

$(H6)$ *$\mathcal{E}_p := X \times L^p([-1, 0], Z)$.*

Under these hypotheses, and for given elements $x \in X$ and $f \in L^p([-1, 0], Z)$, the following initial value problem will be called an (abstract) delay equation (with the history parameter $1 \le p < \infty$)

$$\begin{cases} u'(t) = Bu(t) + \Phi u_t, \\ u(0) = 0, \\ u_0 = f. \end{cases} \tag{1.4.3}$$

Definition 12. *We say that a function $u : [-1, \infty) \to X$ is a classical solution of (1.4.3) if*

(i) $u \in C([-1, \infty), X) \cap C^1([0, \infty), X)$,

(ii) $u(t) \in \mathcal{D}(B)$ and $u_t \in W^{1,p}([-1, 0], Z)$ for all $t \ge 0$,

(iii) u satisfies (1.4.3) for all $t \ge 0$.

Lemma 1. *[10] Let $u : [-1, \infty) \to Z$ be a function that belongs to $W^{1,p}_{loc}([-1, \infty), Z)$. Then the history function $h_u : t \to u_t$ of u is continuously differentiable from $\mathbb{R}^+$ into $L^p([-1, 0], Z)$ with derivative*

$$\frac{d}{dt} h_u(t) = \frac{d}{d\sigma} u_t.$$

By means of Lemma 1, we can now transform classical solutions of (1.4.3) into classical solutions of an abstract Cauchy problem.

Corollary 1. *[10] Let $u : [-1, \infty) \to X$ be a classical solution of (1.4.3). Then the function*

$$\mathcal{U} : t \mapsto \begin{pmatrix} u(t) \\ u_t \end{pmatrix} \in \mathcal{E}_p \tag{1.4.4}$$

from $\mathbb{R}_+$ *into* $\mathcal{E}_p$ *is continuously differentiable with derivative*

$$\mathcal{U}'(t) = A\mathcal{U}(t),$$

where

$$\mathcal{A} := \begin{pmatrix} B & \Phi \\ 0 & \dfrac{d}{d\sigma} \end{pmatrix},$$

where $\dfrac{d}{d\sigma}$ *denotes the distributional derivative, with domain*

$$\mathcal{D}(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{D}(B) \times W^{1,p}([-1,0], Z) : f(0) = x \right\}.$$

Thus every classical solution u *of (1.4.3) yields a classical solution of the abstract Cauchy problem*

$$\begin{cases} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t), t \geq 0, \\ \mathcal{U}(0) = \begin{pmatrix} x \\ f \end{pmatrix}, \end{cases} \tag{1.4.5}$$

on $\mathcal{E}_p$.

By adding the following to our standing hypotheses: $(H7)$ $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the operator on $\mathcal{E}_p$ defined as

$$\mathcal{A} = \begin{pmatrix} B & \Phi \\ 0 & \dfrac{d}{d\sigma} \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{D}(B) \times W^{1,p}([-1,0], Z) : f(0) = x \right\}.$$

We need to show the closedness of the operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$.

Lemma 2. *[10] Under Hypotheses* $(H1)-(H7)$, *the operator* $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ *is closed and densely defined on* $\mathcal{E}_p$.

Corollary 2. *[10] The abstract Cauchy problem (1.4.5) associated to the operator* $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ *on the space* $\mathcal{E}_p$ *is well-posed if and only if* $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ *is the generator of a strongly continuous semigroup* $(T(t))_{t\geq 0}$ *on* $\mathcal{E}_p$. *In this case, the classical and mild solutions of (1.4.5) are given* *by the functions*

$$\mathcal{U}(t) := T(t)\begin{pmatrix} x \\ f \end{pmatrix},$$

for $t \geq 0$.

Now we introduce the following notation.

Definition 13. *By* $\pi_1 : \mathcal{E}_p \to X$, *we denote the canonical projection from* $\mathcal{E}_p$ *onto* X. *Similarly, by* $\pi_2 : \mathcal{E}_p \to L^p([-1,0], Z)$ *we denote the canonical projection from* $\mathcal{E}_p$ *onto* $L^p([-1,0], Z)$.

Proposition 4. *[10] For every classical solution* $\mathcal{U}$ *of (1.4.5), the function*

$$u(t) := \begin{cases} (\pi_1 \circ \mathcal{U})(t) & \text{if } t \geq 0, \\ f(t) & \text{if } t \in [-1, 0) \end{cases}$$

is a classical solution of (1.4.3) and $(\pi_2 \circ \mathcal{U})(t) = u_t$ *for all* $t \geq 0$.

At present, we transfer the notions of well-posedness and of mild solution, known from abstract Cauchy problems and semigroups to (1.4.3).

Proposition 5. *[10] Let* u *be a mild solution of (1.4.3). Then* u *satisfies* $\int_0^t u(s)ds \in \mathcal{D}(B)$, $\int_0^t u_s ds \in W^{1,p}([-1,0], Z)$, *and the integral equation*

$$u(t) = \begin{cases} x + B\displaystyle\int_0^t u(s)ds + \Phi \int_0^t u_s ds, & \text{for } t \geq 0, \\ f(t), & \text{for a.e. } t \in [-1, 0). \end{cases}$$

1.5 Stability of semigroup

In this section we start by introducing some definitions about strong, exponential and polynomial stability of a C_0-semigroup. Then we collect some results about the stability of C_0-semigroup.

Definition 14. *Assume that* A *is the generator of a strongly continuous semigroup of contractions* $(T(t))_{t\geq 0}$ *on* X. *We say that the* C_0*-semigroup* $(T(t))_{t\geq 0}$ *is*

- *Strongly stable if*

$$\lim_{t\to+\infty} \|T(t)u\|_X = 0, \quad \forall u \in X.$$

- *Uniformly stable if*

$$\lim_{t\to+\infty} \|T(t)\|_{\mathcal{L}(X)} = 0.$$

- *Exponentially stable if there exist two positive constants* M *and* ϵ *such that*

$$\|T(t)\,u\|_X \leq Me^{-\epsilon t}\,\|u\|_X\,, \quad \forall t>0, \forall u \in X.$$

- *Polynomially stable if there exist two positive constants* C *and* α *such that*

$$\|T(t)\,u\|_X \leq Ct^{-\alpha}\,\|u\|_X\,, \;\; \forall t>0, \forall u \in X.$$

Proposition 6. *Assume that* A *is the generator of a strongly continuous semigroup of contractions* $(T(t))_{t\geq 0}$ *on* X. *The following statements are equivalent*

- $(T(t))_{t\geq 0}$ *is uniformly stable.*
- $(T(t))_{t\geq 0}$ *is exponentially stable.*

1.6 The theorems of Stampacchia and Lax–Milgram

Definition 15. *A bilinear form* $a : H\times H \to \mathbb{R}$ *is said to be*
(i) Continuous if there is a constant C *such that*

$$|a(u,v)| \leq C\,|u|\,|v|\,, \;\forall u,v \in H.$$

(ii) Coercive if there is a constant $\alpha > 0$ *such that*

$$a(v,v) \geq \alpha\,|v|^2\,, \;\forall v \in H.$$

Theorem 5. *[14] (Stampacchia) Assume that* $a(u,v)$ *is a continuous coercive bilinear form on* H. *Let* $K \subset H$ *be a nonempty closed and convex subset. Then, given any* $\phi \in H^*$, *there exists a unique element* $u \in K$ *such that*

$$a\,(u,v-u) \geq \langle \phi, v-u\rangle\,, \;\forall v \in K. \tag{1.6.1}$$

Moreover, if a *is symmetric, then* u *is characterized by the property*

$$u \in K \;\; and \;\; \frac{1}{2}a\,(u,u) - \langle \phi, u\rangle = min\left\{\frac{1}{2}a\,(v,v) - \langle \phi, v\rangle\right\}, \;\; v \in K. \tag{1.6.2}$$

Corollary 3. *[14] (Lax–Milgram) Assume that* $a(u,v)$ *is a continuous coercive bilinear form* *on H. Then, given any $\phi \in H^*$, there exists a unique element $u \in K$ such that*

$$a(u,v) = \langle \phi, v \rangle, \ \forall v \in H. \quad (1.6.3)$$

Moreover, if a is symmetric, then u is characterized by the property

$$u \in H \text{ and } \frac{1}{2}a(u,u) - \langle \phi, u \rangle = min \left\{ \frac{1}{2}a(v,v) - \langle \phi, v \rangle \right\}, \ v \in H. \quad (1.6.4)$$

1.7 Faedo-Galerkin method

Definition 16. *Let H be a separable Hilbert space and f a family of finite dimensional vector spaces satisfying the axioms:*

1. $V_n \subset V$, $\dim V_n < \infty$.
2. $V_n \to V$, *when* $V_n < \infty$.

In the following sense: there exists V_n dense subspace in V, such that for all $u \in V_n$, we can find a sequence $\{u_n\}_{n \in N^*}$ satisfying:
for all $n, u_n \in V_n$ and $u_n \to u$ in V when $n \to \infty$.

The space V_n is called a Galerkin approximation of order n.

The scheme of the method of Faedo-Galerkin

Let (P) to be the exact problem for which we want to show the existence of a solution in a function space built on a separable Hilbert space V. Let u to be the unique solution of the problem (P).

After having made a choice of a Galerkin approximation V_n of V it is necessary to define an approximate problem (P_n) in finite-dimensional space (V_n) having a unique solution (u_n). Then, the course of the study is then as follows:
Step 1: We define the solution u_n of the problem (P_n).
Step 2: We establish estimates on u_n (called a priori estimate) to show that u_n is uniformly bounded.
Step 3: By using the results that u_n is uniformly bounded, it is possible to extract from $\{u_n\}_{n \in N^*}$ a subsequence $\{u'_n\}_{n \in N^*}$ which has a limit in the weak topology of the spaces involved in the estimations of step 2.

Let u to be the obtained limit.
Step 4: We show that u is the solution of the problem (P).
Step 5: Results of strong convergences.

The objective is to build an approximation process which ultimately provides us with a proof of the existence of solution, this process amounts to approaching $u_n(x,t)$ as a linear combination of functions of the bases w_i such that

$$u_n(x,t) = \sum_{i=1}^{n} r_{in}(t)\, w_i, \qquad (x,t) \in \Omega \times [0,T].$$

For more detail see [41].

Chapter 2

Well posedness and exponential stability in a wave equation with strong damping and strong delay

2.1 Introduction

In this chapter, we consider a wave equation with a strong damping and a strong constant (respectively, distributed) delay. We prove the well-posedness and establish an exponential decay result under a suitable assumptions on the weight of the damping and the weight of the delay. Thus, we consider the following problem with strong damping and strong delay

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) - \mu_1 \Delta u_t(x,t) - \mu_2 \Delta u_t(x,t-\tau) = 0, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ u_t\left(x,t-\tau\right) = f_0\left(x,t-\tau\right), & t \in \left(0,\tau\right), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{2.1.1}$$

where Ω is a bounded and regular domain of $\mathbb{R}^n$, $\tau > 0$ represents the time delay, μ_1, μ_2 are real numbers such that $|\mu_2| < \mu_1$ and u_0, u_1, f_0 are given data. Our equation can be regarded as a Kelvin-Voight linear model for a viscoelastic material in the presence of a delay response.

In the second part of this chapter, the constant delay term in (2.1.1) is replaced by the distributed delay term of the form $-\int_{\tau_1}^{\tau_2} \mu_2(s)\Delta u_t(x,t-s)ds$, where $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function and $\tau_1 < \tau_2$ are two positive constants. We establish well-posedness and exponential decay results under suitable conditions on the weights of the constant (respectively, distributed) delay and the weight of the damping terms.

2.2 Well-posedness

To the best of knowledge, problem (2.1.1) has not been discussed before. We establish the existence of a unique solution by the use of the semigroup theory. For this purpose, we introduce, similarly to [59], the new variable

$$z(x,\rho,t)=u_t(x,t-\tau\rho),\qquad \rho\in(0,1),\ x\in\Omega,\ t>0.$$

Consequently, we have

$$\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0,\qquad \rho\in(0,1),\ x\in\Omega,\ t>0.$$

So, problem (2.1.1) takes the form

$$\begin{cases} u_{tt}(x,t)-\Delta u(x,t)-\mu_1\Delta u_t(x,t)-\mu_2\Delta z(x,1,t)=0, & \text{in}\,\Omega\times(0,\infty),\\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0, & \text{in } \Omega\times(0,1)\times(0,\infty),\\ u(x,t)=z\left(x,\rho,t\right)=0, & \text{on } \partial\Omega\times(0,\infty),\\ u_t\left(x,t-\tau\right)=f_0\left(x,t-\tau\right), & t\in\left(0,\tau\right),\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & \text{in } \Omega. \end{cases}\tag{2.2.1}$$

We consider the following Hilbert space

$$\mathcal{H}=H_0^1\left(\Omega\right)\times L^2\left(\Omega\right)\times L^2\left(\left(0,1\right);H_0^1\left(\Omega\right)\right),$$

equiped with the inner product

$$\left\langle \phi,\widetilde{\phi}\right\rangle_{\mathcal{H}}=\int_\Omega\left(\nabla u.\nabla\widetilde{u}+v\widetilde{v}\right)dx+\tau\left|\mu_2\right|\int_\Omega\int_0^1\nabla z.\nabla\widetilde{z}d\rho dx,$$

for all $\phi=(u,v,z)^T$, $\widetilde{\phi}\left(\widetilde{u},\widetilde{v},\widetilde{z}\right)^T\in\mathcal{H}$.

For $\phi=(u,v,z)^T$, where $v=u_t$, system (2.2.1) can be rewritten as

$$\begin{cases} \phi'+\mathcal{A}\phi=0,\\ \phi\left(0\right)=\phi_0=\left(u_0,u_1,z\left(.-\tau\rho\right)\right), \end{cases}\tag{2.2.2}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$\mathcal{A}\phi = \begin{bmatrix} -v \\ -\Delta u - \mu_1 \Delta v - \mu_2 \Delta z(.,1) \\ \frac{1}{\tau} z_\rho \end{bmatrix}$$

with domain

$$D(\mathcal{A}) = \left\{ (u,v,z) \in \mathcal{H}/u + \mu_1 v + \mu_2 z(.,1) \in H^2(\Omega), z, z_\rho \in L^2\left((0,1), H_0^1(\Omega)\right), z(.,0) = v \right\}.$$

We have the following existence and uniqueness result:

Theorem 6. *Assume that*

$$|\mu_2| \leq \mu_1. \tag{2.2.3}$$

Then for any $\phi_0 \in \mathcal{H}$, problem (2.2.2) has a unique solution

$$\phi \in C([0,+\infty), \mathcal{H}).$$

Moreover if $\phi_0 \in D(\mathcal{A})$, the solution of (2.2.2) satisfies

$$\phi \in C([0,+\infty), D(\mathcal{A})) \cap C^1([0,+\infty), \mathcal{H}).$$

Proof. We will use Hille-Yoside theorem [15, 39]. For this purpose, we start by showing that $\mathcal{A}$ is monotone. So, for $\phi \in D(\mathcal{A})$, we have

$$\begin{aligned} \langle \mathcal{A}\phi, \phi \rangle_{\mathcal{H}} &= -\int_\Omega \nabla v.\nabla u dx + \int_\Omega v \left[-\Delta u - \mu_1 \Delta v - \mu_2 \Delta z(.,1)\right] dx \\ &+ |\mu_2| \int_0^1 \int_\Omega \nabla z.\nabla z_\rho dx d\rho \\ &= \mu_1 \int_\Omega |\nabla v|^2 dx + \mu_2 \int_\Omega \nabla v.\nabla z(1,.) + \frac{|\mu_2|}{2} \int_\Omega |\nabla z(.,1)|^2 dx \\ &- \frac{|\mu_2|}{2} \int_\Omega |\nabla v|^2 dx. \end{aligned} \tag{2.2.4}$$

Using Young's inequality for the second term of (2.2.4), we arrive at

$$\langle \mathcal{A}\phi, \phi \rangle_{\mathcal{H}} \geq (\mu_1 - |\mu_2|) \int_\Omega |\nabla v|^2 dx \geq 0$$

by virtue of (2.2.3). Hence, $\mathcal{A}$ is monotone.

Next, we show that $\mathcal{A}$ is maximal. That is, for each $F = (f,g,h)^T \in \mathcal{H}$ we have to find

$\phi \in D(\mathcal{A})$ such that

$$\phi + \mathcal{A}\phi = F \tag{2.2.5}$$

i.e.

$$\begin{cases} u - v = f, \\ v - \Delta u - \mu_1 \Delta v - \mu_2 \Delta z(.,1) = g, \\ \tau z + z_\rho = \tau h. \end{cases} \tag{2.2.6}$$

The first and the third equations of (2.2.6) give "formally"

$$z(.,\rho) = (u - f)e^{-\rho\tau} + \tau e^{-\rho\tau} \int_0^\rho h(\gamma,.)e^{\gamma\tau} d\gamma. \tag{2.2.7}$$

Clearly, we have $z(.,0) = u - f = v$. Replacing in the second equation of (2.2.6), we get

$$u - k\Delta u = G, \tag{2.2.8}$$

where

$$\begin{cases} k = 1 + \mu_1 + \mu_2 e^{-\tau} > 0, \\ G = g + f - (\mu_1 + \mu_2 e^{-\tau})\, \Delta f + \tau\mu_2 e^{-\tau} \int_0^1 \Delta h(\gamma,.)e^{\gamma\tau} d\gamma \in H^{-1}(\Omega). \end{cases}$$

Over $H_0^1(\Omega)$ we define the bilinear form

$$B(u,w) = \int_\Omega uw + k \int_\Omega \nabla u . \nabla w$$

and the linear form

$$L(w) = \langle G, w \rangle_{H^{-1} \times H_0^1}.$$

A simple calculation shows that B and L satisfy the conditions of Lax-Milgram theorem and thus, the exists a unique $u \in H_0^1(\Omega)$ satisfying

$$B(u,w) = L(w), \ \forall w \in H_0^1(\Omega). \tag{2.2.9}$$

Consequently, $v = u - f \in H_0^1(\Omega)$ and

$$z(.,\rho), z_\rho(.,\rho) \in H_0^1(\Omega).$$

Using (2.2.7), we get $z \in L^2((0,1), H_0^1(\Omega))$. Thus, (2.2.5) has a unique solution $\phi = (u,v,z) \in \mathcal{H}$.
Replacing Δf by $\Delta(u-v)$ and $\tau e^{-\tau} \int_0^1 \Delta h(\gamma,x)\, e^{\gamma\tau} d\gamma$ by $\Delta z(x,1) - e^{-\tau}\Delta v$ in the right-

hand side of (2.2.9) and using Green's formula, we obtain

$$\int_{\Omega} uv + \int_{\Omega} \nabla \left(u + \mu_1 v + \mu_2 z\left(x, 1\right)\right) . \nabla w = \int_{\Omega} \left(f + g \right) w, \, \forall w \in H_0^1\left(\Omega\right).$$

The standard elliptic regularity theory [15], gives

$$u + \mu_1 v + \mu_2 z\left(., 1\right) \in H^2\left(\Omega\right).$$

Therefore

$$\Delta \left(u + \mu_1 v + \mu_2 z\left(., 1\right)\right) = g + f - u \in L^2\left(\Omega\right).$$

Consequently $\phi \in D(\mathcal{A})$. This shows that $\mathcal{A}$ is maximal. Thus, Hille-Yosida theorem guarantees the existence of a unique local solution to problem (2.2.2). This completes the proof of theorem 6. □

2.3 Exponential stability

In this section we state and prove our stability result. First, we introduce the energy associated to the system (2.2.1)

$$E\left(t\right) := \frac{1}{2} \int_{\Omega} u_t^2\left(x, t\right) dx + \frac{1}{2} \int_{\Omega} \left|\nabla u(x, t)\right|^2 dx + \frac{\tau \left|\mu_2\right|}{2} \int_{\Omega} \int_0^1 \left|\nabla z\left(x, \rho, t\right)\right|^2 d\rho dx. \tag{2.3.1}$$

Our main result is given by the following theorem

Theorem 7. *Assume that* $|\mu_2| < \mu_1$ *and* $(u_0, u_1, f_0\left(., \tau.\right)) \in D(\mathcal{A})$. *Then the solution of (2.2.1) satisfies, for two positive constants* η,ω, *the estimate*

$$E(t) \leq \eta e^{-\omega t}, \;\; \forall t \geq 0.$$

The proof will be established through the following three Lemmas:

Lemma 3. *Assume that*

$$|\mu_2| < \mu_1. \tag{2.3.2}$$

Then the energy $E\left(t\right)$ *satisfies, along the solution* $\left(u, z\right)$ *of (2.2.1), the estimate*

$$E'\left(t\right) \leq -\left(\mu_1 - |\mu_2|\right) \int_{\Omega} \left|\nabla u_t(x, t)\right|^2 dx \leq 0. \tag{2.3.3}$$

Proof. Multiplying the first equation of (2.2.1) by u_t integrating over Ω, we obtain

$$\begin{aligned}&\int_\Omega u_{tt}(x,t)u_t(x,t)\,dx-\int_\Omega \Delta u(x,t)u_t(x,t)\,dx\\&-\mu_1\int_\Omega \Delta u_t(x,t)u_t(x,t)\,dx-\mu_2\int_\Omega \Delta z(x,1,t)u_t(x,t)\,dx=0.\end{aligned}$$

Integration by part and recaling the boundary conditions, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2(x,t)\,dx+\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u(x,t)|^2\,dx\\&+\mu_1\int_\Omega |\nabla u_t(x,t)|^2\,dx+\mu_2\int_\Omega \nabla z(x,1,t).\nabla u_t(x,t)\,dx=0.\end{aligned}\tag{2.3.4}$$

Multiplying the second equation of (2.2.1) by $|\mu_2|\,z$, integrating over $(0,1)\times\Omega$, using integration by parts, we get

$$\tau|\mu_2|\int_\Omega\int_0^1 \nabla z_t(x,\rho,t).\nabla z(x,\rho,t)\,d\rho dx+|\mu_2|\int_\Omega\int_0^1 \nabla z_\rho(x,\rho,t).\nabla z(x,\rho,t)\,d\rho dx=0,$$

which can be written as

$$\frac{\tau|\mu_2|}{2}\frac{d}{dt}\int_\Omega\int_0^1 |\nabla z(x,\rho,t)|^2\,d\rho dx+\frac{|\mu_2|}{2}\int_\Omega |\nabla z(x,1,t)|^2\,dx-\frac{|\mu_2|}{2}\int_\Omega |\nabla u_t(x,t)|^2\,dx=0.\tag{2.3.5}$$

Summing up (2.3.4) and (2.3.5), we get

$$\begin{aligned}E'(t)=&-\left(\mu_1-\frac{|\mu_2|}{2}\right)\int_\Omega |\nabla u_t(x,t)|^2\,dx-\mu_2\int_\Omega \nabla z(x,1,t).\nabla u_t(x,t)\,dx\\&-\frac{|\mu_2|}{2}\int_\Omega |\nabla z(x,1,t)|^2\,dx.\end{aligned}$$

Young's inequality and (2.3.2) yield then the desired result. □

Lemma 4. *The functional*

$$F_1(t):=\int_\Omega u_t u\,dx$$

satisfies, along the solution of (2.2.1), the estimate

$$F_1'(t)\le -\frac{1}{2}\int_\Omega |\nabla u|^2\,dx+C\int_\Omega |\nabla u_t|^2\,dx+\mu_2^2\int_\Omega |\nabla z(x,1,t)|^2\,dx,\tag{2.3.6}$$

for some positive constant C.

Proof. The differentiation of $F_1(t)$ gives

$$\begin{aligned} F_1'(t) &= \int_\Omega u_t^2 dx + \int_\Omega u\Delta\left(u + \mu_1 u_t + \mu_2 z(x,1,t)\right) dx \\ &= \int_\Omega u_t^2 dx - \int_\Omega |\nabla u|^2 dx - \mu_1 \int_\Omega \nabla u.\nabla u_t dx - \mu_2 \int_\Omega \nabla u.\nabla z(x,1,t)\, dx. \end{aligned}$$

Young's and Poincaré's inequalities leads to the desired estimate. □

Lemma 5. *The fonctional*

$$F_2(t) := \tau \int_0^1 \int_\Omega e^{-\rho\tau} |\nabla z(x,\rho,t)|^2 dx d\rho \tag{2.3.7}$$

satisfies, along the solution of (2.2.1), the estimate

$$F_2'(t) \leq \int_\Omega |\nabla u_t(x,t)|^2 dx - e^{-\tau} \int_\Omega |\nabla z(x,1,t)|^2 dx - \tau e^{-\tau} \int_0^1 \int_\Omega |\nabla z(x,\rho,t)|^2 dx d\rho. \tag{2.3.8}$$

Proof. A direct differentiation of (2.3.7) using the second equation of (2.2.1), gives

$$\begin{aligned} F_2'(t) &= -2 \int_0^1 \int_\Omega e^{-\rho\tau} \nabla z(x,\rho,t) . \nabla z_\rho(x,\rho,t)\, dx d\rho \\ &= -\int_0^1 \int_\Omega e^{-\rho\tau} \frac{d}{d\rho} |\nabla z(x,\rho,t)|^2 dx d\rho \\ &= -\int_0^1 \int_\Omega \frac{d}{d\rho}\left(e^{-\rho\tau} |\nabla z(x,\rho,t)|^2\right) dx d\rho - \tau \int_0^1 \int_\Omega e^{-\rho\tau} |\nabla z(x,\rho,t)|^2 dx d\rho \\ &= -\int_\Omega e^{-\tau} |\nabla z(x,1,t)|^2 dx - \tau \int_0^1 \int_\Omega e^{-\rho\tau} |\nabla z(x,\rho,t)|^2 dx d\rho + \int_\Omega |\nabla u_t(x,t)|^2 dx. \end{aligned}$$

Thus, (2.3.8) follows immediately. □

Proof of theorem 7

To complete the proof of Theorem 7, we define the Lyapunov function

$$\mathcal{L}(t) := NE(t) + \varepsilon F_1(t) + F_2(t), \tag{2.3.9}$$

where N and ε are positive constants to be chosen carefully. By inserting (2.3.3), (2.3.6) and (2.3.8) in (2.3.9), we obtain

$$
\begin{aligned}
\mathcal{L}'(t) \leq & -N(\mu_1 - |\mu_2|) \int_\Omega |\nabla u_t(x,t)|^2 \, dx \\
& - \frac{\varepsilon}{2} \int_\Omega |\nabla u(x,t)|^2 \, dx + \varepsilon C \int_\Omega |\nabla u_t(x,t)|^2 \, dx + \varepsilon \mu_2^2 \int_\Omega |\nabla z(x,1,t)|^2 \, dx \\
& + \int_\Omega |\nabla u_t(x,t)|^2 \, dx - e^{-\tau} \int_\Omega |\nabla z(x,1,t)|^2 \, dx - \tau e^{-\tau} \int_0^1 \int_\Omega |\nabla z(x,\rho,t)|^2 \, dx d\rho.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathcal{L}'(t) \leq & -(N(\mu_1 - |\mu_2|) - \varepsilon C - 1) \int_\Omega |\nabla u_t|^2 \, dx - \frac{\varepsilon}{2} \int_\Omega |\nabla u(x,t)|^2 \, dx \\
& - \tau e^{-\tau} \int_0^1 \int_\Omega |\nabla z(x,\rho,t)|^2 \, dx d\rho - \left[e^{-\tau} - \varepsilon \mu_2^2 \right] \int_\Omega |\nabla z(x,1,t)|^2 \, dx.
\end{aligned}
$$

It suffices to choose ε so small such that

$$
e^{-\tau} - \varepsilon \mu_2^2 > 0,
$$

then pick N large enough such that

$$
N(\mu_1 - |\mu_2|) - \varepsilon C - 1 > 0
$$

and $\mathcal{L} \sim E$. Thus we arrive at

$$
\mathcal{L}'(t) \leq -\omega \mathcal{L}(t),
$$

which yields, by integration,

$$
\mathcal{L}(t) \leq \mathcal{L}(0) e^{-\omega t}, \ \forall t \geq 0.
$$

The use of $\mathcal{L} \sim E$ again gives

$$
E(t) \leq \eta e^{-\omega t}, \ \forall t \geq 0.
$$

Hence, the desired result.

2.4 Wave equation with internal distributed delay

In this section we will extend the result obtained in the section 3 to the case of distributed delay, namely, we consider the following problem:

$$\begin{cases} u_{tt}-\Delta u\left(x,t\right)-\mu_1\Delta u_t\left(x,t\right)-\int_{\tau_1}^{\tau_2}\mu_2\left(s\right)\Delta u_t\left(x,t-s\right)ds=0, & \text{in } \Omega\times(0,+\infty),\\ u=0, & \text{on } \Gamma\times(0,+\infty),\\ u(x,0)=u_0\left(x\right),u_t(x,0)=u_1\left(x\right), & \text{in } \Omega,\\ u_t\left(x,-t\right)=f_0\left(x,-t\right), & 0<t\leq\tau_2, \end{cases} \tag{2.4.1}$$

where $\tau_1<\tau_2$ are two positive constants and $\mu_2:[\tau_1,\tau_2]\longrightarrow\mathbb{R}$ is a bounded function.

As in section 2, we introduce the new variable z, given by

$$z\left(x,\rho,t,s\right)=u_t\left(x,t-\rho s\right).$$

It is easy to check that

$$sz_t\left(x,\rho,t,s\right)+z_\rho\left(x,\rho,s,t\right)=0. \tag{2.4.2}$$

Thus, system (2.4.1) becomes

$$\begin{cases} u_{tt}-\Delta u\left(x,t\right)-\mu_1\Delta u_t\left(x,t\right)-\int_{\tau_1}^{\tau_2}\mu_2\left(s\right)\Delta z\left(x,1,s,t\right)ds=0, \text{ in } \Omega\times(0,+\infty),\\ sz_t\left(x,\rho,s,t\right)+z_\rho\left(x,\rho,s,t\right)=0, \text{ in } \Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,+\infty),\\ u=0, \text{ on } \Gamma\times[0,+\infty),\\ u\left(x,0\right)=u_0\left(x\right), \ u_t\left(x,0\right)=u_1\left(x\right), \text{ in } \Omega,\\ z\left(x,\rho,s,0\right)=f_0\left(x,-\rho s\right), \text{ in } \Omega\times(0,1)\times(\tau_1,\tau_2). \end{cases} \tag{2.4.3}$$

2.4.1 Well-posedness

In this subsection, we prove by means of the semigroup theory, the well-posedness of system (2.4.3). We introduce the following Hilbert space:

$$\mathcal{H}=H_0^1\left(\Omega\right)\times L^2\left(\Omega\right)\times L^2\left(\left(0,1\right)\times\left(\tau_1,\tau_2\right);H_0^1\left(\Omega\right)\right),$$

equiped with the inner product

$$\left\langle\phi,\widetilde{\phi}\right\rangle_{\mathcal{H}}=\int_\Omega\nabla u.\nabla\widetilde{u}+\int_\Omega v.\widetilde{v}+\int_\Omega\int_{\tau_1}^{\tau_2}s\left|\mu_2\left(s\right)\right|\int_0^1\nabla z\left(x,\rho,s\right).\nabla\widetilde{z}\left(x,\rho,s\right)d\rho dsdx,$$

for $\phi = (u, v, z)^T$, $\widetilde{\phi} = (\widetilde{u}, \widetilde{v}, \widetilde{z})^T \in \mathcal{H}$. Let $v = u_t$, then system (2.4.3) takes the form

$$\begin{cases} \phi' + \mathcal{A}_1 \phi = 0, \\ \phi(0) = \phi_0 = (u_0, u_1, f_0(. - \rho s)), \end{cases}$$

where $\mathcal{A}_1 : D(\mathcal{A}_1) \in \mathcal{H} \to \mathcal{H}$ is the operator defined by

$$\mathcal{A}_1 \phi = \begin{bmatrix} -v \\ -\Delta u(x,t) - \mu_1 \Delta v(x,t) - \int_{\tau_1}^{\tau_2} |\mu_2(s)| \Delta z(x,1,t,s)\, ds \\ \frac{1}{s} z_\rho(x, \rho, t, s) \end{bmatrix}$$

with domain

$$D(\mathcal{A}_1) = \left\{ \phi \in \mathcal{H}, \left(u + \mu_1 v + \left(\int_{\tau_1}^{\tau_2} \mu_2(s) \right) z \right) \in H^2(\Omega), z, z_\rho \in L^2\left((0,1), H_0^1(\Omega)\right), z(0,.) = v \right\}.$$

For the well-posedness, we have the following result:

Theorem 8. *Assume that*

$$\mu_1 > \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds. \tag{2.4.4}$$

Then, for any $\phi_0 \in \mathcal{H}$, problem (2.3.4) has a unique solution satisfying

$$\phi \in C(]0, +\infty[, \mathcal{H}).$$

Moreover, if $\phi_0 \in D(\mathcal{A}_1)$ then

$$\phi = (u, v, z) \in C^1(]0, +\infty[, \mathcal{H}) \cap C(]0, +\infty[, D(\mathcal{A}_1)).$$

Proof. It suffices to show that $\mathcal{A}_1$ is monotone and maximal and use Hille-Yosida theorem.

First for $\phi \in D(\mathcal{A}_1)$, We have

$$\begin{aligned} \langle \mathcal{A}_1 \phi, \phi \rangle_{\mathcal{H}} = & -\int_\Omega \nabla u(x,t) \nabla v(x,t)\, dx \\ & - \int_\Omega v(x,t) \left[\Delta u(x,t) + \mu_1 \Delta v(x,t) + \int_{\tau_1}^{\tau_2} \mu_2(s) \Delta z(x,1,t,s)\, ds \right] dx \\ & + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 \nabla z . \nabla z_\rho(x, \rho, t, s)\, d\rho ds dx \end{aligned}$$

$$= \mu_1 \int_\Omega |\nabla v(x,t)|^2 + \int_\Omega \nabla v \left(\int_{\tau_1}^{\tau_2} \mu_2(s) \nabla z(x,1,t,s)\, ds \right) dx$$
$$+ \frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 \frac{\partial}{\partial \rho} \left(|\nabla z(x,\rho,s,t)|^2 \right) d\rho ds dx.$$

The use of Cauchy Schwarz inequality gives

$$\begin{aligned} \langle \mathcal{A}_1 \phi, \phi \rangle_{\mathcal{H}} \geq{}& \mu_1 \int_\Omega |\nabla v(x,t)|^2 - \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_\Omega |\nabla v(x,t)|^2\, dx \\ & - \frac{1}{2} \int_\Omega \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| \, |\nabla z(x,1,t,s)|^2\, ds \right) dx \\ & + \frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, |\nabla z(x,1,t,s)|^2\, ds dx - \frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, |\nabla z(x,0,t,s)|^2\, ds dx. \end{aligned}$$

Taking into account that $z(.,0,.,.) = v$, we get

$$\langle \mathcal{A}_1 \phi, \phi \rangle_{\mathcal{H}} \geq \frac{1}{2} \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \right) \int_\Omega |\nabla v(x,t)|^2 \geq 0,$$

by virtue of (2.4.4). Therefore, $\mathcal{A}_1$ is monotone.

Next, we show that $\mathcal{A}_1$ is maximal. That is, for each $F = (f,g,h) \in \mathcal{H}$, we have to find $\phi \in D(\mathcal{A}_1)$ such that

$$\phi' + \mathcal{A}_1 \phi = F,$$

i.e.,

$$\left\{ \begin{aligned} u - v &= f, \\ v - \Delta u(x,t) - \mu_1 \Delta v(x,t) - \textstyle\int_{\tau_1}^{\tau_2} |\mu_2(s)| \, \Delta z(x,1,t,s)\, ds &= g, \\ s z(x,\rho,t,s) + z_\rho(x,\rho,t,s) &= sh. \end{aligned} \right. \tag{2.4.5}$$

Using similar arguments as for the first problem, we get

$$z(\rho,.,.) = (u-f)\, e^{-\rho s} + \tau e^{-\rho s} \int_0^\rho h(\gamma,.)\, e^{\gamma s} d\gamma.$$

Thus, $z(.,0,.) = v$ and

$$z, z_\rho \in H_0^1(\Omega).$$

Replacing v by $u-f$ in the second equation of (2.4.5), we get

$$u - \widetilde{k} \Delta u = \widetilde{G}, \tag{2.4.6}$$

where

$$\widetilde{k}=1+\mu_1+\int_{\tau_1}^{\tau_2} e^{-s}\left|\mu_2(s)\right| ds$$

and

$$\begin{aligned}\widetilde{G}=g+f-\left(\mu_1+\int_{\tau_1}^{\tau_2} e^{-s}\left|\mu_2(s)\right| ds\right)\Delta f\\ +\tau\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right| e^{-s}\int_0^1 \Delta h(\gamma,x)\, e^{\gamma s} d\gamma ds\in H^{-1}(\Omega).\end{aligned}$$

The variational formulation corresponding to (2.4.6) takes the form

$$\widetilde{B}(u,w)=\widetilde{L}(w), \tag{2.4.7}$$

where $\widetilde{B}$ is the bilinear form defined over $H_0^1(\Omega)$ by

$$\widetilde{B}(u,w)=\int_\Omega uw+\left(1+\mu_1+\int_{\tau_1}^{\tau_2} e^{-s}\left|\mu_2(s)\right| ds\right)\int_\Omega \nabla u\nabla w$$

and $\widetilde{L}$ is the linear form defined over $H_0^1(\Omega)$ by

$$\widetilde{L}(w)=\left\langle \widetilde{G},w\right\rangle_{H^{-1}\times H_0^1}.$$

One can easily see that $\widetilde{B}$ and $\widetilde{L}$ satisfy the conditions of the Lax-Milgram theorem. Consequently, (2.4.7) has a unique solution $u\in H_0^1(\Omega)$. Therefore, $v=u-f\in H_0^1(\Omega)$ and $z,z_\rho\in H_0^1(\Omega)$.

Replacing Δf by $\Delta(u-f)$ and $e^s\int_0^1\Delta h(\gamma,x)\, e^{\gamma s}d\gamma$ by $\Delta z(x,1,s)-e^s\Delta v$ in (2.4.7), we arrive at

$$\int_\Omega uw+\int_\Omega\nabla\left(u+\mu_1 v+\int_{\tau_1}^{\tau_2}\mu_2(s)\, e^{-s}z(x,1,s)\, ds\right).\nabla w=\int(f+g)\, w,\forall w\in H_0^1(\Omega)$$

which gives, by the standard elliptic theory,

$$u+\mu_1 v+\int_{\tau_1}^{\tau_2}\mu_2(s)\, e^{-s}z(x,1,s)\, ds\in H^2(\Omega).$$

Therefore,

$$\Delta\left(u+\mu_1 v+\int_{\tau_1}^{\tau_2}\mu_2(s)\, e^{-s}z(.,1,s)\, ds\right)=f+g-u\in L^2(\Omega).$$

Consequently, (2.4.5) has a unique solution $(u, v, z) \in D(\mathcal{A}_1)$. This shows that $\mathcal{A}_1$ is maximal. Thus, the Hille-Yosida theorem guarantees the existence of a unique solution to problem (2.4.5). This completes the proof of Theorem 8. □

2.4.2 Exponential stability

In this subsection, we prove our decay result.

Theorem 9. *Suppose that μ_1 and μ_2 satisfy (2.4.4) and that $(u_0, u_1, f_0) \in D(\mathcal{A}_1)$. Then the solution of (2.4.3) satisfies, for two positive constants η, ω the estimate*

$$E(t) \leq \eta e^{-\omega t}, \ \forall t \geq 0.$$

The proof will be established through the following Lemmas:

First, we define the energy associated with the solution of (2.4.3) by

$$E(t) = \frac{1}{2}\left[\int_\Omega u_t^2 dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s.\,|\mu_2(s)|\,|\nabla z(x, \rho, t, s)|^2 \, ds d\rho dx\right]. \quad (2.4.8)$$

Lemma 6. *Suppose that μ_1, μ_2 satisfy (2.4.4). Then the energy given by (2.4.8) satisfies, along the solution of (2.4.3), the estimate*

$$E'(t) \leqslant -\left(\mu_1 - \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)\right) \int_\Omega |\nabla u_t(x, t)|^2 dx < 0. \quad (2.4.9)$$

Proof 1. *A differentiation of $E(t)$ gives*

$$\begin{aligned} E'(t) &= \int_\Omega u_{tt} u_t dx + \int_\Omega \nabla u . \nabla u_t dx \\ &+ \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s\mu_2(s) \nabla z(x, \rho, s, t) . \nabla z_t(x, \rho, s, t)\, ds dx d\rho. \end{aligned}$$

Using (2.4.3) and integrating by parts, we get

$$\begin{aligned} E'(t) &= -\mu_1 \int_\Omega |\nabla u_t(x, t)|^2 dx - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, \nabla z(x, 1, t, s) . \nabla u_t(x, t)\, dx \\ &- \frac{1}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \frac{d}{d\rho} |\nabla z(x, \rho, t, s)|^2 ds dx d\rho \\ &= -\mu_1 \int_\Omega |\nabla u_t|^2 dx - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, \nabla z(x, 1, t, s) . \nabla u_t(x, t)\, dx \\ &- \frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, |\nabla z(x, 1, t, s)|^2 ds dx + \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) \int_\Omega |\nabla u_t|^2 dx. \end{aligned}$$

Young's inequality leads to the desired estimate.

Lemma 7. *The functional*

$$I_1 = \int_\Omega u_t u dx$$

satisfies, along the solution of (2.4.3), the estimate

$$I_1'(t) \leqslant -\frac{1}{2}\int_\Omega |\nabla u|^2 dx + c\int_\Omega |\nabla u_t|^2 dx + c\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| |\nabla z(x,1,t,s)|^2 dsdx, \quad (2.4.10)$$

for a positive constant c.

Proof 2. *The differentiaton of $I_1(t)$, using $(2.4.3)_1$, yields*

$$\begin{aligned} I_1'(t) &= \int_\Omega \left(\Delta u + \mu_1 \Delta u_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\Delta z(x,1,t,s)\, ds\right) udx + \int_\Omega u_t^2 dx \\ &= -\int_\Omega |\nabla u|^2 dx - \mu_1 \int_\Omega \nabla u_t . \nabla u dx \\ &- \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) \nabla z(x,1,t,s) . \nabla u(x,t)\, dsdx + \int_\Omega u_t^2 dx. \end{aligned}$$

Young's inequality gives

$$\begin{aligned} I_1'(t) &\leqslant -\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \mu_1^2 \int_\Omega |\nabla u_t|^2 dx \\ &+ \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| |\nabla z(x,1,t,s)|^2 dxds + \int_\Omega u_t^2 dx. \end{aligned}$$

Then Poincaré's inequality leads to the desired estimate.

Lemma 8. *The functional*

$$I_2(t) = \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu_2(s)| |\nabla z(x,\rho,t,s)|^2 dsd\rho dx \quad (2.4.11)$$

satisfies, along the solution of (2.4.3), the estimate

$$\begin{aligned} I_2'(t) &\leqslant -e^{-\tau_2} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| |\nabla z(x,1,t,s)|^2 dsdx \\ &+ \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) \int_\Omega |\nabla u_t(x,t)|^2 dsdx \\ &- e^{-\tau_2} \int_0^1 \int_\Omega \int_{\tau_1}^{\tau_2} s |\mu_2(s)| |\nabla z(x,\rho,t,s)|^2 dsd\rho dx. \end{aligned}$$

Proof. The differentiation of $I_2(t)$ yeilds

$$I_2'(t) = 2\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| \nabla z(x,\rho,t,s) \nabla z_t(x,\rho,t,s)\, d\rho ds dx.$$

Using the second equation of (2.4.3), we have

$$\begin{aligned} I_2'(t) &= -\int_\Omega \int_{\tau_1}^{\tau_2} \int_0^1 e^{-s\rho} |\mu_2(s)| \frac{d}{d\rho} |\nabla z(x,\rho,t,s)|^2 \, d\rho ds dx, \\ &= -\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 \frac{d}{d\rho}\left(e^{-s\rho} |\nabla z(x,\rho,t,s)|^2\right) d\rho ds dx \\ &- \int_\Omega \int_{\tau_1}^{\tau_2} s |\mu_2(s)| \int_0^1 e^{-s\rho} |\nabla z(x,\rho,t,s)|^2 \, d\rho ds dx. \end{aligned}$$

Thus,

$$\begin{aligned} I_2'(t) &= -\int_\Omega \int_{\tau_1}^{\tau_2} e^{-s} |\mu_2(s)| |\nabla z(x,1,t,s)|^2 \, ds dx + \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) \int_\Omega |\nabla u_t|^2 \, dx \\ &- \int_\Omega \int_{\tau_1}^{\tau_2} s |\mu_2(s)| \int_0^1 e^{-s\rho} |\nabla z|^2 (x,\rho,t,s)\, ds d\rho \end{aligned}$$

and the desired estimate follows immediately. □

2.4.3 Proof of theorem 9

To complete the proof of theorem, we define the Lyapunov functional

$$\mathcal{L}(t) = NE(t) + I_1(t) + MI_2(t),$$

where N and M are positive constants to be specified later.

Differentiating $\mathcal{L}$ and using (2.4.9)–(2.4.11), we obtain

$$\begin{aligned} \mathcal{L}'(t) \leqslant &-\left[N\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) - c - M\left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)\right] \int_\Omega |\nabla u_t(x,t)|^2 \, dx \\ &- \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 \, dx - \left(Me^{-\tau_2} - c\right) \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| |\nabla z(x,1,t,s)|^2 \, dx ds \\ &- e^{-\tau_2} M \int_0^1 \int_\Omega \int_{\tau_1}^{\tau_2} s |\mu_2(s)| \nabla z^2(x,\rho,t,s)\, ds d\rho dx. \end{aligned}$$

It suffices to choose M large enough such that

$$Me^{-\tau_2} - c > 0,$$

then, pick N large enough so that

$$N\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) - c - M\left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) > 0$$

and $\mathcal{L} \sim E$. Thus, Poincaré's inequality leads to

$$\mathcal{L}'(t) \leq -\lambda E(t), \ \forall t \geq 0,$$

for a positive constant λ. The fact that $\mathcal{L} \sim E$ yields

$$\mathcal{L}'(t) \leq -w\mathcal{L}(t), \ \forall t \geq 0.$$

An integration over $(0, t)$ gives

$$\mathcal{L}(t) \leqslant \mathcal{L}(0)\, e^{-wt}, \ \forall t > 0.$$

The use of $\mathcal{L} \sim E$ again leads to the desired inequality.

Chapter 3

Global existence combined with exponential decay of solutions for coupled kirchhoff system with a distributed delay term

3.1 Introduction

This chapter deals with the proof of the global existence of solutions of nonlinear viscoelastic Kirchhoff system with a distributed delay and general coupling terms in a bounded domain. The current study is performed by using the energy method along with Faedo-Galerkin method and under some suitable conditions in coupling terms parameters. In addition, we prove the stability result by using the multiplier method.

In the present chapter, we are interested in the following nonlinear viscoelastic Kirchhoff system with a distributed delay and general coupling terms

$$\begin{cases} |u_t|^l u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t g_1(t-s)\Delta u(s)ds \\ \quad -k_1\Delta u_t - \int_{\tau_1}^{\tau_2} \mu_1(\varrho)\Delta u_t(x,t-\varrho)d\varrho + \alpha v = 0, \\ |v_t|^l v_{tt} - M(\|\nabla v\|^2)\Delta v - \Delta v_{tt} + \int_0^t g_2(t-s)\Delta v(s)ds \\ \quad -k_2\Delta v_t - \int_{\tau_1}^{\tau_2} \mu_2(\varrho)\Delta v_t(x,t-\varrho)d\varrho + \alpha u = 0, \end{cases} \tag{3.1.1}$$

where

$$(x,\varrho,t) \in \Omega \times (\tau_1,\tau_2) \times (0,\infty).$$

With the initial data and boundary conditions

$$\begin{cases} (u(x,0),v(x,0)) = (u_0(x),v_0(x)), & \text{in } \Omega, \\ (u_t(x,0),v_t(x,0)) = (u_1(x),v_1(x)), & \text{in } \Omega, \\ (u_t(x,-t),v_t(x,-t)) = (f_0(x,t),g_0(x,t)), & \text{in } \Omega \times (0,\tau_2), \\ u(x,t) = v(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{3.1.2}$$

where Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, l, α is positive constants. Here, Δ denotes the Laplacien operator, the second integral represents the distributed delay and $\mu_1, \mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ are a bounded functions, where τ_1, τ_2 are two real numbers satisfying $0 \leq \tau_1 < \tau_2$, and g_1, g_2 are the relaxation functions. M is a smooth function defined by

$$M \ : \ \mathbb{R}_+ \to \mathbb{R}_+ \tag{3.1.3}$$

$$r \mapsto M(r) = a + br^{\gamma}, \tag{3.1.4}$$

with $a, b > 0$.

3.2 Preliminaries

In this section, we present some materials which will be used in order to prove our main results. We have the following assumptions:

(A1) $g_i : \mathbb{R}_+ \to \mathbb{R}_+, \ \ i = 1,2$ are C^1 functions satisfying

$$g(0) > 0, \ \ a - \int_0^\infty g_i(s)ds = k > 0, \ \ i = 1,2. \tag{3.2.1}$$

(A2) There exists a positive constants ξ_i satisfying

$$g_i'(t) \leq -\xi_i g_i\left(t\right), \ \ i = 1,2, \ \ t \geq 0. \tag{3.2.2}$$

(A3) Consider that $0 < l \leq \gamma$ satisfying

$$\begin{cases} \gamma \leq \frac{2}{n-2} \ \ if \ \ n > 2, \\ \gamma < \infty \ \ \ if \ \ n \leq 2. \end{cases} \tag{3.2.3}$$

Let us introduce the following notations

$$(g \circ \phi)(t) := \int_0^t g(t-s)\|\phi(t) - \phi(s)\|^2 ds.$$

As in [59], taking the following new variables

$$\begin{cases} z(x,\rho,\varrho,t) = u_t(x,t-\varrho\rho), \\ y(x,\rho,\varrho,t) = v_t(x,t-\varrho\rho), \end{cases}$$

then we obtain

$$\begin{cases} \varrho z_t(x,\rho,\varrho,t) + z_\rho(x,\rho,\varrho,t) = 0, \\ z(x,0,\varrho,t) = u_t(x,t), \end{cases} \tag{3.2.4}$$

and

$$\begin{cases} \varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = 0, \\ y(x,0,\varrho,t) = v_t(x,t). \end{cases} \tag{3.2.5}$$

Consequently, the problem (3.1.1) is equivalent to

$$\begin{cases} |u_t|^l u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t g_1(t-s)\Delta u(s)ds \\ \quad -k_1 \Delta u_t - \int_{\tau_1}^{\tau_2} \mu_1(\varrho)\Delta z(x,1,\varrho,t)d\varrho + \alpha v = 0, \\ |v_t|^l v_{tt} - M(\|\nabla v\|^2)\Delta v - \Delta v_{tt} + \int_0^t g_2(t-s)\Delta v(s)ds \\ \quad -k_2 \Delta v_t - \int_{\tau_1}^{\tau_2} \mu_2(\varrho)\Delta y(x,1,\varrho,t)d\varrho + \alpha u = 0, \\ \varrho z_t(x,\rho,\varrho,t) + z_\rho(x,\rho,\varrho,t) = 0, \\ \varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = 0, \end{cases} \tag{3.2.6}$$

where

$$(x,\rho,\varrho,t) \in \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty).$$

The system together with the initial data and boundary conditions

$$\begin{cases} (u(x,0),v(x,0)) = (u_0(x),v_0(x)), & \text{in } \Omega, \\ (u_t(x,0),v_t(x,0)) = (u_1(x),v_1(x)), & \text{in } \Omega, \\ (u_t(x,-t),v_t(x,-t)) = (f_0(x,t),g_0(x,t)), & \text{in } \Omega \times (0,\tau_2), \\ u(x,t) = v(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ z(x,\rho,\varrho,0) = f_0(x,\rho\varrho), & \text{in } \Omega \times (0,1) \times (0,\tau_2), \\ y(x,\rho,\varrho,0) = g_0(x,\rho\varrho), & \text{in } \Omega \times (0,1) \times (0,\tau_2). \end{cases}$$

We need the following lemma.

Lemma 9. *Assume that*

$$\int_{\tau_1}^{\tau_2} |\mu_i(\varrho)|\, d\varrho < k_i, \; i = 1,2. \tag{3.2.7}$$

The energy functional E, defined by

$$\begin{aligned} E(t) &= \frac{1}{l+2}\Big(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\Big)+\frac{b}{2(\gamma+1)}\Big(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\Big) \\ &+\frac{1}{2}\Big(a-\int_0^t g_1(s)ds\Big)\|\nabla u\|^2+\frac{1}{2}\Big(a-\int_0^t g_2(s)ds\Big)\|\nabla v\|^2 \\ &+\frac{1}{2}\Big(\|\nabla u_t\|^2+\|\nabla v_t\|^2\Big)+\frac{1}{2}(g_1\circ\nabla u)(t)+\frac{1}{2}(g_2\circ\nabla v)(t) \\ &+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\varrho\Big(\mu_1(\varrho)\|\nabla z\|^2+\mu_2(\varrho)\|\nabla y\|^2\Big)d\varrho d\rho+\alpha\int_\Omega uvdx, \end{aligned} \tag{3.2.8}$$

satisfies

$$\begin{aligned} E'(t) &\leq -\lambda\Big(\|\nabla u_t\|^2+\|\nabla v_t\|^2\Big)+\frac{1}{2}(g_1'\circ\nabla u)(t)+\frac{1}{2}(g_2'\circ\nabla v)(t) \\ &-\frac{1}{2}g_1(t)\|\nabla u(t)\|^2-\frac{1}{2}g_2(t)\|\nabla v(t)\|^2\leq 0, \end{aligned} \tag{3.2.9}$$

where λ is a positive constant.

Proof. Multiplying the equation $(3.2.6)_1$ by u_t and the second equation $(3.2.6)_2$ by v_t, then integration by parts over Ω, and using (3.1.2), we get

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{1}{l+2}\|u_t\|_{l+2}^{l+2}+\frac{b}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}+\frac{1}{2}\Big(a-\int_0^t g_1(s)ds\Big)\|\nabla u\|^2 \\ &+\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}(g_1\circ\nabla u)(t)\Big\}-\frac{1}{2}(g_1'\circ\nabla u)(t)+\frac{1}{2}g_1(t)\|\nabla u(t)\|^2 \\ &+k_1\|\nabla u_t(t)\|^2+\int_\Omega\nabla u_t\int_{\tau_1}^{\tau_2}\mu_1(\varrho)\nabla z(x,1,\varrho,t)d\varrho dx+\alpha\int_\Omega u_tvdx=0, \\ &\frac{d}{dt}\Big\{\frac{1}{l+2}\|v_t\|_{l+2}^{l+2}+\frac{b}{2(\gamma+1)}\|\nabla v\|^{2(\gamma+1)}+\frac{1}{2}\Big(a-\int_0^t g_2(s)ds\Big)\|\nabla v\|^2 \\ &+\frac{1}{2}\|\nabla v_t\|^2+\frac{1}{2}(g_2\circ\nabla v)(t)\Big\}-\frac{1}{2}(g_2'\circ\nabla v)(t)+\frac{1}{2}g_2(t)\|\nabla v(t)\|^2 \\ &+k_2\|\nabla v_t(t)\|^2+\int_\Omega\nabla v_t\int_{\tau_1}^{\tau_2}\mu_2(\varrho)\nabla y(x,1,\varrho,t)d\varrho dx+\alpha\int_\Omega v_tudx=0. \end{aligned} \tag{3.2.10}$$

Now, multiplying the equation $(3.2.6)_3$ by $-\Delta z|\mu_1(\varrho)|$, and integrating over $\Omega\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}
&\frac{d}{dt}\quad \frac{1}{2}\int_{\Omega}\int_{0}^{1}\int_{\tau_1}^{\tau_2}\varrho|\mu_1(\varrho)|(\nabla z)^2 d\varrho d\rho dx\\
=&\quad -\int_{\Omega}\int_{0}^{1}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\nabla z\nabla z_{\rho} d\varrho d\rho dx\\
=&\quad -\frac{1}{2}\int_{\Omega}\int_{0}^{1}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\frac{d}{d\rho}(\nabla z)^2 d\varrho d\rho dx\\
=&\quad \frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\Big((\nabla z(x,0,\varrho,t))^2-(\nabla z(x,1,\varrho,t))^2\Big)d\varrho dx\\
=&\quad \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|d\varrho\int_{\Omega}|\nabla u_t|^2dx-\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|(\nabla z\,(x,1,\varrho,t))^2 d\varrho dx\\
=&\quad \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|d\varrho\right)\|\nabla u_t\|^2-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\|\nabla z\,(x,1,\varrho,t)\,\|^2 d\varrho.
\end{aligned} \tag{3.2.11}$$

Similarly, by multiplying the equation $(3.2.6)_4$ by $-\Delta y|\mu_2(\varrho)|$, and integrating the result over $\Omega\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}
&\frac{d}{dt}\quad \frac{1}{2}\int_{\Omega}\int_{0}^{1}\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|(\nabla y)^2 d\varrho d\rho dx\\
=&\quad \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\right)\|\nabla v_t\|^2-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\|\nabla y\,(x,1,\varrho,t)\,\|^2 d\varrho.
\end{aligned} \tag{3.2.12}$$

By summing (3.2.10)-(3.2.12), we obtain

$$\begin{aligned}
E'(t) \quad =&\quad \frac{1}{2}(g_1'\circ\nabla u)(t)-\frac{1}{2}g_1(t)\|\nabla u(t)\|^2-k_1\|\nabla u_t(t)\|^2\\
&-\int_{\Omega}\nabla u_t\int_{\tau_1}^{\tau_2}\mu_1(\varrho)\nabla z(x,1,\varrho,t)d\varrho dx\\
&+\frac{1}{2}(g_2'\circ\nabla v)(t)-\frac{1}{2}g_2(t)\|\nabla v(t)\|^2-k_2\|\nabla v_t(t)\|^2\\
&-\int_{\Omega}\nabla v_t\int_{\tau_1}^{\tau_2}\mu_2(\varrho)\nabla y(x,1,\varrho,t)d\varrho dx\\
&+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|d\varrho\right)\|\nabla u_t\|^2-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\|\nabla z\,(x,1,\varrho,t)\,\|^2 d\varrho\\
&+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\right)\|\nabla v_t\|^2-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\|\nabla y\,(x,1,\varrho,t)\,\|^2 d\varrho.
\end{aligned} \tag{3.2.13}$$

And by Young's and Cauchy-Schwartz inequalities, we have

$$\int_{\Omega} \nabla u_t \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \nabla z(x,1,\varrho,t) d\varrho dx \leq \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho\right) \|\nabla u_t\|^2 + \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \|\nabla z(x,1,\varrho,t)\|^2 d\varrho. \tag{3.2.14}$$

Similarly, we get

$$\int_{\Omega} \nabla v_t \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \nabla y(x,1,\varrho,t) d\varrho dx \leq \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho\right) \|\nabla v_t\|^2 + \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \|\nabla y(x,1,\varrho,t)\|^2 d\varrho. \tag{3.2.15}$$

Using (3.2.14) and (3.2.15), we get

$$\begin{aligned} E'(t) \leq & -\left(k_1 - \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)|\, d\varrho\right) \|\nabla u_t\|^2 - \left(k_2 - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, d\varrho\right) \|\nabla v_t\|^2 \\ & + \frac{1}{2}(g_1' \circ \nabla u)(t) + \frac{1}{2}(g_2' \circ \nabla v)(t) - \frac{1}{2} g_1(t) \|\nabla u(t)\|^2 - \frac{1}{2} g_2(t) \|\nabla v(t)\|^2. \end{aligned} \tag{3.2.16}$$

Using hypothesis (3.2.7), we obtain (3.2.8) and (3.2.9). The Proof is complete. □

3.3 Global existence

Theorem 10. *Assume that (3.2.1)-(3.2.3) hold. Then given $(u_0, v_0) \in (H^2(\Omega) \cap H_0^1(\Omega))^2$, $(u_1, v_1) \in (H_0^1(\Omega))^2$ and $(f_0, g_0) \in (H^1(\Omega, (0,1), (\tau_1, \tau_2)))^2$, there exists a weak solution (u, v, z, y) of problem (3.2.6)-(3.1.2) such that*

$$(u, v, z, y) \in L^\infty(\mathbb{R}_+, \mathcal{H}_1), \;\; u_t, v_t \in L^\infty(\mathbb{R}_+, H_0^1(\Omega)), \;\; u_{tt}, v_{tt} \in L^2(\mathbb{R}_+, H_0^1(\Omega)),$$

where

$$\mathcal{H}_1 = (H^2(\Omega) \cap H_0^1(\Omega))^2 \times (H^1(\Omega, (0,1), (\tau_1, \tau_2)))^2.$$

Proof. Let u_j, v_j, z_j, y_j be the Galerkin basis. For $n \geq 1$, let

$$\begin{aligned} W_n &= span\{u_1, u_2, ..., u_n\}, \\ K_n &= span\{v_1, v_2, ..., v_n\}. \end{aligned}$$

We define for $1 \leq j \leq n$ the sequences $z_j(x, \tau, p)$, $y_j(x, \tau, p)$ by

$$z_j(x, 0, p) = u_j(x), \ \ y_j(x, 0, p) = v_j(x).$$

Then, we can extend $z_j(x, 0, p), y_j(x, 0, p)$ over $L^2((0,1) \times (0,1) \times (\tau_1, \tau_2))$ and denote

$$\begin{aligned} Z_n &= span\{z_1, z_2, ..., z_n\}, \\ Y_n &= span\{y_1, y_2, ..., y_n\}. \end{aligned}$$

Given initial data $u_0, v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1, v_1 \in H_0^1(\Omega)$ and $f_0, g_0 \in L^2((\Omega) \times (0,1) \times (\tau_1, \tau_2))$, we define the approximations

$$\begin{aligned} u_m &= \sum_{j=1}^{n} g_{jm}(t) u_j(x), \\ v_m &= \sum_{j=1}^{n} h_{jm}(t) v_j(x), \\ z_m &= \sum_{j=1}^{n} f_{jm}(t) z_j(x, \tau, p), \\ y_m &= \sum_{j=1}^{n} k_{jm}(t) y_j(x, \tau, p), \end{aligned} \tag{3.3.1}$$

which satisfy the following approximate problem

$$\begin{aligned} &(|u_{mt}|^l u_{mtt}, u_j) + M(\|\nabla u_m(t)\|)(\nabla u_m, \nabla u_j) + (\nabla u_{mtt}, \nabla u_j) + \alpha(v_m, u_j) \\ &\quad - \int_0^t g_1(t-s)(\nabla u_m(s), \nabla u_j) ds + k_1(\nabla u_{mt}, \nabla u_j) \\ &\quad + \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| (\nabla z_m(x, 1, \varrho, t), \nabla u_j) d\varrho = 0, \end{aligned}$$

$$(|v_{mt}|^l v_{mtt}, v_j) + M(\|\nabla v_m(t)\|)(\nabla v_m, \nabla v_j) + (\nabla v_{mtt}, \nabla v_j) + \alpha(u_m, v_j)$$

$$-\int_0^t g_2(t-s)(\nabla v_m(s), \nabla v_j)ds + k_2(\nabla v_{mt}, \nabla v_j)$$
$$+\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|(\nabla y_m(x,1,\varrho,t), \nabla v_j)d\varrho = 0,$$

$$(\varrho z_{mt}(x,\rho,\varrho,t), z_j) + (z_{m\rho}(x,\rho,\varrho,t), z_j) = 0,$$
$$(\varrho y_{mt}(x,\rho,\varrho,t), y_j) + (y_{m\rho}(x,\rho,\varrho,t), y_j) = 0, \tag{3.3.2}$$

with initial conditions

$$u_m(0) = u_0^m, u_{mt}(0) = u_1^m,$$
$$v_m(0) = v_1^m, v_{mt}(0) = v_1^m,$$
$$z_m(0) = z_0^m, y_m(0) = y_0^m, \tag{3.3.3}$$

which satisfies

$$u_0^m \to u_0, \;\; in \;\; H^2(\Omega) \cap H_0^1(\Omega),$$
$$u_1^m \to u_1, \;\; in \;\; H_0^1(\Omega),$$
$$v_0^m \to v_0, \;\; in \;\; H^2(\Omega) \cap H_0^1(\Omega),$$
$$v_1^m \to v_1, \;\; in \;\; H_0^1(\Omega),$$
$$z_0^m \to z_0, \;\; in \;\; L^2(\Omega \times (0,1) \times (\tau_1,\tau_2)),$$
$$y_0^m \to y_0, \;\; in \;\; L^2(\Omega \times (0,1) \times (\tau_1,\tau_2)). \tag{3.3.4}$$

Noting that $\frac{l}{2(l+1)} + \frac{1}{2(l+1)} + \frac{1}{2} = 1$, by applying the generalized Hölder inequality, we find

$$\begin{aligned}(|u_{mt}|^l u_{mtt}, u_j) &= \int_\Omega |u_{mt}|^l u_{mtt} u_j dx \\ &\leq \left(\int_\Omega |u_{mt}|^{2(l+1)} dx\right)^{\frac{1}{2(l+1)}} \|u_{mtt}\|_{2(l+1)} \|u_j\|_2.\end{aligned} \tag{3.3.5}$$

Since (3.2.3) holds, according to the Sobolev embedding the nonlinear terms (for more detail see ([27])
$(|u_{mt}|^l u_{mtt}, u_j)$ and $(|v_{mt}|^l v_{mtt}, v_j)$ in (3.3.2) make sense.

First estimate.

Since the sequences $u_0^m, v_0^m, u_1^m, v_1^m, z_0^m(.,.,0)$ and $y_0^m(.,.,0)$ converge and from (3.2.9) with employing Gronwall's lemma, we find $C_1 > 0$ independent of m such that

$$E_m(t) + \beta \int_{\tau_1}^{\tau_2} \varrho\Big(|\mu_1(\varrho)| \|\nabla z_m(x,1,\varrho,t)\|^2 + |\mu_2(\varrho)| \|\nabla y_m(x,1,\varrho,t)\|^2 \Big) d\varrho \leq C_1, \quad (3.3.6)$$

where

$$\begin{aligned} E_m(t) &= \frac{1}{l+2}\Big(\|u_{mt}\|_{l+2}^{l+2} + \|v_{mt}\|_{l+2}^{l+2} \Big) + \alpha \int_\Omega u_m v_m dx \\ &+ \frac{b}{2(\gamma+2)}\Big(\|\nabla u_m\|^{2(\gamma+2)} + \|\nabla v_m\|^{2(\gamma+2)} \Big) \\ &+ \frac{1}{2}\Big(a - \int_0^t g_1(s)ds \Big) \|\nabla u_m\|^2 + \frac{1}{2}\Big(a - \int_0^t g_2(s)ds \Big) \|\nabla v_m\|^2 \\ &+ \frac{1}{2}\Big(\|\nabla u_{mt}\|^2 + \|\nabla v_{mt}\|^2 \Big) + \frac{1}{2}(g_1 \circ \nabla u_m)(t) + \frac{1}{2}(g_2 \circ \nabla v_m)(t) \\ &+ \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} \varrho\Big(|\mu_1(\varrho)| \|\nabla z_m\|^2 + |\mu_2(\varrho)| \|\nabla y_m\|^2 \Big) d\varrho d\rho. \end{aligned} \quad (3.3.7)$$

Using (3.3.6) and (3.2.3), gives

$$\begin{aligned} u_m, v_m &\quad \text{are bounded in } L^\infty_{loc}(\mathbb{R}_+, H_0^1(\Omega)), \\ u_{mt}, v_{mt} &\quad \text{are bounded in } L^\infty_{loc}(\mathbb{R}_+, H_0^1(\Omega)), \\ z_m(x,\rho,\varrho,t), y_m(x,\rho,\varrho,t) &\quad \text{are bounded in } L^\infty_{loc}(\mathbb{R}_+, H_0^1(\Omega \times (0,1) \times (\tau_1,\tau_2)). \end{aligned} \quad (3.3.8)$$

The second estimate.

By multiplying $(3.3.2)_1$, $(3.3.2)_2$ by g_{jmtt}, h_{jmtt} respectively, by summing j from 1 to n, then

$$\begin{aligned} &\int_\Omega |u_{mt}|^l |u_{mtt}|^2 dx + \int_\Omega M(\|\nabla u_m(t)\|) \nabla u_m \nabla u_{mtt} dx + \int_\Omega |\nabla u_{mtt}|^2 dx \\ &+ \alpha \int_\Omega v_m u_{mtt} dx - \int_\Omega \int_0^t g_1(t-s) \nabla u_m(s) \nabla u_{mtt} ds dx + k_1 \int_\Omega \nabla u_{mt} \nabla u_{mtt} dx \\ &+ \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \nabla z_m(x,1,\varrho,t) \nabla u_{mtt} d\varrho dx = 0, \end{aligned}$$

$$\int_\Omega |v_{mt}|^l |v_{mtt}|^2 dx + \int_\Omega M(\|\nabla v_m(t)\|) \nabla v_m \nabla v_{mtt} dx + \int_\Omega |\nabla v_{mtt}|^2 dx$$

$$+\alpha\int_{\Omega}u_m v_{mtt}dx-\int_{\Omega}\int_0^t g_2(t-s)\nabla v_m(s)\nabla v_{mtt}dsdx+k_2\int_{\Omega}\nabla v_{mt}\nabla v_{mtt}dx$$
$$+\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\nabla y_m\left(x,1,\varrho,t\right)\nabla v_{mtt}d\varrho dx=0. \tag{3.3.9}$$

Differentiating $(3.3.2)_3$, $(3.3.2)_4$ with respect to t, we get

$$\begin{aligned}&(\varrho z_{mtt}(x,\rho,\varrho,t)+z_{mt\rho}(x,\rho,\varrho,t),z_j)=0,\\&(\varrho y_{mtt}(x,\rho,\varrho,t)+y_{mt\rho}(x,\rho,\varrho,t),y_j)=0.\end{aligned} \tag{3.3.10}$$

Multiplying $(3.3.10)_1$ by z_{jmt} and $(3.3.10)_2$ by y_{jmt}, summing over j from 1 to n, we have

$$\begin{aligned}&\frac{1}{2}\varrho\frac{d}{dt}\|z_{mt}\|^2+\frac{1}{2}\frac{d}{d\rho}\|z_{mt}\|^2=0,\\&\frac{1}{2}\varrho\frac{d}{dt}\|y_{mt}\|^2+\frac{1}{2}\frac{d}{d\rho}\|y_{mt}\|^2=0.\end{aligned} \tag{3.3.11}$$

By integrating over $(0,1)$ with respect to ρ, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\varrho\|z_{mt}\|^2d\rho+\frac{1}{2}\|z_{mt}(x,1,\varrho,t)\|^2-\frac{1}{2}\|u_{mtt}(x,t)\|^2=0,\\&\frac{1}{2}\frac{d}{dt}\int_0^1\varrho\|y_{mt}\|^2d\rho+\frac{1}{2}\|y_{mt}(x,1,\varrho,t)\|^2-\frac{1}{2}\|v_{mtt}(x,t)\|^2=0.\end{aligned} \tag{3.3.12}$$

Summing (3.3.9), (3.3.12) and using the fact that $M(r)\geq a$, we get

$$\begin{aligned}&\int_{\Omega}|u_{mt}|^l|u_{mtt}|^2dx+\|\nabla u_{mtt}\|^2+\frac{1}{2}\frac{d}{dt}\int_0^1\varrho\|z_{mt}\|^2d\rho+\frac{1}{2}\|z_{mt}(x,1,\varrho,t)\|^2\\\leq\ &\frac{1}{2}\|u_{mtt}\|^2-\int_{\Omega}a\nabla u_m\nabla u_{mtt}dx-\alpha\int_{\Omega}v_m u_{mtt}dx\\&+\int_{\Omega}\int_0^t g_1(t-s)\nabla u_m(s)\nabla u_{mtt}dsdx-k_1\int\nabla u_{mt}\nabla u_{mtt}dx\\&-\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\nabla z_m\left(x,1,\varrho,t\right)\nabla u_{mtt}d\varrho dx,\end{aligned}$$

$$\begin{aligned}&\int_{\Omega}|v_{mt}|^l|v_{mtt}|^2dx+\|\nabla v_{mtt}\|^2+\frac{1}{2}\frac{d}{dt}\int_0^1\varrho\|y_{mt}\|^2d\rho+\frac{1}{2}\|y_{mt}(x,1,\varrho,t)\|^2\\\leq\ &\frac{1}{2}\|v_{mtt}(x,t)\|^2-\int_{\Omega}a\nabla v_m\nabla v_{mtt}dx-\alpha\int_{\Omega}u_m v_{mtt}dx\end{aligned}$$

$$+\int_\Omega\int_0^t g_2(t-s)\nabla v_m(s)\nabla v_{mtt}dsdx - k_2\int \nabla v_{mt}\nabla v_{mtt}dx$$
$$-\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\nabla y_m\left(x,1,\varrho,t\right)\nabla v_{mtt}d\varrho dx. \tag{3.3.13}$$

Now, we estimate the right hand side of (3.3.13):

Using integration by parts, Young's and Poincaré inequalities, we have

$$\left|-\alpha\int_\Omega v_m u_{mtt}dx\right| \le C_*\alpha\bigg(\eta\|\nabla u_{mtt}\|^2 + \frac{1}{4\eta}\|\nabla v_m\|^2\bigg). \tag{3.3.14}$$

Similarly, we get

$$\left|-\alpha\int_\Omega u_m v_{mtt}dx\right| \le C_*\alpha\bigg(\frac{1}{4\eta}\|\nabla u_m\|^2 + \eta\|\nabla v_{mtt}\|^2\bigg). \tag{3.3.15}$$

Also, by Young's inequality, we get

$$\left|\int_\Omega a\nabla u_m\nabla u_{mtt}dx\right| \le \eta\|\nabla u_{mtt}\|^2 + \frac{a^2}{4\eta}\|\nabla u_m\|^2,$$
$$\left|\int_\Omega a\nabla v_m\nabla v_{mtt}dx\right| \le \eta\|\nabla v_{mtt}\|^2 + \frac{a^2}{4\eta}\|\nabla v_m\|^2. \tag{3.3.16}$$

Similarly, we get

$$\left|-k_1\int_\Omega \nabla u_{mt}\nabla u_{mtt}dx\right| \le \eta\|\nabla u_{mtt}\|^2 + \frac{k_1^2}{4\eta}\|\nabla u_{mt}\|^2,$$
$$\left|-k_2\int_\Omega \nabla v_{mt}\nabla v_{mtt}dx\right| \le \eta\|\nabla v_{mtt}\|^2 + \frac{k_2^2}{4\eta}\|\nabla v_{mt}\|^2, \tag{3.3.17}$$

and we have

$$\left|\int_\Omega\int_0^t g_1(t-s)\nabla u_m(s)\nabla u_{mtt}dsdx\right| \le \eta\|\nabla u_{mtt}\|^2 + \frac{(a-k)g_1(0)}{4\eta}\int_0^t \|\nabla u_m(s)\|^2 ds,$$

$$\left|\int_\Omega\int_0^t g_2(t-s)\nabla v_m(s)\nabla v_{mtt}dsdx\right| \le \eta\|\nabla v_{mtt}\|^2 + \frac{(a-k)g_2(0)}{4\eta}\int_0^t \|\nabla v_m(s)\|^2 ds. \tag{3.3.18}$$

Similarly, we get

$$\begin{aligned}
&\left|\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \nabla z_m\left(x,1,\varrho,t\right) \nabla u_{mtt} d\varrho dx\right| \\
\leq\ & \eta k_1 \|\nabla u_{mtt}\|^2 + \frac{1}{4\eta}\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \|\nabla z_m\left(x,1,\varrho,t\right)\|^2 d\varrho, \\
&\left|\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \nabla y_m\left(x,1,\varrho,t\right) \nabla v_{mtt} d\varrho dx\right| \\
\leq\ & \eta k_2 \|\nabla v_{mtt}\|^2 + \frac{1}{4\eta}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \|\nabla y_m\left(x,1,\varrho,t\right)\|^2 d\varrho.
\end{aligned} \tag{3.3.19}$$

Substituting (3.3.14)-(3.3.19) into (3.3.13), and by using (3.2.9), yields

$$\begin{aligned}
&\int_\Omega |u_{mt}|^l |u_{mtt}|^2 dx + \left(1 - \left\{\eta(k_1+3+\alpha C_*)\right\}\right)\|\nabla u_{mtt}\|^2 \\
&+\frac{1}{2}\frac{d}{dt}\int_0^1 \varrho \|z_{mt}\|^2 d\rho + \frac{1}{2}\|z_{mt}(x,1,\varrho,t)\|^2 \\
\leq\ & C_2 + \frac{1}{4\eta}(a-k) g_1(0) C_1 T, \\
&\int_\Omega |v_{mt}|^l |v_{mtt}|^2 dx + \left(1 - \left\{\eta(k_2+3+\alpha C_*)\right\}\right)\|\nabla v_{mtt}\|^2 \\
&+\frac{1}{2}\frac{d}{dt}\int_0^1 \varrho \|y_{mt}\|^2 d\rho + \frac{1}{2}\|y_{mt}(x,1,\varrho,t)\|^2 \\
\leq\ & C_2 + \frac{1}{4\eta}(a-k) g_2(0) C_1 T,
\end{aligned} \tag{3.3.20}$$

where C_2 is a positive constant that depends on $\eta, \alpha, a, C_*, C_1$.
Integrating (3.3.20) over $(0, t)$, we get

$$\begin{aligned}
&\int_0^t \int_\Omega |u_{mt}(\sigma)|^l |u_{mtt}(\sigma)|^2 dx d\sigma \\
&+\left(1 - \left\{\eta(k_1+3+\alpha C_*)\right\}\right)\int_0^t \|\nabla u_{mtt}(\sigma)\|^2 d\sigma \\
&+\frac{1}{2}\int_0^1 \varrho \|z_{mt}\|^2 d\rho + \frac{1}{2}\int_0^t \|z_{mt}(x,1,\varrho,\sigma)\|^2 d\sigma \\
\leq\ & \left(C_2 + \frac{1}{4\eta}(a-k) g_1(0) C_1 T\right) T,
\end{aligned}$$

$$\begin{aligned}
&\int_0^t\int_\Omega |v_{mt}(\sigma)|^l|v_{mtt}(\sigma)|^2dxd\sigma\\
&+\left(1-\left\{\eta(k_2+3+\alpha C_*)\right\}\right)\int_0^t\|\nabla v_{mtt}(\sigma)\|^2d\sigma\\
&+\frac{1}{2}\int_0^1\varrho\|y_{mt}\|^2d\rho+\frac{1}{2}\int_0^t\|y_{mt}(x,1,\varrho,\sigma)\|^2d\sigma\\
\leq\ &\left(C_2+\frac{1}{4\eta}(a-k)g_2(0)C_1T\right)T. \qquad (3.3.21)
\end{aligned}$$

At this point, we choose $\eta>0$ such that

$$\left(1-\left\{\eta(k_i+2+C_*\alpha)\right\}\right)>0,\quad \text{for } i=1,2 \qquad (3.3.22)$$

we obtain the second estimate

$$\int_0^t\left(\|\nabla u_{mtt}(\sigma)\|^2+\|\nabla v_{mtt}(\sigma)\|^2\right)d\sigma+\frac{1}{2}\int_0^1\varrho\left(\|z_{mt}\|^2+\|y_{mt}\|^2\right)d\rho\leq C_3. \qquad (3.3.23)$$

We observe from (3.2.9) and (3.3.23) that there exist subsequences (u_k) of (u_m) and (v_k) of (v_m) such that

$$\begin{aligned}
(u_k,v_k)&\rightharpoonup (u,v)\ \textit{weakly star in}\ L^\infty(0,T,H_0^1(\Omega)),\\
(u_{kt},v_{kt})&\rightharpoonup (u_t,v_t)\ \textit{weakly star in}\ L^\infty(0,T,H_0^1(\Omega)),\\
(u_{ktt},v_{ktt})&\rightharpoonup (u_{tt},v_{tt})\ \textit{weakly star in}\ L^2(0,T,H_0^1(\Omega)),\\
(z_k,y_k)&\rightharpoonup (z,y)\ \textit{weakly star in}\ L^\infty(0,T,L^2(\Omega\times(0,1)\times(\tau_1,\tau_2)),\\
(z_{kt},y_{kt})&\rightharpoonup (z_t,y_t)\ \textit{weakly star in}\ L^\infty(0,T,L^2(\Omega\times(0,1)\times(\tau_1,\tau_2)).
\end{aligned} \qquad (3.3.24)$$

Now, we will treat the nonlinear term. From (3.2.9), we get

$$\begin{aligned}
\||u_{kt}|^lu_{kt}\|_{L^2(0,T,L^2(\Omega))}&=\int_0^T\|u_{kt}\|_{2(l+1)}^{2(l+1)}dt\\
&\leq C_*^{2(l+1)}\int_0^T\|u_{kt}\|_2^{2(l+1)}dt\leq C_4, \qquad (3.3.25)
\end{aligned}$$

where C_4 depends only on C_*,C_1,T,l.

On the other hand, from the Aubin-Lions theorem (see Lions [41]), we deduce that there exists a subsequence of (u_k), still denoted by (u_k), such that

$$u_{kt} \to u_t \text{ stongly in } L^2(0,T,L^2(\Omega)), \tag{3.3.26}$$

which implies

$$u_{kt} \to u_t \text{ almost every where in } \Omega \times \mathbb{R}_+. \tag{3.3.27}$$

Hence

$$|u_{kt}|^l u_{kt} \to |u_t|^l u_t \text{ almost every where in } \Omega \times \mathbb{R}_+. \tag{3.3.28}$$

Thus, using (3.3.26), (3.3.28) and the Lions lemma, we derive

$$|u_{kt}|^l u_{kt} \rightharpoonup |u_t|^l u_t \text{ weakly in } L^2(0,T,L^2(\Omega)). \tag{3.3.29}$$

Similarly

$$|v_{kt}|^l v_{kt} \rightharpoonup |v_t|^l v_t \text{ weakly in } L^2(0,T,L^2(\Omega)) \tag{3.3.30}$$

and

$$(z_k, y_k) \to (z, y) \text{ stongly in } L^2(0,T,L^2(\Omega \times (0,1) \times (\tau_1,\tau_2)), \tag{3.3.31}$$

which implies

$$(z_k, y_k) \to (z, y) \text{ almost every where in } \Omega \times (0,1) \times (\tau_1,\tau_2) \times \mathbb{R}_+. \tag{3.3.32}$$

By multiplying (3.3.2) by $\phi(t) \in \mathcal{D}(0,T)$ and by integrating over $(0,T)$, it follows that

$$\begin{aligned}
&-\frac{1}{l+1}\int_0^T (|u_{mt}|^l u_{mtt}, u_j)\phi'(t)dt + \int_0^T M(\|\nabla u_m(t)\|)(\nabla u_m, \nabla u_j)\phi(t)dt \\
&+\int_0^T (\nabla u_{mtt}, \nabla u_j)\phi(t)dt + \alpha\int_0^T (v_m, u_j)\phi(t)dt \\
&\quad -\int_0^T\int_0^t g_1(t-s)(\nabla u_m(s), \nabla u_j)\phi(t)dsdt + k_1\int_0^T (\nabla u_{mt}, \nabla u_j)\,\phi(t)dt \\
&+\int_0^T\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)|(\nabla z_m(x,1,\varrho,t), \nabla u_j)\phi(t)d\varrho dt = 0, \\
&-\frac{1}{l+1}\int_0^T (|v_{mt}|^l v_{mtt}, v_j)\phi'(t)dt + \int_0^T M(\|\nabla v_m(t)\|)(\nabla v_m, \nabla v_j)\phi(t)dt \\
&+\int_0^T (\nabla v_{mtt}, \nabla v_j)\phi(t)dt + \alpha\int_0^T (u_m, v_j)\phi(t)dt \\
&\quad -\int_0^T\int_0^t g_2(t-s)(\nabla v_m(s), \nabla v_j)\phi(t)dsdt + k_2\int_0^T (\nabla v_{mt}, \nabla v_j)\,\phi(t)dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|(\nabla y_m(x,1,\varrho,t),\nabla v_j)\phi(t)d\varrho dt=0,\\
&\int_0^T(\varrho z_{mt}(x,\rho,\varrho,t)+z_{m\rho}(x,\rho,\varrho,t),z_j)\phi(t)dt=0,\\
&\int_0^T(\varrho y_{mt}(x,\rho,\varrho,t)+y_{m\rho}(x,\rho,\varrho,t),y_j)\phi(t)dt=0,
\end{aligned}
\tag{3.3.33}
$$

for all $j=1,...,m$.

The convergence of (3.3.24), (3.3.28) and (3.3.30) is sufficient to pass to the limit in (3.3.33). The proof of the theorem is complete. □

3.4 Exponential decay

In this section, we aim to study the asymptotic behavior of the system (3.2.6)-(3.1.2). Thus, we use the Lyapunov method. We use c throughout this section to denote a generic positive constant.

Lemma 10. *The functional*

$$
\begin{aligned}
F_1(t):=&\frac{1}{l+1}\int_\Omega\Big(|u_t|^lu_tu+|v_t|^lv_tv\Big)dx+\int_\Omega\Big(\nabla u_t\nabla u+\nabla v_t\nabla v\Big)dx\\
&+\frac{k_1}{2}\|\nabla u\|^2+\frac{k_2}{2}\|\nabla v\|^2
\end{aligned}
\tag{3.4.1}
$$

satisfies

$$
\begin{aligned}
F_1(t) \leq\ & \frac{1}{l+2}\Big(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\Big)+\frac{1}{2}\Big(\|\nabla u_t\|^2+\|\nabla v_t\|^2\Big)\\
&+\Big(\frac{(l+1)^{-1}}{l+2}C_*^{l+2}+\frac{c}{2}\Big)\Big(\|\nabla u\|^{l+2}+\|\nabla v\|^{l+2}\Big)
\end{aligned}
\tag{3.4.2}
$$

and

$$
\begin{aligned}
F_1'(t) \leq\ & \frac{1}{l+1}\Big(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\Big)+\Big(\|\nabla u_t\|^2+\|\nabla v_t\|^2\Big)\\
&-\frac{k}{2}\Big(\|\nabla u\|^2+\|\nabla v\|^2\Big)-b\|\nabla u\|^{2(\gamma+1)}-b\|\nabla v\|^{2(\gamma+1)}\\
&+c\int_{\tau_1}^{\tau_2}\Big(|\mu_1(\varrho)|\|\nabla z(x,1,\varrho,t)\|^2+|\mu_2(\varrho)|\|\nabla y(x,1,\varrho,t)\|^2\Big)d\varrho\\
&+c\big(\,(g_1\circ\nabla u)+(g_2\circ\nabla v)\,\big)-2\alpha\int_\Omega uvdx.
\end{aligned}
\tag{3.4.3}
$$

Proof. 1) Using Young's and Poincaré inequalities, we get

$$\begin{aligned}
|F_1(t)| &\leq \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\|u\|^{l+2} + \frac{1}{l+2}\|v_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\|v\|^{l+2} \\
&+\frac{1}{2}\Big(\|\nabla u_t\|^2 + \|\nabla u\|^2\Big) + \frac{1}{2}\Big(\|\nabla v_t\|^2 + \|\nabla v\|^2\Big) + \frac{k_1}{2}\|\nabla u\|^2 + \frac{k_2}{2}\|\nabla v\|^2 \\
&\leq \frac{1}{l+2}\Big(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\Big) + \frac{1}{2}\Big(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\Big) \\
&+\Big(\frac{(l+1)^{-1}}{l+2}C_*^{l+2} + \frac{c}{2}\Big)\Big(\|\nabla u\|^{l+2} + \|\nabla v\|^{l+2}\Big).
\end{aligned}$$

2) Differentiating $F_1(t)$ with respect to t and using the first and second equations of (3.2.6), we get

$$\begin{aligned}
F_1'(t) &= \int_\Omega \left[|u_t|^l u_{tt}\right] u\, dx + \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} + \int_\Omega \left[|v_t|^l v_{tt}\right] v\, dx + \frac{1}{l+1}\|v_t\|_{l+2}^{l+2} \\
&- \int_\Omega \Delta u_{tt} u\, dx + \|\nabla u_t\|^2 - \int_\Omega \Delta v_{tt} v\, dx + \|\nabla v_t\|^2 + k_1\int_\Omega \nabla u_t \nabla u dx + k_2 \int_\Omega \nabla v_t \nabla v dx \\
&= \frac{1}{l+1}(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}) + \int_\Omega \left[|u_t|^l u_{tt} - \Delta u_{tt}\right] u\, dx + \int_\Omega \left[|v_t|^l v_{tt} - \Delta v_{tt}\right] v\, dx \\
&+ \|\nabla u_t\|^2 + \|\nabla v_t\|^2 + k_1 \int_\Omega \nabla u_t \nabla u dx + k_2 \int_\Omega \nabla v_t \nabla v dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{l+1}(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}) + \|\nabla u_t\|^2 + \|\nabla v_t\|^2 + k_1 \int_\Omega \nabla u_t \nabla u dx + k_2 \int_\Omega \nabla v_t \nabla v dx \\
&+ \int_\Omega \left[-\alpha v + M(\|\nabla u\|^2)\Delta u - \int_0^t g_1(t-s)\Delta u(s)\, ds + k_1 \Delta u_t + \int_{\tau_1}^{\tau_2} \mu_1(\varrho)\, \Delta z(x,1,\varrho,t) d\varrho\right] u\, dx \\
&+ \int_\Omega \left[-\alpha u + M(\|\nabla v\|^2)\Delta v - \int_0^t g_2(t-s)\Delta v(s)\, ds + k_2 \Delta v_t + \int_{\tau_1}^{\tau_2} \mu_2(\varrho)\, \Delta y(x,1,\varrho,t) d\varrho\right] v\, dx \\
&= \frac{1}{l+1}(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}) - M(\|\nabla u\|^2)\|\nabla u\|^2 + \int_\Omega \nabla u(t) \int_0^t g_1(t-s)\nabla u(s) ds dx \\
&- \int_\Omega \nabla u \int_{\tau_1}^{\tau_2} \mu_1(\varrho)\, \nabla z(x,1,\varrho,t) d\varrho\, dx - M(\|\nabla v\|^2)\|\nabla v\|^2 + \int_\Omega \nabla v(t) \int_0^t g_2(t-s)\nabla v(s)\, ds\, dx \\
&- \int_\Omega \nabla v \int_{\tau_1}^{\tau_2} \mu_2(\varrho)\, \nabla y(x,1,\varrho,t) d\varrho\, dx + \|\nabla u_t\|^2 + \|\nabla v_t\|^2 - 2\alpha \int_\Omega uv\, dx.
\end{aligned}$$

As $M(r) = a + br^\gamma$, we get

$$\begin{aligned}
F_1'(t) &= \frac{1}{l+1}(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}) - a\|\nabla u\|^2 - b\|\nabla u\|^{2(\gamma+1)} - a\|\nabla v\|^2 - b\|\nabla v\|^{2(\gamma+1)} \\
&+ \|\nabla u_t\|^2 + \|\nabla v_t\|^2 + \int_\Omega \nabla u(t) \int_0^t g_1(t-s)\nabla u(s) ds dx
\end{aligned}$$

$$
\begin{aligned}
&-\int_\Omega \nabla u \int_{\tau_1}^{\tau_2} \mu_1(\varrho)\, \nabla z(x,1,\varrho,t) d\varrho\, dx + \int_\Omega \nabla v(t) \int_0^t g_2(t-s) \nabla v(s)\, ds\, dx \\
&-\int_\Omega \nabla v \int_{\tau_1}^{\tau_2} \mu_2(\varrho)\, \nabla y(x,1,\varrho,t) d\varrho\, dx - 2\alpha \int_\Omega uv\, dx.
\end{aligned} \tag{3.4.4}
$$

By using Young and Cauchy-Schwartz inequalities and (3.2.1), we obtain the flowing estimates for $\varepsilon_1 > 0$

$$
\begin{aligned}
&\left| -\int_\Omega \nabla u(t) \int_{\tau_1}^{\tau_2} \mu_1(\varrho)\, \nabla z(x,1,\varrho,t) d\varrho\, dx \right| \\
&\leq \frac{\varepsilon_1}{2} \| \nabla u(t) \|^2 + \frac{1}{2\varepsilon_1} \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)|\, d\varrho \right) \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)|\, \| \nabla z(x,1,\varrho,t) \|^2 d\varrho.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&\left| -\int_\Omega \nabla v(t) \int_{\tau_1}^{\tau_2} \mu_2(\varrho)\, \nabla y(x,1,\varrho,t) d\varrho\, dx \right| \\
&\leq \frac{\varepsilon_1}{2} \| \nabla v(t) \|^2 + \frac{1}{2\varepsilon_1} \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, d\varrho \right) \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, \| \nabla y(x,1,\varrho,t) \|^2 d\varrho.
\end{aligned}
$$

And

$$
\begin{aligned}
&\int_\Omega \nabla u(t) \int_0^t g_1(t-s) \nabla u(s) ds dx \\
&\leq \int_\Omega \int_0^t g_1(t-s) \left| \nabla u(t) \left(\nabla u(s) - \nabla u(t) \right) \right| ds dx + \left(\int_0^t g_1(s) ds \right) \| \nabla u(t) \|^2 \\
&\leq \frac{\varepsilon_1}{2} \| \nabla u(t) \|^2 + \frac{1}{2\varepsilon_1} \left(\int_0^t g_1(s) ds \right) \int_\Omega \int_0^t g_1(t-s) \left| \nabla u(s) - \nabla u(t) \right|^2 ds dx \\
&+ \| \nabla u(t) \|^2 \left(\int_0^t g_1(s) ds \right) \\
&\leq \left(\frac{\varepsilon_1}{2} + \left(\int_0^t g_1(s) ds \right) \right) \| \nabla u(t) \|^2 + \frac{1}{2\varepsilon_1} \left(\int_0^t g_1(s) ds \right) (g_1 \circ \nabla u)(t).
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
\int_\Omega \nabla v(t) \int_0^t g_2(t-s) \nabla v(s) ds dx \leq & \left(\frac{\varepsilon_1}{2} + \left(\int_0^t g_2(s) ds \right) \right) \| \nabla v(t) \|^2 \\
& + \frac{1}{2\varepsilon_1} \left(\int_0^t g_2(s) ds \right) (g_2 \circ \nabla v)(t).
\end{aligned}
$$

Combining all above estimates with (3.4.4), we obtain

$$
\begin{aligned}
F_1'(t) \leq\ & \frac{1}{l+1}\Big(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\Big)+\Big(\|\nabla u_t\|^2+\|\nabla v_t\|^2\Big)\\
&-b\|\nabla u\|^{2(\gamma+1)}-b\|\nabla v\|^{2(\gamma+1)}\\
&-\Big(a-\int_0^t g_1(s)ds-\varepsilon_1\Big)\|\nabla u\|^2-\Big(a-\int_0^t g_2(s)ds-\varepsilon_1\Big)\|\nabla v\|^2\\
&+c\int_{\tau_1}^{\tau_2}\Big(|\mu_1(\varrho)|\|\nabla z(x,1,\varrho,t)\|^2+|\mu_2(\varrho)|\|\nabla y(x,1,\varrho,t)\|^2\Big)d\varrho\\
&+c\big((g_1\circ\nabla u)+(g_2\circ\nabla v)\big)-2\alpha\int_\Omega uvdx. \qquad (3.4.5)
\end{aligned}
$$

By using (3.2.1)

$$
\begin{aligned}
F_1'(t) \leq\ & \frac{1}{l+1}\Big(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\Big)+\Big(\|\nabla u_t\|^2+\|\nabla v_t\|^2\Big)\\
&-b\|\nabla u\|^{2(\gamma+1)}-b\|\nabla v\|^{2(\gamma+1)}-(k-\varepsilon_1)\|\nabla u\|^2-(k-\varepsilon_1)\|\nabla v\|^2\\
&+c\int_{\tau_1}^{\tau_2}\Big(|\mu_1(\varrho)|\|\nabla z(x,1,\varrho,t)\|^2+|\mu_2(\varrho)|\|\nabla y(x,1,\varrho,t)\|^2\Big)d\varrho\\
&+c\big((g_1\circ\nabla u)+(g_2\circ\nabla v)\big)-2\alpha\int_\Omega uvdx. \qquad (3.4.6)
\end{aligned}
$$

Taking $\varepsilon_1=\frac{k}{2}$, we obtain (3.4.3). □

Lemma 11. *The functional*

$$
\begin{aligned}
F_2(t) :=\ & \int_\Omega\Big(\Delta u_t-\frac{1}{l+1}|u_t|^l u_t\Big)\int_0^t g_1(t-s)(u(t)-u(s))dsdx\\
&+\int_\Omega\Big(\Delta v_t-\frac{1}{l+1}|v_t|^l v_t\Big)\int_0^t g_2(t-s)(v(t)-v(s))dsdx
\end{aligned}
$$

satisfies,

$$
\begin{aligned}
F_2(t) \leq\ & \frac{1}{l+2}\Big(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\Big)+\frac{1}{2}\Big(\|\nabla u_t\|^2+\|\nabla v_t\|^2\Big)\\
&+\Big(\frac{(l+1)^{-1}}{l+2}(a-k)^{l+2}C_*^{l+2}2^{2l+1}\Big)\Big(\|\nabla u\|^{2(l+1)}+\|\nabla v\|^{2(l+1)}\Big)\\
&+\frac{1}{2}(a-k)\Big\{1+\frac{(l+1)^{-1}}{l+2}(a-k)^l C_*^{l+2}\Big\}\Big((g_1\circ\nabla u)+(g_2\circ\nabla v)\Big). \qquad (3.4.7)
\end{aligned}
$$

And for any $\varepsilon_2>0$

$$\begin{aligned}
F_2'(t) \leq & -\frac{1}{l+1}(a-k)\Big[\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\Big] \\
& +\Big(\varepsilon_2(a-k)^2+\alpha C_*\frac{\varepsilon_2}{2}\Big)\Big(\|\nabla u\|^2+\|\nabla v\|^2\Big) \\
& +\frac{\varepsilon_2}{2}M(\|\nabla u\|^2)\|\nabla u\|^2+\frac{\varepsilon_2}{2}M(\|\nabla v\|^2)\|\nabla v\|^2 \\
& -\Big((a-k)-\frac{\varepsilon_2}{2}(2+k_1)\Big)\|\nabla u_t\|^2+\frac{\varepsilon_2 k_1}{2}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\|\nabla z(x,1,\varrho,t)\|^2 d\varrho \\
& -\Big((a-k)-\frac{\varepsilon_2}{2}(2+k_2)\Big)\|\nabla v_t\|^2+\frac{\varepsilon_2 k_2}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\|\nabla y(x,1,\varrho,t)\|^2 d\varrho \\
& +\Big\{\frac{M(\|\nabla u\|^2)}{2\varepsilon_2}+\frac{1}{2\varepsilon_2}\Big(2+k_1+\alpha C_*\Big)+\varepsilon_2\Big\}(a-k)(g_1\circ\nabla u) \\
& +\Big\{\frac{M(\|\nabla v\|^2)}{2\varepsilon_2}+\frac{1}{2\varepsilon_2}\Big(2+k_2+\alpha C_*\Big)+\varepsilon_2\Big\}(a-k)(g_2\circ\nabla v) \\
& -\frac{cg_1(0)}{\varepsilon_2}(g_1'\circ\nabla u)-\frac{cg_2(0)}{\varepsilon_2}(g_2'\circ\nabla v). \qquad (3.4.8)
\end{aligned}$$

Proof. 1) We use Young's inequality with the conjugate exponents $p'=\frac{l+2}{l+1}$ and $q'=l+2$, and by using Hölder inequality, we get

$$\begin{aligned}
& \Big|-\int_\Omega \frac{1}{l+1}|u_t|^l u_t \int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx\Big| \\
\leq & \frac{1}{l+1}\Big|\int_\Omega (|u_t|^l u_t)(\int_0^t g_1(t-s)(u(t)-u(s))\,ds)\,dx\Big| \\
\leq & \frac{1}{l+1}\Big[\frac{1}{p'}\int_\Omega \Big||u_t|^l u_t\Big|^{p'}dx+\frac{1}{q'}\int_\Omega\Big|\int_0^t g_1(t-s)(u(t)-u(s))\,ds\Big|^{q'}dx\Big] \\
\leq & \frac{1}{l+1}\Big[\frac{1}{p'}\int_\Omega (|u_t|^{l+1})^{p'}dx+\frac{1}{q'}\int_\Omega(\int_0^t g_1(t-s)|u(t)-u(s)|ds)^{q'}dx\Big] \\
\leq & \frac{1}{l+2}\|u_t\|_{l+2}^{l+2}+\frac{(l+1)^{-1}}{l+2}\int_\Omega\Big[\int_0^t\Big(g_1(t-s)\Big)^{\frac{l+1}{l+2}}\Big((g_1(t-s))^{\frac{1}{l+2}}|u(t)-u(s)|\Big)ds\Big]^{l+2}dx.
\end{aligned} \qquad (3.4.9)$$

We have by Hölder's inequality

$$\begin{aligned}
& \int_\Omega\Big[\int_0^t\Big(g_1(t-s)\Big)^{\frac{l+1}{l+2}}\Big((g_1(t-s))^{\frac{1}{l+2}}|u(t)-u(s)|\Big)ds\Big]^{l+2}dx \\
\leq & \int_\Omega\Big[\Big(\int_0^t\Big((g_1(t-s))^{\frac{l+1}{l+2}}\Big)^p ds\Big)^{\frac{1}{p}}\Big(\int_0^t\Big((g_1(t-s))^{\frac{1}{l+2}}|u(t)-u(s)|\Big)^q ds\Big)^{\frac{1}{q}}\Big]^{l+2}dx \\
\leq & \int_\Omega\Big[\Big(\int_0^t g_1(t-s)\,ds\Big)^{\frac{l+1}{l+2}}\Big(\int_0^t g_1(t-s)|u(t)-u(s)|^{l+2}ds\Big)^{\frac{1}{l+2}}\Big]^{l+2}dx
\end{aligned}$$

$$
\begin{aligned}
&\le \Big(\int_0^t g_1(t-s)\,ds\Big)^{l+1}\int_0^t g_1(t-s)\|u(t)-u(s)\|_{l+2}^{l+2}\,ds\\
&\le (a-k)^{l+1}c_s^{l+2}\int_0^t \sqrt{g_1(t-s)}\sqrt{g_1(t-s)}\|\nabla u(t)-\nabla u(s)\|^{l+1}\|\nabla u(t)-\nabla u(s)\|\,ds\\
&\le (a-k)^{l+1}c_s^{l+2}\Big(\frac{1}{2}\int_0^t g_1(t-s)\|\nabla u(t)-\nabla u(s)\|^{2l+2}\,ds+\frac{1}{2}\int_0^t g_1(t-s)\|\nabla u(t)-\nabla u(s)\|^{2}\,ds\Big)\\
&\le (a-k)^{l+1}c_s^{l+2}\Big(\frac{1}{2}\int_0^t g_1(t-s)\|2\nabla u(t)\|^{2l+2}\,ds+\frac{1}{2}(g_1 o\nabla u)(t)\Big)\\
&\le (a-k)^{l+1}c_s^{l+2}\Big(2^{2l+1}(a-k)\|\nabla u(t)\|^{2(l+1)}+\frac{1}{2}(g_1 o\nabla u)(t)\Big).
\end{aligned}
\tag{3.4.10}
$$

Combining (3.4.10) with (3.4.9), we obtain

$$
\begin{aligned}
\Big|-\int_\Omega \frac{1}{l+1}|u_t|^l u_t\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx\Big| &\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2}\\
&+\frac{(l+1)^{-1}}{l+2}\Big[(a-k)^{l+1}c_s^{l+2}\Big(2^{2l+1}(a-k)\|\nabla u(t)\|^{2(l+1)}+\frac{1}{2}(g_1 o\nabla u)(t)\Big)\Big].
\end{aligned}
\tag{3.4.11}
$$

Young's inequality give

$$
\begin{aligned}
&\Big|-\int_\Omega \nabla u_t\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))dsdx\Big|\\
\le\ &\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}(a-k)(g_1\circ\nabla u_t).
\end{aligned}
\tag{3.4.12}
$$

Similarly, we get

$$
\begin{aligned}
&\Big|-\int_\Omega \frac{1}{l+1}|v_t|^l v_t\int_0^t g_2(t-s)(v(t)-v(s))dsdx\Big| \le \frac{1}{l+2}\|v_t\|_{l+2}^{l+2}\\
&+\frac{(l+1)^{-1}}{l+2}\Big[(a-k)^{l+1}C_*^{l+2}\Big(2^{2l+1}(a-k)\|\nabla v\|^{2(l+1)}+\frac{1}{2}(g_2\circ\nabla v)\Big)\Big]
\end{aligned}
\tag{3.4.13}
$$

and

$$
\begin{aligned}
&\Big|-\int_\Omega \nabla v_t\int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))dsdx\Big|\\
\le\ &\frac{1}{2}\|\nabla v_t\|^2+\frac{1}{2}(a-k)(g_2\circ\nabla v_t).
\end{aligned}
\tag{3.4.14}
$$

Combining (3.4.11)-(3.4.14), we deduce (3.4.7).

2) Differentiating F_2, using integrating by parts and (3.1.2), we get

$$
\begin{aligned}
F_2'(t) &= \int_\Omega \left[\Delta u_{tt} - |u_t|^l u_{tt}\right] \int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx \\
&+ \int_\Omega \left(\Delta u_t - \frac{1}{l+1}|u_t|^l u_t\right) \left(\int_0^t (g_1'(t-s)(u(t)-u(s)) + g_1(t-s)u_t(t))ds\right) dx \\
&+ \int_\Omega \left(\Delta v_{tt} - |v_t|^l v_{tt}\right) \int_0^t g_2(t-s)(v(t)-v(s))\,ds\,dx \\
&+ \int_\Omega \left(\Delta v_t - \frac{1}{l+1}|v_t|^l v_t\right) \left(\int_0^t (g_2'(t-s)(v(t)-v(s)) + g_2(t-s)v_t(t))ds\right) dx \\
&= \alpha \int_\Omega v(t) \int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx \\
&+ \int_\Omega M(\|\nabla u\|^2)\nabla u(t) \int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&- \int_\Omega \int_0^t g_1(t-s)\nabla u(s)\,ds \int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&+ k_1 \int_\Omega \nabla u_t \int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&+ \int_\Omega \int_{\tau_1}^{\tau_2} \mu_1(\varrho)\,\nabla z(x,1,t,\varrho)d\varrho \int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&- \int_\Omega \nabla u_t \int_0^t g_1'(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&- \frac{1}{l+1} \int_\Omega |u_t|^l u_t \int_0^t g_1'(t-s)(u(t)-u(s))\,ds\,dx \\
&- \|\nabla u_t\|^2 \int_0^t g_1(s)\,ds - \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} \int_0^t g_1(s)\,ds \\
&+ \alpha \int_\Omega u(t) \int_0^t g_2(t-s)(v(t)-v(s))\,ds\,dx \\
&+ \int_\Omega M(\|\nabla v\|^2)\nabla v(t) \int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))\,ds\,dx \\
&- \int_\Omega \int_0^t g_2(t-s)\nabla v(s)\,ds \int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))\,ds\,dx \\
&+ k_2 \int_\Omega \nabla v_t \int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))\,ds\,dx \\
&+ \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(\varrho)\,\nabla y(x,1,t,\varrho)d\varrho \int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))\,ds\,dx \\
&- \int_\Omega \nabla v_t \int_0^t g_2'(t-s)(\nabla v(t)-\nabla v(s))\,ds\,dx \\
&- \frac{1}{l+1} \int_\Omega |v_t|^l v_t \int_0^t g_2'(t-s)(v(t)-v(s))\,ds\,dx \\
&- \|\nabla v_t\|^2 \int_0^t g_2(s)\,ds - \frac{1}{l+1}\|v_t\|_{l+2}^{l+2} \int_0^t g_2(s)\,ds
\end{aligned}
$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 - \|\nabla u_t\|^2 \int_0^t g_1(s)\,ds - \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} \int_0^t g_1(s)\,ds \tag{3.4.15}$$
$$- \|\nabla v_t\|^2 \int_0^t g_2(s)\,ds - \frac{1}{l+1}\|v_t\|_{l+2}^{l+2} \int_0^t g_2(s)\,ds$$

where

$$I_1 = \int_\Omega M(\|\nabla u\|^2)\nabla u(t) \int_0^t g_1(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx$$
$$+ \int_\Omega M(\|\nabla v\|^2)\nabla v(t) \int_0^t g_2(t-s)(\nabla v(t) - \nabla v(s))\,ds\,dx,$$
$$I_2 = -\int_\Omega \int_0^t g_1(t-s)\nabla u(s)\,ds \int_0^t g_1(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx$$
$$- \int_\Omega \int_0^t g_2(t-s)\nabla v(s)\,ds \int_0^t g_2(t-s)(\nabla v(t) - \nabla v(s))\,ds\,dx,$$
$$I_3 = \int_\Omega \int_{\tau_1}^{\tau_2} \mu_1(\varrho)\,\nabla z(x,1,t,\varrho)d\varrho \int_0^t g_1(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx$$
$$+ \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(\varrho)\,\nabla y(x,1,t,\varrho)d\varrho \int_0^t g_2(t-s)(\nabla v(t) - \nabla v(s))\,ds\,dx,$$
$$I_4 = -\int_\Omega \nabla u_t \int_0^t g_1'(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx - \int_\Omega \nabla v_t \int_0^t g_2'(t-s)(\nabla v(t) - \nabla v(s))\,ds\,dx,$$
$$I_5 = -\frac{1}{l+1}\int_\Omega |u_t|^l u_t \int_0^t g_1'(t-s)(u(t) - u(s))\,ds\,dx - \frac{1}{l+1}\int_\Omega |v_t|^l v_t \int_0^t g_2'(t-s)(v(t) - v(s))\,ds\,dx,$$
$$I_6 = \alpha \int_\Omega v(t) \int_0^t g_1(t-s)(u(t) - u(s))\,ds\,dx + \alpha \int_\Omega u(t) \int_0^t g_2(t-s)(v(t) - v(s))\,ds\,dx,$$
$$I_7 = k_1 \int_\Omega \nabla u_t \int_0^t g_1(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx + k_2 \int_\Omega \nabla v_t \int_0^t g_2(t-s)(\nabla v(t) - \nabla v(s))\,ds\,dx.$$

In what follows we will estimate $I_1, ..., I_7$.
For I_1, we use Hölder's and Young's inequalities with $p = q = 2$, we get

$$|I_1| \le M(\|\nabla u\|^2)\Big[\frac{\varepsilon_2}{2}\int_\Omega |\nabla u(t)|^2 dx + \frac{1}{2\varepsilon_2}\int_\Omega \Big(\int_0^t g_1(t-s)|\nabla u(t) - \nabla u(s)|ds\Big)^2 dx\Big]$$
$$+ M(\|\nabla v\|^2)\Big[\frac{\varepsilon_2}{2}\int_\Omega |\nabla v(t)|^2 dx + \frac{1}{2\varepsilon_2}\int_\Omega \Big(\int_0^t g_2(t-s)|\nabla v(t) - \nabla v(s)|ds\Big)^2 dx\Big]$$
$$\le M(\|\nabla u\|^2)\Big[\frac{\varepsilon_2}{2}\int_\Omega |\nabla u(t)|^2 dx + \frac{1}{2\varepsilon_2}\Big(\int_0^t g_1(s)ds\Big)\int_\Omega \int_0^t g_1(t-s)|\nabla u(t) - \nabla u(s)|^2\,ds\,dx\Big]$$
$$+ M(\|\nabla v\|^2)\Big[\frac{\varepsilon_2}{2}\int_\Omega |\nabla v(t)|^2 dx + \frac{1}{2\varepsilon_2}\Big(\int_0^t g_2(s)ds\Big)\int_\Omega \int_0^t g_2(t-s)|\nabla v(t) - \nabla v(s)|^2\,ds\,dx\Big]$$
$$\le M(\|\nabla u\|^2)\Big(\frac{\varepsilon_2}{2}\|\nabla u(t)\|^2 + \frac{1}{2\varepsilon_2}(a-k)(g_1 o \nabla u)(t)\Big)$$

$$+ M(\|\nabla v\|^2)\Big(\frac{\varepsilon_2}{2}\|\nabla v(t)\|^2 + \frac{1}{2\varepsilon_2}(a-k)(g_2 o\nabla v)(t)\Big). \tag{3.4.16}$$

Similarly,

$$\begin{aligned}
|I_2| &\le \frac{\varepsilon_2}{2}\int_\Omega \Big(\int_0^t g_1(t-s)|\nabla u(s)|\,ds\Big)^2 dx + \frac{1}{2\varepsilon_2}\int_\Omega \Big(\int_0^t g_1(t-s)|\nabla u(t)-\nabla u(s)|\,ds\Big)^2 dx\\
&+ \frac{\varepsilon_2}{2}\int_\Omega \Big(\int_0^t g_2(t-s)|\nabla v(s)|\,ds\Big)^2 dx + \frac{1}{2\varepsilon_2}\int_\Omega \Big(\int_0^t g_2(t-s)|\nabla v(t)-\nabla v(s)|\,ds\Big)^2 dx\\
&\le \frac{\varepsilon_2}{2}\int_\Omega \Big(\int_0^t g_1(t-s)(|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|)\,ds\Big)^2 dx + \frac{1}{2\varepsilon_2}\Big(\int_0^t g_1(s)\,ds\Big)(g_1 o\nabla u)(t)\\
&+ \frac{\varepsilon_2}{2}\int_\Omega \Big(\int_0^t g_2(t-s)(|\nabla v(s)-\nabla v(t)|+|\nabla v(t)|)\,ds\Big)^2 dx + \frac{1}{2\varepsilon_2}\Big(\int_0^t g_2(s)\,ds\Big)(g_2 o\nabla v)(t)\\
&\le \varepsilon_2\|\nabla u(t)\|^2\Big(\int_0^t g_1(t)\,ds\Big)^2 + (\varepsilon_2+\frac{1}{2\varepsilon_2})\Big(\int_0^t g_1(s)\,ds\Big)(g_1 o\nabla u)(t)\\
&+ \varepsilon_2\|\nabla v(t)\|^2\Big(\int_0^t g_2(t)\,ds\Big)^2 + (\varepsilon_2+\frac{1}{2\varepsilon_2})\Big(\int_0^t g_2(s)\,ds\Big)(g_2 o\nabla v)(t)\\
&\le \varepsilon_2\|\nabla u(t)\|^2(a-k)^2 + (\varepsilon_2+\frac{1}{2\varepsilon_2})(a-k)(g_1 o\nabla u)(t) + \varepsilon_2\|\nabla v(t)\|^2(a-k)^2\\
&+ (\varepsilon_2+\frac{1}{2\varepsilon_2})(a-k)(g_2 o\nabla v)(t),
\end{aligned} \tag{3.4.17}$$

$$\begin{aligned}
|I_3| &\le \frac{\varepsilon_2}{2}\int_\Omega \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)|\,\nabla z(x,1,t,\varrho)d\varrho\right)^2 dx + \frac{1}{2\varepsilon_2}\int_\Omega \left(\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)^2 dx\\
&+ \frac{\varepsilon_2}{2}\int_\Omega \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\,\nabla y(x,1,t,\varrho)d\varrho\right)^2 dx + \frac{1}{2\varepsilon_2}\int_\Omega \left(\int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))\,ds\right)^2 dx\\
&\le \frac{\varepsilon_2}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)|\,d\varrho\right)\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)|\,|\nabla z(x,1,t,\varrho)|^2\,d\varrho dx\\
&+ \frac{1}{2\varepsilon_2}\left(\int_0^t g_1(s)ds\right)\int_\Omega\int_0^t g_1(t-s)\,|\nabla u(t)-\nabla u(s)|^2\,dsdx\\
&+ \frac{\varepsilon_2}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\,d\varrho\right)\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\,|\nabla y(x,1,t,\varrho)|^2\,d\varrho dx\\
&+ \frac{1}{2\varepsilon_2}\left(\int_0^t g_2(s)ds\right)\int_\Omega\int_0^t g_2(t-s)\,|\nabla v(t)-\nabla v(s)|^2\,dsdx\\
&\le \frac{\varepsilon_2}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)|\,d\varrho\right)\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)|\,\|\nabla z(x,1,t,\varrho)\|^2\,d\varrho\\
&+ \frac{\varepsilon_2}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\,d\varrho\right)\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\,\|\nabla y(x,1,t,\varrho)\|^2\,d\varrho
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2\varepsilon_2}\left(\int_0^t g_1(s)ds\right)\int_\Omega\int_0^t g_1(t-s)\left|\nabla u(t)-\nabla u(s)\right|^2 dsdx\\
&+\frac{1}{2\varepsilon_2}\left(\int_0^t g_2(s)ds\right)\int_\Omega\int_0^t g_2(t-s)\left|\nabla v(t)-\nabla v(s)\right|^2 dsdx\\
&\leq\frac{\varepsilon_2}{2}\left(k_1\int_{\tau_1}^{\tau_2}\left|\mu_1(\varrho)\right|\left\|\nabla z(x,1,t,\varrho)\right\|^2 d\varrho+k_2\int_{\tau_1}^{\tau_2}\left|\mu_2(\varrho)\right|\left\|\nabla y(x,1,t,\varrho)\right\|^2 d\varrho\right)\\
&+\frac{(a-k)}{2\varepsilon_2}(g_1o\nabla u)(t)+\frac{(a-k)}{2\varepsilon_2}(g_2o\nabla v)(t),
\end{aligned}
\tag{3.4.18}
$$

$$
\begin{aligned}
|I_4|&\leq\frac{\varepsilon_2}{2}\int_\Omega|\nabla u_t|^2 dx+\frac{1}{2\varepsilon_2}\int_\Omega\Big(\int_0^t|g_1'(t-s)||\nabla u(t)-\nabla u(s)|\,ds\Big)^2 dx\\
&+\frac{\varepsilon_2}{2}\int_\Omega|\nabla v_t|^2 dx+\frac{1}{2\varepsilon_2}\int_\Omega\Big(\int_0^t|g_2'(t-s)||\nabla v(t)-\nabla v(s)|\,ds\Big)^2 dx\\
&\leq\frac{\varepsilon_2}{2}\|\nabla u_t\|^2+\frac{1}{2\varepsilon_2}\int_0^t(-g_1'(t-s))\,ds\int_\Omega\int_0^t(-g_1'(t-s))|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\\
&+\frac{\varepsilon_2}{2}\|\nabla v_t\|^2+\frac{1}{2\varepsilon_2}\int_0^t(-g_2'(t-s))\,ds\int_\Omega\int_0^t(-g_2'(t-s))|\nabla v(t)-\nabla v(s)|^2\,ds\,dx\\
&\leq\frac{\varepsilon_2}{2}\|\nabla u_t\|^2-\frac{g_1(0)}{2\varepsilon_2}(g_1'o\nabla u)(t)+\frac{\varepsilon_2}{2}\|\nabla v_t\|^2-\frac{g_2(0)}{2\varepsilon_2}(g_2'o\nabla u)(t).
\end{aligned}
\tag{3.4.19}
$$

As in [54], we can estimate I_5 as follows

$$
\begin{aligned}
&-\frac{1}{l+1}\int_\Omega|u_t|^l u_t\int_0^t g_1'(t-s)(u(t)-u(s))\,ds\,dx\\
&\leq\delta\int_\Omega|u_t|^{2(l+1)}dx+\frac{c}{\delta}\int_\Omega\left(\int_0^t\sqrt{-g_1'(t-s)}\sqrt{-g_1'(t-s)}\left|u(s)-u(t)\right|ds\right)^2 dx\\
&\leq\delta c_{2(l+1)}^{2(l+1)}\left(\int_\Omega|\nabla u_t|^2 dx\right)^{(l+1)}dx-\frac{cg_1(0)}{\delta}\int_\Omega\int_0^t g_1'(t-s)\left|u(s)-u(t)\right|^2 dsdx\\
&\leq\delta c_{2(l+1)}^{2(l+1)}\left[\frac{2(\gamma+1)}{\gamma}E(0)\right]^l\int_\Omega|\nabla u_t|^2 dx-\frac{cg_1(0)}{\delta}\left(g_1'o\nabla u\right)(t).
\end{aligned}
$$

Similary

$$
\begin{aligned}
&-\frac{1}{l+1}\int_\Omega|v_t|^l v_t\int_0^t g_2'(t-s)(v(t)-v(s))\,ds\,dx\\
&\leq\delta c_{2(l+1)}^{2(l+1)}\left[\frac{2(\gamma+1)}{\gamma}E(0)\right]^l\int_\Omega|\nabla v_t|^2 dx-\frac{cg_2(0)}{\delta}\left(g_2'o\nabla v\right)(t).
\end{aligned}
$$

Consequently

$$\begin{aligned}|I_5| \leq \delta c_{2(l+1)}^{2(l+1)} \left[\frac{2(\gamma+1)}{\gamma}E(0)\right]^l \|\nabla u_t\|^2 - \frac{cg_1(0)}{\delta}\left(g_1' o\nabla u\right)(t) \\ + \delta c_{2(l+1)}^{2(l+1)} \left[\frac{2(\gamma+1)}{\gamma}E(0)\right]^l \|\nabla v_t\|^2 - \frac{cg_1(0)}{\delta}\left(g_2' o\nabla v\right)(t).\end{aligned} \tag{3.4.20}$$

And

$$|I_6| \leq \alpha C_* \Big(\frac{\varepsilon_2}{2}\left(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\right) + \frac{(a-k)}{2\varepsilon_2}[(g_1 o\nabla u)(t) + (g_2 o\nabla v)(t)]\Big), \tag{3.4.21}$$

$$\begin{aligned}|I_7| \leq k_1 \frac{\varepsilon_2}{2}\|\nabla u_t\|^2 + \frac{k_1}{2\varepsilon_2}(a-k)(g_1 o\nabla u)(t) \\ + k_2 \frac{\varepsilon_2}{2}\|\nabla v_t\|^2 + \frac{k_2}{2\varepsilon_2}(a-k)(g_2 o\nabla v)(t).\end{aligned} \tag{3.4.22}$$

Combining (3.4.15) and (3.4.16)-(3.4.22) and taking

$$\delta = \frac{\varepsilon_2}{2c_{2(l+1)}^{2(l+1)}\left[\frac{2(\gamma+1)}{\gamma}E(0)\right]^l},$$

we obtain

$$\begin{aligned}F_2'(t) &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 - \|\nabla u_t\|^2 \int_0^t g_1(s)\,ds - \frac{1}{l+1}\|u_t\|_{l+2}^{l+2}\int_0^t g_1(s)\,ds \\ &- \|\nabla v_t\|^2 \int_0^t g_2(s)\,ds - \frac{1}{l+1}\|v_t\|_{l+2}^{l+2}\int_0^t g_2(s)\,ds \\ &\leq M(\|\nabla u\|^2)\Big(\frac{\varepsilon_2}{2}\|\nabla u(t)\|^2 + \frac{1}{2\varepsilon_2}(a-k)(g_1 o\nabla u)(t)\Big) \\ &+ M(\|\nabla v\|^2)\Big(\frac{\varepsilon_2}{2}\|\nabla v(t)\|^2 + \frac{1}{2\varepsilon_2}(a-k)(g_2 o\nabla v)(t)\Big) \\ &+ \varepsilon_2\|\nabla u(t)\|^2(a-k)^2 + (\varepsilon_2 + \frac{1}{2\varepsilon_2})(a-k)(g_1 o\nabla u)(t) \\ &+ \varepsilon_2\|\nabla v(t)\|^2(a-k)^2 + (\varepsilon_2 + \frac{1}{2\varepsilon_2})(a-k)(g_2 o\nabla v)(t) \\ &+ \frac{\varepsilon_2}{2}\left(k_1\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\,\|\nabla z(x,1,t,\varrho)\|^2\,d\varrho + k_2\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,\|\nabla y(x,1,t,\varrho)\|^2\,d\varrho\right) \\ &+ \frac{(a-k)}{2\varepsilon_2}(g_1 o\nabla u)(t) + \frac{(a-k)}{2\varepsilon_2}(g_2 o\nabla v)(t) \\ &+ \frac{\varepsilon_2}{2}\|\nabla u_t\|^2 - \frac{g_1(0)}{2\varepsilon_2}(g_1' o\nabla u)(t) + \frac{\varepsilon_2}{2}\|\nabla v_t\|^2 - \frac{g_2(0)}{2\varepsilon_2}(g_2' o\nabla u)(t) \\ &+ \frac{\varepsilon_2}{2}\|\nabla u_t\|^2 - \frac{cg_1(0)}{\varepsilon_2}\left(g_1' o\nabla u\right)(t) + \frac{\varepsilon_2}{2}\|\nabla v_t\|^2 - \frac{cg_1(0)}{\varepsilon_2}\left(g_2' o\nabla v\right)(t) \\ &+ \alpha C_*\Big(\frac{\varepsilon_2}{2}\left(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\right) + \frac{(a-k)}{2\varepsilon_2}[(g_1 o\nabla u)(t) + (g_2 o\nabla v)(t)]\Big)\end{aligned}$$

$$
\begin{aligned}
&+ k_1\frac{\varepsilon_2}{2}\|\nabla u_t\|^2 + \frac{k_1}{2\varepsilon_2}(a-k)(g_1 o\nabla u)(t) + k_2\frac{\varepsilon_2}{2}\|\nabla v_t\|^2 + \frac{k_2}{2\varepsilon_2}(a-k)(g_2 o\nabla v)(t)\\
&- \|\nabla u_t\|^2\left(\int_0^t g_1(s)\,ds\right) - \frac{1}{l+1}\|u_t\|_{l+2}^{l+2}\left(\int_0^t g_1(s)\,ds\right) - \|\nabla v_t\|^2\left(\int_0^t g_2(s)\,ds\right)\\
&- \frac{1}{l+1}\|v_t\|_{l+2}^{l+2}\left(\int_0^t g_2(s)\,ds\right).
\end{aligned}
\tag{3.4.23}
$$

Simple calculations and using assumption **(A1)**, we find

$$
\begin{aligned}
F_2'(t) \;\leq\; & -\frac{1}{l+1}(a-k)\left[\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right]\\
&+\left(\varepsilon_2(a-k)^2 + \alpha C_*\frac{\varepsilon_2}{2}\right)\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\\
&+\frac{\varepsilon_2}{2}M(\|\nabla u\|^2)\|\nabla u\|^2 + \frac{\varepsilon_2}{2}M(\|\nabla v\|^2)\|\nabla v\|^2\\
&-\left((a-k) - \frac{\varepsilon_2}{2} - \frac{\varepsilon_2}{2} - k_1\frac{\varepsilon_2}{2}\right)\|\nabla u_t\|^2 + k_1\frac{\varepsilon_2}{2}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\|\nabla z(x,1,\varrho,t)\|^2 d\varrho\\
&-\left((a-k) - \frac{\varepsilon_2}{2} - \frac{\varepsilon_2}{2} - k_2\frac{\varepsilon_2}{2}\right)\|\nabla v_t\|^2 + k_2\frac{\varepsilon_2}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\|\nabla y(x,1,\varrho,t)\|^2 d\varrho\\
&+\left\{\frac{M(\|\nabla u\|^2)}{2\varepsilon_2} + \left(\varepsilon_2 + \frac{1}{2\varepsilon_2}\right) + \frac{1}{2\varepsilon_2} + \frac{\alpha C_*}{2\varepsilon_2} + k_1\frac{1}{2\varepsilon_2}\right\}(a-k)(g_1\circ\nabla u)\\
&+\left\{\frac{M(\|\nabla v\|^2)}{2\varepsilon_2} + \left(\varepsilon_2 + \frac{1}{2\varepsilon_2}\right) + \frac{1}{2\varepsilon_2} + \frac{\alpha C_*}{2\varepsilon_2} + k_2\frac{1}{2\varepsilon_2}\right\}(a-k)(g_2\circ\nabla v)\\
&-\frac{cg_1(0)}{\varepsilon_2}(g_1'\circ\nabla u) - \frac{cg_2(0)}{\varepsilon_2}(g_2'\circ\nabla v).
\end{aligned}
\tag{3.4.24}
$$

Then, we obtain (3.4.8). □

In the next Lemma we introduce the functional used later to obtain the stability result:

Lemma 12. *The functional*

$$
F_3(t) := \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho}\left(|\mu_1(\varrho)|\,|\nabla z(x,\rho,\varrho,t)|^2 + |\mu_2(\varrho)|\,|\nabla y(x,\rho,\varrho,t)|^2\right)d\varrho d\rho dx.
$$

Satisfies,

$$
F_3(t) \leq \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} \varrho\left(|\mu_1(\varrho)|\,|\nabla z(x,\rho,\varrho,t)|^2 + |\mu_2(\varrho)|\,|\nabla y(x,\rho,\varrho,t)|^2\right)d\varrho d\rho dx, \tag{3.4.25}
$$

and

$$
\begin{aligned}
F_3'(t) \leq & -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho\Big(|\mu_1(\varrho)| \|\nabla z(x,\rho,\varrho,t)\|^2 + |\mu_2(\varrho)| \|\nabla y(x,\rho,\varrho,t)\|^2\Big) d\varrho d\rho \\
& + k_0\Big(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\Big) \\
& - \eta_1 \int_{\tau_1}^{\tau_2} \Big(|\mu_1(\varrho)| \|\nabla z(x,1,\varrho,t)\|^2 + |\mu_2(\varrho)| \|\nabla y(x,1,\varrho,t)\|^2\Big) d\varrho, \qquad (3.4.26)
\end{aligned}
$$

where $\eta_1 > 0$.

Proof. By differentiating F_3, and use the equations $(3.2.6)_3$, $(3.2.6)_4$, we get

$$
\begin{aligned}
F_3'(t) = & -2 \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\varrho\rho} |\mu_1(\varrho)| \nabla z \nabla z_\rho (x,\rho,\varrho,t)\, d\varrho d\rho dx \\
& -2 \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\varrho\rho} |\mu_1(\varrho)| \nabla y \nabla y_\rho (x,\rho,\varrho,t)\, d\varrho d\rho dx \\
= & - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} |\mu_1(\varrho)|\, |\nabla z|^2\, d\varrho d\rho dx \\
& - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \Big[e^{-\varrho} |\nabla z(x,1,\varrho,t)|^2 - |\nabla z(x,0,\varrho,t)|^2\Big] d\varrho dx \\
& - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} |\mu_2(\varrho)|\, |\nabla y|^2\, d\varrho d\rho dx \\
& - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \Big[e^{-\varrho} |\nabla y(x,1,\varrho,t)|^2 - |\nabla y(x,0,\varrho,t)|^2\Big] d\varrho dx.
\end{aligned}
$$

Using the equality $z(x,0,\varrho,t) = u_t(x,t)$, $y(x,0,\varrho,t) = v_t(x,t)$ and $e^{-\varrho} \leq e^{-\rho\varrho} \leq 1$, for any $0 < \rho < 1$, we get

$$
\begin{aligned}
F_3'(t) = & - \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} \Big(|\mu_1(\varrho)| \|\nabla z\|^2 + |\mu_2(\varrho)| \|\nabla y\|^2\Big) d\varrho d\rho \\
& - \int_{\tau_1}^{\tau_2} e^{-\varrho} \Big(|\mu_1(\varrho)| \|\nabla z(x,1,\varrho,t)\|^2 + |\mu_2(\varrho)| \|\nabla y(x,1,\varrho,t)\|^2\Big) d\varrho \\
& + \Big(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho\Big) \|\nabla u_t\|^2 + \Big(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho\Big) \|\nabla v_t\|^2.
\end{aligned}
$$

As $-e^{-\varrho}$ is a increasing function, we have $-e^{-\varrho} \leq -e^{-\tau_2}$, for any $\varrho \in [\tau_1, \tau_2]$.
Then, setting $\eta_1 = e^{-\tau_2}$ and $k_0 = \max(k_1, k_2)$, we obtain (3.4.26). □

Theorem 11. *Assume (3.2.1)-(3.2.3), there exist positive constants* ζ_1 *and* ζ_2 *such that the* *energy functional (3.2.8) satisfies*

$$E(t) \leq \zeta_2 e^{-\zeta_1 t}, \forall t \geq t_0. \tag{3.4.27}$$

Proof. We define a Lyapunov functional

$$\mathcal{L}(t) := NE(t) + F_1(t) + N_2F_2(t) + N_3F_3(t), \tag{3.4.28}$$

where $N, N_2, N_3 > 0$.
First, if we let

$$\mathcal{K}(t) = F_1(t) + N_2F_2(t) + N_3F_3(t)$$

then, by (3.4.2), (3.4.7), (3.4.25), we get

$$|\mathcal{K}(t)| \leq cE(t).$$

Consequently,

$$|\mathcal{K}(t)| = |\mathcal{L}(t) - NE(t)| \leq cE(t)$$

which yield

$$(N-c)E(t) \leq \mathcal{L}(t) \leq (N+c)E(t). \tag{3.4.29}$$

On the other hand, by differentiating (3.4.28) and using (3.2.9), (3.4.3), (3.4.8), (3.4.26) and (3.2.2), we have

$$\begin{aligned}
\mathcal{L}'(t) \leq\ & -N\lambda\Big(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\Big) + \frac{N}{2}(g_1' \circ \nabla u)(t) + \frac{N}{2}(g_2' \circ \nabla v)(t) \\
& - \frac{N}{2}g_1(t)\|\nabla u(t)\|^2 - \frac{N}{2}g_2(t)\|\nabla v(t)\|^2 \\
& + \frac{1}{l+1}\Big(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\Big) + \Big(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\Big) \\
& - \frac{k}{2}\Big(\|\nabla u\|^2 + \|\nabla v\|^2\Big) - b\|\nabla u\|^{2(\gamma+1)} - b\|\nabla v\|^{2(\gamma+1)} \\
& + c\int_{\tau_1}^{\tau_2}\Big(|\mu_1(\varrho)|\|\nabla z(x,1,\varrho,t)\|^2 + |\mu_2(\varrho)|\|\nabla y(x,1,\varrho,t)\|^2\Big)d\varrho \\
& + c\big((g_1 \circ \nabla u) + (g_2 \circ \nabla v)\big) - 2\alpha\int_\Omega uv\,dx \\
& - \frac{N_2}{l+1}(a-k)\Big[\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\Big] \\
& + N_2\Big(\varepsilon_2(a-k)^2 + \alpha C_*\frac{\varepsilon_2}{2}\Big)\Big(\|\nabla u\|^2 + \|\nabla v\|^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&+N_2\frac{\varepsilon_2}{2}M(\|\nabla u\|^2)\|\nabla u\|^2+N_2\frac{\varepsilon_2}{2}M(\|\nabla v\|^2)\|\nabla v\|^2\\
&-N_2\bigg(\ (a-k)-\frac{\varepsilon_2}{2}\left(2+k_1\right)\bigg)\|\nabla u_t\|^2+N_2\frac{\varepsilon_2 k_1}{2}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\|\nabla z(x,1,\varrho,t)\|^2d\varrho\\
&-N_2\bigg(\ (a-k)-\frac{\varepsilon_2}{2}\left(2+k_2\right)\bigg)\|\nabla v_t\|^2+N_2\frac{\varepsilon_2 k_2}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\|\nabla y(x,1,\varrho,t)\|^2d\varrho\\
&+N_2\bigg\{\frac{M(\|\nabla u\|^2)}{2\varepsilon_2}+\frac{1}{2\varepsilon_2}\bigg(2+k_1+\alpha C_*\bigg)+\varepsilon_2\bigg\}(a-k)(g_1\circ\nabla u)\\
&+N_2\bigg\{\frac{M(\|\nabla v\|^2)}{2\varepsilon_2}+\frac{1}{2\varepsilon_2}\bigg(2+k_2+\alpha C_*\bigg)+\varepsilon_2\bigg\}(a-k)(g_2\circ\nabla v)\\
&-N_2\frac{cg_1(0)}{\varepsilon_2}(g_1'\circ\nabla u)-N_2\frac{cg_2(0)}{\varepsilon_2}(g_2'\circ\nabla v)\\
&-N_3\eta_1\int_0^1\int_{\tau_1}^{\tau_2}\varrho\bigg(|\mu_1(\varrho)|\|\nabla z(x,\rho,\varrho,t)\|^2+|\mu_2(\varrho)|\|\nabla y(x,\rho,\varrho,t)\|^2\bigg)d\varrho d\rho\\
&+N_3k_0\bigg(\|\nabla u_t\|^2+\|\nabla v_t\|^2\bigg)\\
&-N_3\eta_1\int_{\tau_1}^{\tau_2}\bigg(|\mu_1(\varrho)|\|\nabla z(x,1,\varrho,t)\|^2+|\mu_2(\varrho)|\|\nabla y(x,1,\varrho,t)\|^2\bigg)d\varrho,
\end{aligned}
$$

then, we have

$$
\begin{aligned}
\mathcal{L}'(t)\ \leq\ &-\frac{1}{l+1}\left\{N_2\left(a-k\right)-1\right\}\left[\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\right]\\
&-\left\{N\lambda-1+N_2\left((a-k)-\frac{\varepsilon_2}{2}(2+k_0)\right)-N_3k_0\right\}\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\\
&-\frac{N}{2}g_1(t)\|\nabla u(t)\|^2-\frac{N}{2}g_2(t)\|\nabla v(t)\|^2\\
&-\left(\frac{k}{2}-N_2\bigg(\varepsilon_2(a-k)^2+\alpha C_*\frac{\varepsilon_2}{2}\bigg)-1\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\\
&+N_2\frac{\varepsilon_2}{2}\left(a+b\|\nabla u\|^{2\gamma}\right)\|\nabla u\|^2+N_2\frac{\varepsilon_2}{2}\left(a+b\|\nabla u\|^{2\gamma}\right)\|\nabla v\|^2\\
&-b\|\nabla u\|^{2(\gamma+1)}-b\|\nabla v\|^{2(\gamma+1)}\\
&+c\big(\ (g_1\circ\nabla u)+(g_2\circ\nabla v)\ \big)-2\alpha\int_\Omega uvdx\\
&+N_2\bigg\{\frac{M(\|\nabla u\|^2)}{2\varepsilon_2}+\frac{1}{2\varepsilon_2}\bigg(2+k_1+\alpha C_*\bigg)+\varepsilon_2\bigg\}(a-k)(g_1\circ\nabla u)\\
&+N_2\bigg\{\frac{M(\|\nabla v\|^2)}{2\varepsilon_2}+\frac{1}{2\varepsilon_2}\bigg(2+k_2+\alpha C_*\bigg)+\varepsilon_2\bigg\}(a-k)(g_2\circ\nabla v)\\
&+\left(\frac{N}{4}+\frac{N}{4}\right)(g_1'\circ\nabla u)(t)+\left(\frac{N}{4}+\frac{N}{4}\right)(g_2'\circ\nabla v)(t)
\end{aligned}
$$

$$
\begin{aligned}
&-N_2\frac{cg_1(0)}{\varepsilon_2}(g_1'\circ\nabla u)-N_2\frac{cg_2(0)}{\varepsilon_2}(g_2'\circ\nabla v)\\
&-N_3\eta_1\int_0^1\int_{\tau_1}^{\tau_2}\varrho\Big(|\mu_1(\varrho)|\|\nabla z(x,\rho,\varrho,t)\|^2+|\mu_2(\varrho)|\|\nabla y(x,\rho,\varrho,t)\|^2\Big)d\varrho d\rho\\
&-\left(N_3\eta_1-c-N_2\frac{\varepsilon_2k_0}{2}\right)\int_{\tau_1}^{\tau_2}\Big(|\mu_1(\varrho)|\|\nabla z(x,1,\varrho,t)\|^2+|\mu_2(\varrho)|\|\nabla y(x,1,\varrho,t)\|^2\Big)d\varrho.
\end{aligned}
$$

By using the inequality

$$
\frac{N}{4}g_i'(t)\leq-\frac{N}{4}\xi_ig_i(t),\ i=1,2,
$$

we arive at

$$
\begin{aligned}
\mathcal{L}'(t)\ \leq\ &-\frac{1}{l+1}\left\{N_2(a-k)-1\right\}\left[\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\right]\\
&-\left\{N\lambda-1+N_2\left((a-k)-\frac{\varepsilon_2}{2}(2+k_0)\right)-N_3k_0\right\}\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\\
&-\left(\frac{k}{2}-N_2\left(\varepsilon_2(a-k)^2+\alpha C_*\frac{\varepsilon_2}{2}+a\frac{\varepsilon_2}{2}\right)\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\\
&-b\left(1-N_2\frac{\varepsilon_2}{2}\right)\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right)\\
&+N_2\left\{\frac{M(\|\nabla u\|^2)}{2\varepsilon_2}+\frac{1}{2\varepsilon_2}\left(2+k_1+\alpha C_*\right)+\varepsilon_2\right\}(a-k)(g_1\circ\nabla u)\\
&+N_2\left\{\frac{M(\|\nabla v\|^2)}{2\varepsilon_2}+\frac{1}{2\varepsilon_2}\left(2+k_2+\alpha C_*\right)+\varepsilon_2\right\}(a-k)(g_2\circ\nabla v)\\
&+c\big((g_1\circ\nabla u)+(g_2\circ\nabla v)\big)-2\alpha\int_\Omega uvdx\\
&-\frac{N}{4}\xi_1(g_1\circ\nabla u)(t)-\frac{N}{4}\xi_2(g_2\circ\nabla v)(t)\\
&+\left(\frac{N}{4}-N_2\frac{cg_1(0)}{\varepsilon_2}\right)(g_1'\circ\nabla u)+\left(\frac{N}{4}-N_2\frac{cg_2(0)}{\varepsilon_2}\right)(g_2'\circ\nabla v)\\
&-N_3\eta_1\int_0^1\int_{\tau_1}^{\tau_2}\varrho\Big(|\mu_1(\varrho)|\|\nabla z(x,\rho,\varrho,t)\|^2+|\mu_2(\varrho)|\|\nabla y(x,\rho,\varrho,t)\|^2\Big)d\varrho d\rho\\
&-\left(N_3\eta_1-c-N_2\frac{\varepsilon_2k_0}{2}\right)\int_{\tau_1}^{\tau_2}\Big(|\mu_1(\varrho)|\|\nabla z(x,1,\varrho,t)\|^2+|\mu_2(\varrho)|\|\nabla y(x,1,\varrho,t)\|^2\Big)d\varrho.
\end{aligned}
$$

Finaly, we obtain

$$
\begin{aligned}
\mathcal{L}'(t)\ \leq\ &-\frac{1}{l+1}\Big\{N_2(a-k)-1\Big\}\left[\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\right]-2\alpha\int_\Omega uvdx\\
&-\left\{\lambda N+N_2\left((a-k)-\frac{\varepsilon_2}{2}(2+k_0)\right)-1-N_3k_0\right\}\Big(\|\nabla u_t\|^2+\|\nabla v_t\|^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&-b\left(1-N_2\frac{\varepsilon_2}{2}\right)\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right)\\
&-N_3\eta_1\int_0^1\int_{\tau_1}^{\tau_2}\varrho\Big(|\mu_1(\varrho)|\|\nabla z\|^2+|\mu_2(\varrho)|\|\nabla y\|^2\Big)d\varrho d\rho\\
&-\left\{\frac{k}{2}-N_2\varepsilon_2\left(\frac{a}{2}+(a-k)^2+\frac{\alpha C_*}{2}\right)\right\}\left[\|\nabla u\|^2+\|\nabla v\|^2\right]\\
&-\left\{\frac{N\xi}{4}-c-N_2(a-k)\left(\frac{1}{2\varepsilon_2}\left(M_0+2+k_0+\alpha C_*\right)+\varepsilon_2\right)\right\}\left[(g_1\circ\nabla u)+(g_2\circ\nabla v)\right]\\
&+\left\{\frac{N}{4}-\frac{cN_2h_1}{\varepsilon_2}\right\}\left[(g_1'\circ\nabla u)+(g_2'\circ\nabla v)\right]\\
&-\left\{N_3\eta_1-c-N_2\varepsilon_2\frac{k_0}{2}\right\}\int_{\tau_1}^{\tau_2}\left(|\mu_1(\varrho)|\|\nabla z(x,1,\varrho,t)\|^2+|\mu_2(\varrho)|\|\nabla y(x,1,\varrho,t)\|^2\right)d\varrho.
\end{aligned}
$$

Where $h_1=\min\Big(g_1(0),g_2(0)\Big)$, $M_0=\max\Big(M(\|\nabla u\|^2),M(\|\nabla v\|^2)\Big)$, $\xi=\min\Big(\xi_1,\xi_2\Big)$. Now, we choose and fixed N_2, such that

$$\alpha_1=N_2(a-k)-1>0.$$

After that, we choose ε_2 such that

$$\alpha_3=1-N_2\frac{\varepsilon_2}{2}>0,$$

and

$$\alpha_4=\left\{\frac{k}{2}-N_2\varepsilon_2\left(\frac{a}{2}+(a-k)^2+\frac{\alpha C_*}{2}\right)\right\}>0.$$

Further, we choose N_3 large enough such that

$$\alpha_6=N_3\eta_1-c-N_2\varepsilon_2\frac{k_0}{2}>0.$$

Finaly, we choose N large enough such that

$$\alpha_2=\lambda N+N_2\left((a-k)-\frac{\varepsilon_2}{2}(2+k_0)\right)-1-N_3k_0>0,$$

$$\alpha_5=\frac{N\xi}{4}-c-N_2(a-k)\left(\frac{1}{2\varepsilon_2}\left(M_0+2+k_0+\alpha C_*\right)+\varepsilon_2\right)>0,$$

and

$$\frac{N}{4}-\frac{cN_2h_1}{\varepsilon_2}>0.$$

Thus, we arrive at

$$\begin{aligned}\mathcal{L}'(t) \quad\leq\quad & -\frac{1}{l+1}\alpha_1\left[\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\right]-\alpha_2\Big(\|\nabla u_t\|^2+\|\nabla v_t\|^2\Big)-2\alpha\int_\Omega uvdx\\ & -b\alpha_3\Big(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\Big)-\alpha_4\left[\|\nabla u\|^2+\|\nabla v\|^2\right]\\ & -\alpha_5\left[(g_1\circ\nabla u)+(g_2\circ\nabla v)\right]-\eta_1\int_0^1\int_{\tau_1}^{\tau_2}\varrho\Big(|\mu_1(\varrho)|\|\nabla z\|^2+|\mu_2(\varrho)|\|\nabla y\|^2\Big)d\varrho d\rho.\end{aligned} \tag{3.4.30}$$

This is equivalent

$$\begin{aligned}\mathcal{L}'(t) \quad\leq\quad & -\frac{1}{l+1}\alpha_1\left[\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\right]-\left(\alpha_2+\frac{\alpha_2}{2}\right)\Big(\|\nabla u_t\|^2+\|\nabla v_t\|^2\Big)-2\alpha\int_\Omega uvdx\\ & -b\alpha_3\Big(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\Big)-\alpha_4\left[\|\nabla u\|^2+\|\nabla v\|^2\right]\\ & -\alpha_5\left[(g_1\circ\nabla u)+(g_2\circ\nabla v)\right]-\eta_1\int_0^1\int_{\tau_1}^{\tau_2}\varrho\Big(|\mu_1(\varrho)|\|\nabla z\|^2+|\mu_2(\varrho)|\|\nabla y\|^2\Big)d\varrho d\rho\\ & +\frac{\alpha_2}{2}\Big(\|\nabla u_t\|^2+\|\nabla v_t\|^2\Big).\end{aligned} \tag{3.4.31}$$

And

$$c_1E(t)\leq\mathcal{L}(t)\leq c_2E(t),\forall t\geq 0, \tag{3.4.32}$$

using (3.2.8) and (3.2.9), estimates (3.4.31), (3.4.32), respectively, we get

$$\mathcal{L}'(t)\leq -m_1E(t)-m_2E'(t),\forall t\geq t_0 \tag{3.4.33}$$

for some $m_1, m_2, c_1, c_2 > 0$.
A combination (3.4.33) with (3.4.32), gives

$$\mathcal{R}'(t)\leq -\lambda_1\mathcal{R}(t), \tag{3.4.34}$$

where

$$\mathcal{R}(t)=\mathcal{L}(t)+m_2E(t)\sim E(t). \tag{3.4.35}$$

Finally, a simple integration of (3.4.34) over (t_0, t), gives

$$\mathcal{R}(t)\leq\mathcal{R}(t_0)e^{-\zeta_1(t-t_0)},\ \ \forall t\geq t_0. \tag{3.4.36}$$

Thanks to (3.4.35), we obtain (3.4.27). This completes the proof. □

Chapter 4

General decay rate for a coupled Lamé system with viscoelastic damping and distributed delay terms

4.1 Introduction

In this chapter, we prove a general energy decay results of a coupled Lamé system with distributed time delay. By assuming a more general of relaxation functions and using some properties of convex functions, we establish the general energy decay results to the system by using an appropriate Lyapunov functional. We are going to study the general decay rate of the following Lamé system in $\Omega \times \mathbb{R}_+$:

$$\begin{cases} u_{tt} - \Delta_e u + \int_0^t g_1(t-s)\,\Delta u(s)\,ds - k_1 \Delta u_t - \int_{\tau_1}^{\tau_2} \mu_1(\varrho)\Delta u_t(x,t-\varrho)d\varrho = f_1(u,v), \\ v_{tt} - \Delta_e v + \int_0^t g_2(t-s)\,\Delta v(s)\,ds - k_2 \Delta v_t - \int_{\tau_1}^{\tau_2} \mu_2(\varrho)\Delta v_t(x,t-\varrho)d\varrho = f_2(u,v). \end{cases} \tag{4.1.1}$$

Equations (4.1.1) are associated with the following boundary and initial conditions

$$\begin{cases} u(x,t) = v(x,t) = 0, \text{ on } \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), x \in \Omega, \\ (u_t(x,-t), v_t(x,-t)) = (f_0(x,t), g_0(x,t)), \text{ in } \Omega \times (0,\tau_2), \end{cases} \tag{4.1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n = 1,2,3)$, with smooth boundary $\partial\Omega$. The elasticity differential operator Δ_e is given by

$$\Delta_e u = \mu \Delta u + (\mu + \lambda) \nabla (divu),$$

and the Lamé constants μ and λ are satisfying the following conditions

$$\mu > 0, \mu + \lambda > 0.$$

The parameters k_1, k_2, τ_1 and τ_2 are positive constants, with $\tau_1 < \tau_2$. The functions μ_1, $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ are bounded. The functions $f_1(u,v)$ and $f_2(u,v)$ which represent the source terms will be specified later.

During this chapter, we have extended the general decay result obtained by Baowei Feng in [29] to the case of distributed term delay, namely, we will make sure that the result is achieved if the distributed delay term exists.

4.2 Preliminaries

In this section, we provide some materials and necessary assumptions wich we need in the prove of our results. We use the standard Lebesgue and Sobolev spaces with their scaler products and norms. For simplicity, we would write $\|.\|$ instead of $\|.\|_2$. Throughout this chapter, we used a generic positive constant c.

For the relaxation functions g_1 and g_2, we assume, for $i = 1, 2$,

(A1) $g_i(t) : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing C^1 functions satisfying

$$g_i(0) > 0 \text{ and } \mu - \int_0^\infty g_i(s)ds = l_i > 0. \tag{4.2.1}$$

We assume further that, for $i = 1, 2$:

(A2) There exist two C^1 functions $G_i : \mathbb{R}_+ \to \mathbb{R}_+$, with $G_i(0) = G_i'(0) = 0$, which are linear or are strictly increasing and strictly convex functions of class $C^2(\mathbb{R}_+)$ on $(0, r], r \leq g_i(0)$, such that

$$g_i'(t) \leq -\xi_i(t) G_i(g_i(t)), \ \forall t \geq 0, \tag{4.2.2}$$

where $\xi_i(t)$ are C^1 functions satisfying

$$\xi_i(t) > 0, \ \xi_i'(t) \leq 0, \ \forall t \geq 0. \tag{4.2.3}$$

(A3) For the source terms f_1 and f_2, we take

$$f_1(u,v) = \alpha|u+v|^{p-1}(u+v) + \beta|u|^{\frac{p-3}{2}}u|v|^{\frac{p+1}{2}}, \ \forall (u,v) \in [\mathbb{R}^n]^2,$$

$$f_2(u,v) = \alpha|u+v|^{p-1}(u+v) + \beta|v|^{\frac{p-3}{2}}v|u|^{\frac{p+1}{2}},\ \forall(u,v)\in[\mathbb{R}^n]^2,$$

with $\alpha, \beta > 0$. Clearly,

$$uf_1(u,v) + vf_2(u,v) = (p+1)F(u,v),\ \forall(u,v)\in[\mathbb{R}^n]^2, \tag{4.2.4}$$

where

$$F(u,v) = \frac{1}{(p+1)}\left[\alpha|u+v|^{p+1} + 2\beta|uv|^{\frac{p+1}{2}}\right],\ \forall(u,v)\in[\mathbb{R}^n]^2, \tag{4.2.5}$$

and

$$f_1(u,v) = \frac{\partial F}{\partial u},\ f_2(u,v) = \frac{\partial F}{\partial v}. \tag{4.2.6}$$

Further, we assume that there is $C > 0$, such that

$$\left|\frac{\partial f_i}{\partial u}(u,v)\right| + \left|\frac{\partial f_i}{\partial v}(u,v)\right| \leq C\left(|u|^{p-1} + |v|^{p-1}\right),\ i=1,2 \quad \text{where} \quad 1\leq p\leq 6.$$

(A4)

$$\text{if} \quad n=1,2;\ p\geq 3, \quad \text{if} \quad n=3\ ; p=3. \tag{4.2.7}$$

So, we have the embedding

$$H_0^1(\Omega)\hookrightarrow L^q(\Omega) \ \text{ for } 2\leq q\leq \frac{2n}{n-2} \text{ if } n\geq 3 \text{ or } q\geq 2 \text{ if } n=1,2$$

and

$$L^r \hookrightarrow L^q \text{ for } q<r.$$

Let c_s the same embedding constant, so we have

$$\|\nu\|_q \leq c_s\|\nabla\nu\|_2,\ \|\nu\|_q\leq c_s\|\nu\|_r \ \text{ for } \nu\in H_0^1(\Omega). \tag{4.2.8}$$

The following remark is proved by Said-Houari el al in [70]

Remark 3. *There exist two constants* $\Lambda_1 > 0$ *and* $\Lambda_2 > 0$ *such that*

$$\int_\Omega |f_i(u,v)|^2 dx \leq \Lambda_i\left(l_1\|\nabla u\|^2 + l_2\|\nabla v\|^2\right)^p,\ i=1,2. \tag{4.2.9}$$

As in many papers, we introduce the following new variables

$$\begin{cases} z(x,\rho,\varrho,t) = u_t(x,t-\varrho\rho), \\ y(x,\rho,\varrho,t) = v_t(x,t-\varrho\rho), \end{cases}$$

then we obtain

$$\begin{cases} \varrho z_t(x,\rho,\varrho,t) + z_\rho(x,\rho,\varrho,t) = 0, \\ z(x,0,\varrho,t) = u_t(x,t), \end{cases} \tag{4.2.10}$$

and

$$\begin{cases} \varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = 0, \\ y(x,0,\varrho,t) = v_t(x,t). \end{cases} \tag{4.2.11}$$

Consequently, the problem (4.1.1) is equivalent to

$$\begin{cases} u_{tt} - \Delta_e u + \int_0^t g_1(t-s)\,\Delta u(s)\,ds - k_1\Delta u_t - \int_{\tau_1}^{\tau_2} \mu_1(\varrho)\Delta z(x,1,\varrho,t)d\varrho = f_1(u,v), \\ v_{tt} - \Delta_e v + \int_0^t g_2(t-s)\,\Delta v(s)\,ds - k_2\Delta v_t - \int_{\tau_1}^{\tau_2} \mu_2(\varrho)\Delta y(x,1,\varrho,t)d\varrho = f_2(u,v), \\ \varrho z_t(x,\rho,\varrho,t) + z_\rho(x,\rho,\varrho,t) = 0, \\ \varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = 0, \end{cases} \tag{4.2.12}$$

with the initial data and boundary conditions

$$\begin{cases} (u(x,0),v(x,0)) = (u_0(x),v_0(x)), & \text{in } \Omega, \\ (u_t(x,0),v_t(x,0)) = (u_1(x),v_1(x)), & \text{in } \Omega, \\ (u_t(x,-t),v_t(x,-t)) = (f_0(x,t),g_0(x,t)), & \text{in } \Omega\times(0,\tau_2), \\ u(x,t) = v(x,t) = 0, & \text{on } \partial\Omega\times(0,\infty), \\ (z(x,\rho,\varrho,0),y(x,\rho,\varrho,0)) = (f_0(x,\rho\varrho),g_0(x,\rho\varrho)), & \text{in } \Omega\times(0,1)\times(0,\tau_2), \end{cases} \tag{4.2.13}$$

where

$$(x,\rho,\varrho,t) \in \Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

We recall the following notations

$$(h*\varphi) = \int_0^t h(t-s)\,\varphi(s)\,dsdx,$$
$$(h\circ\varphi)(t) = \int_0^t h(t-s)\,|\varphi(t)-\varphi(s)|^2\,ds.$$

Thus, we have the following important property

$$\int_\Omega (h*\varphi)\,\varphi_t dx = -\frac{1}{2}h(t)\,\|\varphi(t)\|^2 + \frac{1}{2}(h'\circ\varphi)(t) - \frac{1}{2}\frac{d}{dt}\left[(h\circ\varphi)(t) - \left(\int_0^t h(s)\,ds\right)\|\varphi(t)\|^2\right]. \tag{4.2.14}$$

The energy modified associated to the problem (4.2.12) is defined by

$$\begin{aligned} E(t) &= \frac{1}{2}\left[\|u_t\|^2 + \left(\mu - \int_0^t g_1(s)\,ds\right)\|\nabla u\|^2 + (\lambda+\mu)\|divu\|^2\right] \\ &+ \frac{1}{2}\left[(g_1\circ\nabla u)(t) + \eta\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho\,|\mu_1(\varrho)|\,|\nabla z(x,\rho,\varrho,t)|^2\,d\varrho d\rho dx\right] \\ &+ \frac{1}{2}\left[\|v_t\|^2 + \left(\mu - \int_0^t g_2(s)\,ds\right)\|\nabla v\|^2 + (\lambda+\mu)\|divv\|^2\right] \\ &+ \frac{1}{2}\left[(g_2\circ\nabla v)(t) + \eta\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho\,|\mu_2(\varrho)|\,|\nabla y(x,\rho,\varrho,t)|^2\,d\varrho d\rho dx\right] \\ &- \int_\Omega F(u,v)\,dx. \end{aligned} \tag{4.2.15}$$

First, we prove in the following theorem the result of energy identity.

Lemma 13. *Assume that*

$$\int_{\tau_1}^{\tau_2}|\mu_i(\varrho)|\,d\varrho < k_i,\;\; i=1,2. \tag{4.2.16}$$

Then, the energy modified defined by (4.2.15) *satisfies, along the solution* (u,v,z,y) *of* (4.2.12)*, the estimate*

$$\begin{aligned} \frac{d}{dt}E(t) &\le -\frac{1}{2}g_1(t)\|\nabla u\|^2 + \frac{1}{2}(g_1'\circ\nabla u)(t) - \frac{1}{2}g_2(t)\|\nabla v\|^2 + \frac{1}{2}(g_2'\circ\nabla v)(t) \\ &- \left[k_1 - \left(\frac{\eta+1}{2}\right)\left(\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\,d\varrho\right)\right]\|\nabla u_t\|^2 \\ &- \left(\frac{\eta-1}{2}\right)\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\,|\nabla z(x,1,\varrho,t)|^2\,d\varrho dx \\ &- \left[k_2 - \left(\frac{\eta+1}{2}\right)\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,d\varrho\right)\right]\|\nabla v_t\|^2 \\ &- \left(\frac{\eta-1}{2}\right)\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,|\nabla y(x,1,\varrho,t)|^2\,d\varrho dx \le 0, \end{aligned} \tag{4.2.17}$$

for

$$1 < \eta < \min\left(\frac{2k_1}{\left(\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\,d\varrho\right)}, \frac{2k_2}{\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,d\varrho\right)}\right) - 1. \tag{4.2.18}$$

Proof. First multiplying the equation $(4.2.12)_1$ by u_t and integrating by parts over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\|u_t\|^2 + \mu\|\nabla u\|^2 + (\lambda+\mu)\|divu\|^2\right] - \int_\Omega\nabla u_t\int_0^t g_1(t-s)\nabla u(s)\,ds$$

$$+k_1 \|\nabla u_t\|^2 + \int_\Omega \nabla u_t \int_{\tau_1}^{\tau_2} |\mu_1(\varrho) \nabla z(x,1,\varrho,t)\, d\varrho dx = \int_\Omega f_1(u,v).udx,$$

by using (4.2.14), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|u_t\|^2 + \left(\mu - \int_0^t g_1(s)\,ds\right)\|\nabla u\|^2 + (\lambda+\mu)\|divu\|^2 + (g_1 \circ \nabla u)(t)\right] \\ &= -\frac{1}{2}g_1\|\nabla u\|^2 + \frac{1}{2}(g_1' \circ \nabla u)(t) + \int_\Omega u_t f_1(u,v)\,dx - k_1\|\nabla u_t\|^2 \\ &+ \int_\Omega \nabla u_t \int_{\tau_1}^{\tau_2} \mu_1(\varrho)\nabla z(x,1,\varrho,t)\,d\varrho dx. \end{aligned} \tag{4.2.19}$$

Similarly, multiplying the equation $(4.2.12)_2$ by v_t and integrating over Ω, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|v_t\|^2 + \left(\mu - \int_0^t g_2(s)\,ds\right)\|\nabla v\|^2 + (\lambda+\mu)\|divv\|^2 + (g_2 \circ \nabla v)(t)\right] \\ &= -\frac{1}{2}g_2\|\nabla v\|^2 + \frac{1}{2}(g_2' \circ \nabla v)(t) + \int_\Omega v_t f_2(u,v)\,dx - k_2\|\nabla v_t\|^2 \\ &+ \int_\Omega \nabla v_t \int_{\tau_1}^{\tau_2} \mu_2(\varrho)\nabla y(x,1,\varrho,t)\,d\varrho dx. \end{aligned} \tag{4.2.20}$$

Multiplying the equation $(4.2.12)_3$ by $-\eta|\mu_1(\varrho)|\Delta z(x,\rho,\varrho,t)$ and integrating by parts over $\Omega\times(0,1)\times(\tau_1,\tau_2)$, we obtain

$$\begin{aligned} &\eta\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_1(\varrho)|\nabla z(x,\rho,\varrho,t)\nabla z_t(x,\rho,\varrho,t)\,d\varrho d\rho dx \\ &= -\eta\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)|\nabla z(x,\rho,\varrho,t)\nabla z_\rho(x,\rho,\varrho,t)\,d\varrho d\rho dx, \end{aligned}$$

therefore

$$\begin{aligned} &\frac{d}{dt}\frac{\eta}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_1(\varrho)||\nabla z(x,\rho,\varrho,t)|^2\,d\varrho d\rho dx \\ &= -\frac{\eta}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)|\frac{d}{d\rho}|\nabla z(x,\rho,\varrho,t)|^2\,d\varrho d\rho dx \\ &= -\frac{\eta}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)||\nabla z(x,1,\varrho,t)|^2\,d\varrho dx + \frac{\eta}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)||\nabla u_t(x,t)|^2\,d\varrho dx. \end{aligned} \tag{4.2.21}$$

Multiplying the fourth equation of (4.2.12) by $-\eta|\mu_2(\varrho)|\Delta y(x,\rho,\varrho,t)$ and integrating over $\Omega\times(0,1)\times(\tau_1,\tau_2)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\eta}{2}\int_{\Omega}\int_{0}^{1}\int_{\tau_1}^{\tau_2}\varrho\left|\mu_2\left(\varrho\right)\right|\left|\nabla y\left(x,\rho,\varrho,t\right)\right|^2 d\varrho d\rho dx\right)\\
&=-\frac{\eta}{2}\int_{\Omega}\int_{0}^{1}\int_{\tau_1}^{\tau_2}\left|\mu_2\left(\varrho\right)\right|\frac{d}{d\rho}\left|\nabla y\left(x,\rho,\varrho,t\right)\right|^2 d\varrho d\rho dx \qquad (4.2.22)\\
&=-\frac{\eta}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2\left(\varrho\right)\right|\left|\nabla y\left(x,1,\varrho,t\right)\right|^2 d\varrho dx+\frac{\eta}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2\left(\varrho\right)\right|\left|\nabla v_t\left(x,t\right)\right|^2 d\varrho dx.
\end{aligned}$$

For the source term, we have

$$\begin{aligned}
&\int_{\Omega}u_t f_1\left(u,v\right)dx+\int_{\Omega}v_t f_2\left(u,v\right)dx=\int_{\Omega}u_t\left(\alpha|u+v|^{p-1)}(u+v)+\beta|u|^{\frac{p-3}{2}}u|v|^{\frac{p+1}{2}}\right)\\
&+\int_{\Omega}v_t\left(\alpha|u+v|^{p-1}(u+v)+\beta|v|^{\frac{p-3}{2}}v|u|^{\frac{p+1}{2}}\right)\\
&=\int_{\Omega}\left(\alpha|u+v|^{p-1}(u+v)\left(u_t+v_t\right)+\beta\left(|u|^{\frac{p-3}{2}}u.u_t\right)|v|^{\frac{p+1}{2}}+\beta\left(|v|^{\frac{p-3}{2}}v.v_t\right)|u|^{\frac{p+1}{2}}\right)dx\\
&=\frac{d}{dt}\int_{\Omega}\left(\frac{\alpha}{p+1}|u+v|^{p+1}+\frac{2\beta}{p+1}|uv|^{\frac{p+1}{2}}\right)dx=\frac{d}{dt}\int_{\Omega}F\left(u,v\right)dx.
\end{aligned} \qquad (4.2.23)$$

By collecting the previous equations (4.2.19)-(4.2.23), we get

$$\begin{aligned}
\frac{d}{dt}E\left(t\right)=&-\frac{1}{2}g_1\left\|\nabla u\right\|^2+\frac{1}{2}\left(g_1'\circ\nabla u\right)\left(t\right)-k_1\left\|\nabla u_t\right\|+\int_{\Omega}\nabla u_t\int_{\tau_1}^{\tau_2}\mu_1\left(\varrho\right)\nabla z\left(x,1,\varrho,t\right)d\varrho dx\\
&-\frac{1}{2}g_2\left\|\nabla v\right\|^2+\frac{1}{2}\left(g_2'\circ\nabla v\right)\left(t\right)-k_2\left\|\nabla v_t\right\|+\int_{\Omega}\nabla v_t\int_{\tau_1}^{\tau_2}\mu_2\left(\varrho\right)\nabla y\left(x,1,\varrho,t\right)d\varrho dx\\
&-\frac{\eta}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_1\left(\varrho\right)\right|\left|\nabla z\left(x,1,\varrho,t\right)\right|^2 d\varrho dx+\frac{\eta}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_1\left(\varrho\right)\right|\left|\nabla u_t\left(x,t\right)\right|^2 d\varrho dx\\
&-\frac{\eta}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2\left(\varrho\right)\right|\left|\nabla y\left(x,1,\varrho,t\right)\right|^2 d\varrho dx+\frac{\eta}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2\left(\varrho\right)\right|\left|\nabla v_t\left(x,t\right)\right|^2 d\varrho dx.
\end{aligned} \qquad (4.2.24)$$

Using Young's inequality, we obtain

$$\begin{aligned}
&\int_{\Omega}\nabla u_t\int_{\tau_1}^{\tau_2}\mu_1\left(\varrho\right)\nabla z\left(x,1,\varrho,t\right)d\varrho dx\leq\int_{\Omega}\int_{\tau_1}^{\tau_2}\left(\nabla u_t\sqrt{\left|\mu_1\left(\varrho\right)\right|}\right)\left(\sqrt{\left|\mu_1\left(\varrho\right)\right|}\nabla z\left(x,1,\varrho,t\right)\right)d\varrho dx\\
&\leq\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_1\left(\varrho\right)\right|d\varrho\right)\left\|\nabla u_t\right\|+\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_1\left(\varrho\right)\right|\left|\nabla z\left(x,1,\varrho,t\right)\right|^2 d\varrho dx,
\end{aligned} \qquad (4.2.25)$$

similarly

$$\begin{aligned}\int_{\Omega} \nabla v_t \int_{\tau_1}^{\tau_2} \mu_2(\varrho) \nabla y(x,1,\varrho,t)\, d\varrho dx \le \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, d\varrho\right) \|\nabla v_t\| \\ + \frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, |\nabla y(x,1,\varrho,t)|^2\, d\varrho dx.\end{aligned} \tag{4.2.26}$$

This completes the proof. □

4.3 General decay

In this section we will prove that the solution of problem (4.2.12)-(4.2.13) decay generally to trivial solution. Using the energy method and suitable Lyapunov functinal.

In the following, we will present our main stability result:

Theorem 12. *(Decay rates of energy) Assume that* **(A1)**-**(A3)** *hold. Then, for every $t_0 > 0$ there exist two positive constants α_1 and α_2 such that the energy defined by (4.2.15) satisfies the following decay*

$$E(t) \le \alpha_2 G_4^{-1}\left(\alpha_1 \int_{g^{-1}(r)}^{t} \xi(s)\, ds\right), \ \forall t \ge g^{-1}(r), \tag{4.3.1}$$

where

$$G_4(t) = \int_t^r \frac{1}{sG_0(s)} ds,\ G_0(t) = \min\{G_1'(t), G_2'(t)\},$$

and $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}, g(t) = \max\{g_1(t), g_2(t)\}.$

This theorem will be proved later after providing some remarks:

Remark 4.

1. *In case $\int_0^\infty \xi_i(t)\, dt = \infty$, Theorem 12 ensures $\lim_{t\to\infty} E(t) = 0$.*
2. *From* **(A2)**, *we infer that $\lim_{t\to\infty} g_i(t) = 0$. Then, there exists some $t_1 \ge 0$ large enough such that*
$$g_i(t_1) = r \Rightarrow g_i(t) \le r, \forall t \ge t_1. \tag{4.3.2}$$
As G_i are positive continuous functions and g_i and ξ_i are positive nonincreasing continuous functions, then, for all $0 \le t \le t_1$,
$$0 < g_i(t_1) \le g_i(t) \le g_i(0) \ \text{ and } 0 < \xi_i(t_1) \le \xi_i(t) \le \xi_i(0),$$
which implies for some positive constants a_i and b_i,
$$a_i \le \xi_i(t)\, G_i(g_i(t)) \le b_i.$$

Consequently,

$$g_i'(t) \leq -\xi_i(t) G_i(g_i(t)) \leq -\frac{a_i}{g_i(0)} g_i(0) \leq -\frac{a_i}{g_i(0)} g_i(t), \text{ for } t \in [0, t_1]. \tag{4.3.3}$$

3. *We also mention Johnson's inequality, which is very important for proving our result. If G is a convex function on $[a,b]$, $g : \Omega \to [a,b]$, we have*

$$G\left[\frac{1}{k}\int_\Omega g(x) h(x) dx\right] \leq \frac{1}{k}\int_\Omega G[g(x)] h(x) dx,$$

where h is a function that satisfies

$$h(x) \geq 0 \text{ and } \int_\Omega h(x) dx = k > 0.$$

To prove the desired result, we create a Lyapunov functional equivalent to E. For this, we define some functions that allow us to construct this Lyapunov function.

As in Baowei [29] and I. Mustafa [53, 54], we define

$$C_{\zeta,i} = \int_0^\infty \frac{g_i^2(s)}{\sqrt{\zeta g_i(s) - g_i'(s)}} \text{ and } h_i(t) = \zeta g_i(t) - g_i'(t), \; i = 1, 2, \tag{4.3.4}$$

for any $0 < \zeta < 1$.

Lemma 14. *Let (u, v, z, y) be a solution of the problem (4.2.12). Then, the functional*

$$\varphi(t) = \int_\Omega u(t) u_t(t) dx + \int_\Omega v(t) v_t(t) dx, \tag{4.3.5}$$

satisfies the estimate

$$\begin{aligned}\varphi'(t) \leq & -\frac{l_1}{2}\|\nabla u(t)\|^2 - \frac{l_2}{2}\|\nabla v(t)\|^2 + \|u_t(t)\|^2 + \|v_t(t)\|^2 \\ & -(\lambda+\mu)\|divu(t)\|^2 - (\lambda+\mu)\|divv(t)\|^2 + \frac{3C_{\zeta,1}}{2l_1}(h_1 \circ \nabla u)(t) + \frac{3k_1^2}{2l_1}\|\nabla u_t\|^2 \\ & + \frac{3k_2^2}{2l_2}\|\nabla v_t\|^2 + \frac{3C_{\zeta,2}}{2l_2}(h_2 \circ \nabla v)(t) + (p+1)\int_\Omega F(u(t), v(t)) dx \\ & + \frac{3k_1}{2l_1}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |\nabla z(x,1,\varrho,t)|^2 d\varrho dx \\ & + \frac{3k_2}{2l_2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |\nabla y(x,1,\varrho,t)|^2 d\varrho dx.\end{aligned} \tag{4.3.6}$$

Proof. Taking the derivative of (4.3.5), we obtain

$$\varphi'(t)=\int_\Omega |u_t(t)|^2\,dx+\int_\Omega u(t)\,u_{tt}(t)\,dx+\int_\Omega |v_t(t)|^2\,dx+\int_\Omega v(t)\,v_{tt}(t)\,dx.$$

From problem (4.2.12) and using integration by parts, we get

$$\begin{aligned}
&\varphi'(t)=\|u_t(t)\|^2+\|v_t(t)\|^2\\
&+\int_\Omega u(t)\left(\Delta_e u-\int_0^t g_1(t-s)\,\Delta u(s)\,ds+k_1\Delta u_t+\int_{\tau_1}^{\tau_2}\mu_1(\varrho)\Delta z(x,1,\varrho,t)d\varrho+f_1(u,v)\right)dx\\
&+\int_\Omega v(t)\left(\Delta_e v-\int_0^t g_2(t-s)\,\Delta v(s)\,ds+k_2\Delta v_t+\int_{\tau_1}^{\tau_2}\mu_2(\varrho)\Delta y(x,1,\varrho,t)d\varrho+f_2(u,v)\right)dx\\
&=\|u_t(t)\|^2+\|v_t(t)\|^2-k_1\int_\Omega \nabla u\nabla u_t dx-k_2\int_\Omega \nabla v\nabla v_t dx\\
&-\mu\|\nabla u(t)\|^2-(\lambda+\mu)\|divu(t)\|^2+\int_\Omega \nabla u(t)\int_0^t g_1(t-s)\,\nabla u(s)\,ds\\
&-\int_\Omega \nabla u(t)\int_{\tau_1}^{\tau_2}\mu_1(\varrho)\nabla z(x,1,\varrho,t)d\varrho+\int_\Omega u(t)\,f_1(u,v)\,dx\\
&-\mu\|\nabla v(t)\|^2-(\lambda+\mu)\|divv(t)\|^2+\int_\Omega \nabla v(t)\int_0^t g_2(t-s)\,\nabla v(s)\,ds\\
&-\int_\Omega \nabla v(t)\int_{\tau_1}^{\tau_2}\mu_2(\varrho)\nabla y(x,1,\varrho,t)d\varrho+\int_\Omega v(t)\,f_2(u,v)\,dx\\
&=\|u_t(t)\|^2+\|v_t(t)\|^2-k_1\int_\Omega \nabla u\nabla u_t dx-k_2\int_\Omega \nabla v\nabla v_t dx\\
&-\left(\mu-\int_0^t g_1(s)\,ds\right)\|\nabla u(t)\|^2-(\lambda+\mu)\|divu(t)\|^2\\
&+\int_\Omega \nabla u(t)\int_0^t g_1(t-s)\,(\nabla u(s)-\nabla u(t))\,dsdx\\
&-\int_\Omega \nabla u(t)\int_{\tau_1}^{\tau_2}\mu_1(\varrho)\nabla z(x,1,\varrho,t)d\varrho dx+\int_\Omega u(t)\,f_1(u,v)\,dx\\
&-\left(\mu-\int_0^t g_2(s)\,ds\right)\|\nabla v(t)\|^2-(\lambda+\mu)\|divv(t)\|^2\\
&+\int_\Omega \nabla v(t)\int_0^t g_2(t-s)\,(\nabla v(s)-\nabla v(t))\,dsdx\\
&-\int_\Omega \nabla v(t)\int_{\tau_1}^{\tau_2}\mu_2(\varrho)\nabla y(x,1,\varrho,t)d\varrho dx+\int_\Omega v(t)\,f_2(u,v)\,dx.
\end{aligned}$$

By using Hölder and young's inequalities, we have

$$\begin{aligned}
&\int_{\Omega}\nabla u(t)\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))\,dsdx\\
&\le \frac{l_1}{6}\|\nabla u(t)\|^2+\frac{3}{2l_1}\int_{\Omega}\int_0^t (g_1(t-s)(\nabla u(s)-\nabla u(t))\,ds)^2\,dx\\
&\le \frac{l_1}{6}\|\nabla u(t)\|^2\\
&+\frac{3}{2l_1}\int_{\Omega}\int_0^t\left(\frac{g_1(t-s)}{\sqrt{\zeta g_1(t-s)-g_1'(t-s)}}\sqrt{\zeta g_1(t-s)-g_1'(t-s)}(\nabla u(s)-\nabla u(t))\,ds\right)^2dx\\
&\le \frac{l_1}{6}\|\nabla u(t)\|^2\\
&+\frac{3}{2l_1}\left(\int_0^t\frac{g_1(s)}{\sqrt{\zeta g_1(s)-g_1'(s)}}ds\right)\int_{\Omega}\int_0^t(\zeta g_1(t-s)-g_1'(t-s))\,|\nabla u(s)-\nabla u(t)|^2\,ds\\
&\le \frac{l_1}{6}\|\nabla u(t)\|^2+\frac{3C_{\zeta,1}}{2l_1}(h_1\circ\nabla u)(t).
\end{aligned} \tag{4.3.7}$$

Similarly, we obtain

$$\int_{\Omega}\nabla v(t)\int_0^t g_2(t-s)(\nabla v(s)-\nabla v(t))\,dsdx\le\frac{l_2}{6}\|\nabla v(t)\|^2+\frac{3C_{\zeta,2}}{2l_2}(h_2\circ\nabla v)(t), \tag{4.3.8}$$

$$\begin{aligned}
&\int_{\Omega}\nabla u(t)\int_{\tau_1}^{\tau_2}\mu_1(\varrho)\nabla z(x,1,\varrho,t)d\varrho dx\\
&\le\frac{l_1}{6}\|\nabla u(t)\|^2+\frac{3}{2l_1}\int_{\Omega}\left(\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\nabla z(x,1,\varrho,t)d\varrho\right)^2dx\\
&\le\frac{l_1}{6}\|\nabla u(t)\|^2+\frac{3}{2l_1}\int_{\Omega}\left(\int_{\tau_1}^{\tau_2}\sqrt{|\mu_1(\varrho)|}\sqrt{|\mu_1(\varrho)|}\nabla z(x,1,\varrho,t)d\varrho\right)^2\\
&\le\frac{l_1}{6}\|\nabla u(t)\|^2+\frac{3}{2l_1}\left(\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|d\varrho\right)\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\,|\nabla z(x,1,\varrho,t)|^2\,d\varrho\\
&\le\frac{l_1}{6}\|\nabla u(t)\|^2+\frac{3k_1}{2l_1}\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\,|\nabla z(x,1,\varrho,t)|^2\,d\varrho,
\end{aligned} \tag{4.3.9}$$

and

$$\begin{aligned}
&\int_{\Omega}\nabla v(t)\int_{\tau_1}^{\tau_2}\mu_2(\varrho)\nabla y(x,1,\varrho,t)d\varrho dx\\
&\le\frac{l_2}{6}\|\nabla v(t)\|^2+\frac{3}{2l_2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\right)\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,|\nabla y(x,1,\varrho,t)|^2\,d\varrho\\
&\le\frac{l_2}{6}\|\nabla v(t)\|^2+\frac{3k_2}{2l_2}\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,|\nabla y(x,1,\varrho,t)|^2\,d\varrho.
\end{aligned} \tag{4.3.10}$$

The Young's inequality gives

$$k_1 \int_\Omega \nabla u(t) \nabla u_t(t)\, dx \leq \frac{l_1}{6} \|\nabla u(t)\|^2 + \frac{3k_1^2}{2l_1} \|\nabla u_t(t)\|^2, \tag{4.3.11}$$

and

$$k_2 \int_\Omega \nabla v(t) \nabla v_t(t)\, dx \leq \frac{l_2}{6} \|\nabla v(t)\|^2 + \frac{3k_2^2}{2l_2} \|\nabla v_t(t)\|^2. \tag{4.3.12}$$

For the source term, we have

$$\int_\Omega u(t) f_1(u,v)\, dx + \int_\Omega v(t) f_2(u,v)\, dx = (p+1) \int_\Omega F(u,v)\, dx. \tag{4.3.13}$$

Combining the equations (4.3.7)-(4.3.13), thus, our proof is completed. □

Lemma 15. *Let (u,v,z,y) be a solution of the problem (4.2.12). Then, the functional*

$$\psi(t) = \int_\Omega u_t(t) \int_0^t g_1(t-s)(u(s)-u(t))\, dsdx + \int_\Omega v_t(t) \int_0^t g_2(t-s)(v(s)-v(t))\, dsdx$$
$$= \psi_1(t) + \psi_2(t),$$

satisfies for any $\delta > 0$ the estimate

$$\begin{aligned}
\psi'(t) \leq{}& (\delta + \delta\Lambda_3 l_1) \|\nabla u(t)\|^2 + \delta \|divu\|^2 + \delta\Lambda_3 l_2 \|\nabla v(t)\|^2 \\
&+ \left(\delta - \int_0^t g_1(s)\, ds\right) \|u_t(t)\|^2 + \frac{c[C_{\zeta,1}+1]}{\delta} (h_1 \circ \nabla u)(t) \\
&+ \delta k_1 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |\nabla z(x,1,\varrho,t)|^2 d\varrho dx + \delta k_1 \|\nabla u_t\|^2 \\
&+ (\delta + \delta\Lambda_4 l_2) \|\nabla v(t)\|^2 + \delta \|divv\|^2 + \delta\Lambda_4 l_1 \|\nabla u(t)\|^2 \\
&+ \left(\delta - \int_0^t g_2(s)\, ds\right) \|v_t(t)\|^2 + \frac{c[C_{\zeta,2}+1]}{\delta} (h_2 \circ \nabla v)(t) \\
&+ \delta k_2 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |\nabla y(x,1,\varrho,t)|^2 d\varrho dx + \delta k_1 \|\nabla v_t\|^2,
\end{aligned} \tag{4.3.14}$$

where Λ_3 and Λ_4 are two positive constants.

Proof. First we begin to estimate $\psi_1'(t)$

$$\begin{aligned}
\psi_1'(t) ={}& \int_\Omega u_{tt}(t) \int_0^t g_1(t-s)(u(s)-u(t))\, dsdx \\
&+ \int_\Omega u_t(t) \int_0^t g_1'(t-s)(u(s)-u(t))\, dsdx - \left(\int_0^t g_1(s)\, ds\right) \|u_t(t)\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega}\left(\Delta_e u-\int_0^t g_1(t-s)\,\Delta u(s)\,ds+k_1\Delta u+\int_{\tau_1}^{\tau_2}\mu_1(\varrho)\Delta z(x,1,\varrho,t)d\varrho+f_1(u,v)\right)\\
&\times\left(\int_0^t g_1(t-s)\,(u(s)-u(t))\,dsdx\right)\\
&+\int_{\Omega}u_t(t)\int_0^t g_1'(t-s)\,(u(s)-u(t))\,dsdx-\left(\int_0^t g_1(s)\,ds\right)\|u_t(t)\|^2\\
&=\int_{\Omega}u_t(t)\int_0^t g_1'(t-s)\,(u(s)-u(t))\,dsdx-\left(\int_0^t g_1(s)\,ds\right)\|u_t(t)\|^2\\
&-\mu\int_{\Omega}\nabla u\int_0^t g_1(t-s)\,(\nabla u(s)-\nabla u(t))\,dsdx\\
&-(\lambda+\mu)\int_{\Omega}divu(t)\int_0^t g_1(t-s)\,(divu(s)-divu(t))\,dsdx\\
&+\left(\int_0^t g_1(t-s)\,\nabla u(s)\,ds\right)\left(\int_0^t g_1(t-s)\,(\nabla u(s)-\nabla u(t))\,dsdx\right)\\
&+\int_{\Omega}\left(\int_{\tau_1}^{\tau_2}\mu_1(\varrho)\nabla z(x,1,\varrho,t)d\varrho\int_0^t g_1(t-s)\,(\nabla u(s)-\nabla u(t))\,ds\right)dx\\
&\qquad\qquad+\int_{\Omega}f_1(u,v)\int_0^t g_1(t-s)\,(u(s)-u(t))\,dsdx\\
&\qquad\qquad-k_1\int_{\Omega}\nabla u_t\int_0^t g_1(t-s)\,(\nabla u(s)-\nabla u(t))\,dsdx.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\psi_1'(t)=&\int_{\Omega}u_t(t)\int_0^t g_1'(t-s)\,(u(s)-u(t))\,dsdx-\left(\int_0^t g_1(s)\,ds\right)\|u_t(t)\|^2\\
&-\left(\mu-\int_0^t g_1(s)\,ds\right)\int_{\Omega}\nabla u\int_0^t g_1(t-s)\,(\nabla u(s)-\nabla u(t))\,dsdx\\
&-(\lambda+\mu)\int_{\Omega}divu(t)\int_0^t g_1(t-s)\,(divu(s)-divu(t))\,dsdx\\
&+\left(\int_{\Omega}\int_0^t g_1(t-s)\,(\nabla u(s)-\nabla u(t))\,dsdx\right)^2\\
&-k_1\int_{\Omega}\nabla u_t\int_0^t g_1(t-s)\,(\nabla u(s)-\nabla u(t))\,dsdx\\
&+\int_{\Omega}\left(\int_{\tau_1}^{\tau_2}\mu_1(\varrho)\nabla z(x,1,\varrho,t)d\varrho.\int_0^t g_1(t-s)\,(\nabla u(s)-\nabla u(t))\,ds\right)dx\\
&+\int_{\Omega}f_1(u,v)\int_0^t g_1(t-s)\,(u(s)-u(t))\,dsdx.
\end{aligned}
$$

As in previous proof and by using Young's inequality, we conclude that for any $\delta>0$,

$$\begin{aligned}&\left(\mu-\int_0^t g_1(s)\,ds\right)\int_\Omega \nabla u\int_0^t g_1(t-s)\left(\nabla u(s)-\nabla u(t)\right)dsdx\\&\leq \delta\left\|\nabla u\right\|^2+\frac{c.C_{\zeta,1}}{\delta}\left(h_1\circ\nabla u\right)(t).\end{aligned}\tag{4.3.15}$$

Similarly and by using the fact $\left\|divu\right\|^2\leq c\left\|\nabla u\right\|^2$, we have

$$\begin{aligned}&(\lambda+\mu)\int_\Omega divu(t)\int_0^t g_1(t-s)\left(divu(s)-divu(t)\right)dsdx\\&\leq \delta\left\|divu\right\|^2+\frac{c.C_{\zeta,1}}{\delta}\left(h_1\circ\nabla u\right)(t).\end{aligned}\tag{4.3.16}$$

The same argument and by using Remark 4, we obtain

$$\begin{aligned}&\int_\Omega f_1(u,v)\int_0^t g_1(t-s)\left(u(s)-u(t)\right)dsdx\\&\leq \delta\int_\Omega\left|f_1(u,v)\right|^2dx+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g_1(t-s)\left(u(s)-u(t)\right)ds\right)^2dx\\&\leq \delta\Lambda_1\left(l_1\left\|\nabla u\right\|^2+l_2\left\|\nabla v\right\|^2\right)^p+\frac{c.C_{\zeta,1}}{\delta}\left(h_1\circ\nabla u\right)(t)\\&\leq \delta\Lambda_1\left[\frac{2(p+1)}{p-1}E(0)\right]^{p-1}\left(l_1\left\|\nabla u\right\|^2+l_2\left\|\nabla v\right\|^2\right)+\frac{c.C_{\zeta,1}}{\delta}\left(h_1\circ\nabla u\right)(t)\\&\leq \delta\Lambda_3\left(l_1\left\|\nabla u\right\|^2+l_2\left\|\nabla v\right\|^2\right)+\frac{c.C_{\zeta,1}}{\delta}\left(h_1\circ\nabla u\right)(t),\end{aligned}\tag{4.3.17}$$

where $\Lambda_3=\left[\frac{2(p+1)}{p-1}E(0)\right]^{p-1}$.

From (4.3.4), we have

$$\begin{aligned}&\int_\Omega u_t(t)\int_0^t g_1'(t-s)\left(u(s)-u(t)\right)dsdx\\&=\int_\Omega u_t(t)\int_0^t h_1(t-s)\left(u(s)-u(t)\right)dsdx-\int_\Omega u_t(t)\int_0^t \zeta g_1(t-s)\left(u(s)-u(t)\right)dsdx\\&\leq \delta\left\|u_t(t)\right\|^2+\frac{c}{\delta}\int_\Omega\left(\int_0^t\sqrt{h_1(t-s)}\sqrt{h_1(t-s)}\left(u(s)-u(t)\right)ds\right)^2dx\\&+\frac{c\zeta^2}{\delta}\int_\Omega\left(\int_0^t g_1(t-s)\left(u(s)-u(t)\right)ds\right)^2dx\\&\leq \delta\left\|u_t(t)\right\|^2+\frac{c}{\delta}\left(\int_0^t h_1(s)\,ds\right)\left(h_1\circ u\right)(t)+\frac{c\zeta^2C_{\zeta,1}}{\delta}\left(h_1\circ u\right)(t)\\&\leq \delta\left\|u_t(t)\right\|^2+\frac{c}{\delta}\left(h_1\circ\nabla u\right)(t)+\frac{c\zeta^2C_{\zeta,1}}{\delta}\left(h_1\circ\nabla u\right)(t)\end{aligned}$$

$$\leq \delta \left\| u_t(t) \right\|^2 + \frac{c\left[C_{\zeta,1}+1\right]}{\delta}\left(h_1 \circ \nabla u\right)(t), \tag{4.3.18}$$

and

$$\begin{aligned}
&\int_\Omega \left(\int_{\tau_1}^{\tau_2} \mu_1(\varrho) \nabla z(x,1,\varrho,t) d\varrho . \int_0^t g_1(t-s)\left(\nabla u(s) - \nabla u(t)\right) ds \right) dx \\
&\leq \delta \int_\Omega \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \nabla z(x,1,\varrho,t) d\varrho \right)^2 dx + \frac{1}{4\delta} \int_\Omega \left(\int_0^t g_1(t-s)\left(\nabla u(s) - \nabla u(t)\right) ds \right)^2 dx \\
&\leq \delta \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \left| \nabla z(x,1,\varrho,t) \right|^2 d\varrho dx + \frac{cC_{\zeta,1}}{\delta}\left(h_1 \circ \nabla u\right)(t) \\
&\leq \delta k_1 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \left| \nabla z(x,1,\varrho,t) \right|^2 d\varrho dx + \frac{cC_{\zeta,1}}{\delta}\left(h_1 \circ \nabla u\right)(t).
\end{aligned} \tag{4.3.19}$$

Finally, Young's inequality gives

$$\begin{aligned}
&k_1 \int_\Omega \nabla u_t \int_0^t g_1(t-s)\left(u(s) - u(t)\right) ds dx \\
&\leq \delta k_1 \left\| \nabla u_t \right\|^2 + \frac{k_1}{4\delta} \int_\Omega \left(\int_0^t g_1(t-s)\left(u(s) - u(t)\right) ds \right)^2 dx \\
&\leq \delta k_1 \left\| \nabla u_t \right\| + \frac{cC_{\zeta,1}}{\delta}\left(h_1 \circ \nabla u\right)(t).
\end{aligned} \tag{4.3.20}$$

Then

$$\begin{aligned}
\psi_1'(t) \leq& \left(\delta - \left(\int_0^t g_1(s) ds \right) \right) \left\| u_t(t) \right\|^2 + \left(\delta + \delta \Lambda_3 l_1\right) \left\| \nabla u \right\|^2 + \delta \left\| divu \right\|^2 \\
&+ \delta \Lambda_3 l_2 \left\| \nabla v \right\|^2 + \frac{c\left[C_{\zeta,1}+1\right]}{\delta}\left(h_1 \circ \nabla u\right)(t) + \delta k_1 \left\| \nabla u_t \right\|^2 \\
&+ \delta k_1 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \left| \nabla z(x,1,\varrho,t) \right|^2 d\varrho dx.
\end{aligned} \tag{4.3.21}$$

The same steps can be taken to get the next estimate for $\psi_2'(t)$,

$$\begin{aligned}
\psi_2'(t) \leq& \left(\delta - \left(\int_0^t g_2(s) ds \right) \right) \left\| v_t(t) \right\|^2 + \left(\delta + \delta \Lambda_4 l_2\right) \left\| \nabla v \right\|^2 + \delta \left\| divv \right\|^2 \\
&+ \delta \Lambda_4 l_1 \left\| \nabla u \right\|^2 + \frac{c\left[C_{\zeta,2}+1\right]}{\delta}\left(h_2 \circ \nabla v\right)(t) + \delta k_2 \left\| \nabla v_t \right\|^2 \\
&+ \delta k_2 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \left| \nabla y(x,1,\varrho,t) \right|^2 d\varrho dx,
\end{aligned} \tag{4.3.22}$$

where $\Lambda_4 = \Lambda_2 \left[\frac{2(p+1)}{p-1} E(0)\right]^{(p-1)}$. □

Lemma 16. *Let (u, v, z, y) be a solution of the problem (4.2.12). Then, the functional*

$$I(t) = \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} \left[|\mu_1(\varrho)| |\nabla z(x,\rho,\varrho,t)|^2 + |\mu_2(\varrho)| |\nabla y(x,\rho,\varrho,t)|^2\right] dx d\varrho d\rho, \tag{4.3.23}$$

satisfies the estimate

$$I'(t) \leq -e^{-\tau_2} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |\nabla z(x,1,\varrho,t)|^2 dx d\varrho - e^{-\tau_2} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |\nabla y(x,1,\varrho,t)|^2 dx d\varrho$$
$$+ k_1 \|\nabla u_t(t)\|^2 + k_2 \|\nabla v_t(t)\|^2 - I(t). \tag{4.3.24}$$

Proof. Differentiating (4.3.23) with respect to t, we get

$$\frac{d}{dt} I(t) =$$
$$2 \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} \left[|\mu_1(\varrho)| \nabla z(x,\rho,\varrho,t) \nabla z_t(x,\rho,\varrho,t) + |\mu_2(\varrho)| \nabla y(x,\rho,\varrho,t) \nabla y_t(x,\rho,\varrho,t)\right] dx d\varrho d\rho.$$

By using (4.2.10)-(4.2.11), we have

$$\begin{aligned} \frac{d}{dt} I(t) &= -\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\varrho\rho} \left[|\mu_1(\varrho)| \frac{\partial}{\partial\rho} |\nabla z(x,\rho,\varrho,t)|^2 + |\mu_2(\varrho)| \frac{\partial}{\partial\rho} |\nabla y(x,\rho,\varrho,t)|^2\right] dx d\varrho d\rho \\ &= -\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \left[|\mu_1(\varrho)| \frac{\partial}{\partial\rho} \left(e^{-\varrho\rho} \nabla z(x,\rho,\varrho,t)^2\right) + |\mu_2(\varrho)| \frac{\partial}{\partial\rho} \left(e^{-\varrho\rho} |\nabla y(x,\rho,\varrho,t)|^2\right)\right] dx d\varrho d\rho \\ &\quad - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} \left[|\mu_1(\varrho)| |\nabla z(x,\rho,\varrho,t)|^2 + |\mu_2(\varrho)| |\nabla y(x,\rho,\varrho,t)|^2\right] dx d\varrho d\rho. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{d}{dt} I(t) = &-\int_\Omega \int_{\tau_1}^{\tau_2} e^{-\varrho} |\mu_1(\varrho)| |\nabla z(x,1,\varrho,t)|^2 dx d\varrho + \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho\right) \|\nabla u_t(x,\varrho,t)\|^2 \\ &- \int_\Omega \int_{\tau_1}^{\tau_2} e^{-\varrho} |\mu_2(\varrho)| |\nabla y(x,1,\varrho,t)|^2 dx d\varrho + \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho\right) \|\nabla v_t(x,\varrho,t)\|^2 \\ &- \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} \left[|\mu_1(\varrho)| |\nabla z(x,\rho,\varrho,t)|^2 + |\mu_2(\varrho)| |\nabla y(x,\rho,\varrho,t)|^2\right] dx d\varrho d\rho. \end{aligned}$$

Since $e^{-\varrho}$ is deceasing function over (τ_1, τ_2), the desired estimate (4.3.24) follows immediately from (4.2.16). □

The following lemmas are needed to prove the general decay when the functions $G_i(t)$ $(i = 1, 2)$

are nonlinear. The proof can be found in Mustafa [53].

Lemma 17. *The functional*

$$\theta_1(t) = \int_\Omega \int_0^t \sigma_1(t-s) |\nabla u(s)|^2 \, ds dx,$$

where $\sigma_1(t) = \int_t^\infty g_1(s)\, ds$, *satisfies*

$$\theta_1'(t) \le -\frac{1}{2} (g_1 \circ \nabla u)(t) + 3(\mu - l_1) \|\nabla u\|^2 . \tag{4.3.25}$$

Lemma 18. *The functional*

$$\theta_2(t) = \int_\Omega \int_0^t \sigma_2(t-s) |\nabla v(s)|^2 \, ds dx,$$

where $\sigma_2(t) = \int_t^\infty g_2(s)\, ds$, *satisfies*

$$\theta_2'(t) \le -\frac{1}{2} (g_2 \circ \nabla v)(t) + 3(\mu - l_2) \|\nabla v\|^2 . \tag{4.3.26}$$

Now, we define the following functional

$$\mathcal{F}(t) = NE(t) + N_1\phi(t) + N_2\psi(t) + I(t), \tag{4.3.27}$$

where N, N_1 and N_2 are positive constants. It is easy to prove $F(t)$ and $E(t)$ are equivalent, namely, there exist two positive constants κ_1 and κ_2 such that

$$\kappa_1 E(t) \le \mathcal{F}(t) \le \kappa_2 E(t). \tag{4.3.28}$$

By Young's inequality, we get

$$\begin{aligned}
\mathcal{F}(t) \le & \left(\frac{N}{2} + \frac{N_1}{2} + \frac{N_2}{2}\right) \left[\|u_t\|^2 + \|v_t\|^2\right] \\
& + \left(\frac{N}{2}\left(\mu - \int_0^t g_1(s)\, ds\right) + c\frac{N_1}{2} + c\frac{N_2}{2}\right) \|\nabla u\|^2 \\
& + \left(\frac{N}{2}\left(\mu - \int_0^t g_2(s)\, ds\right) + c\frac{N_1}{2} + c\frac{N_2}{2}\right) \|\nabla v\|^2 \\
& + \left(\frac{N}{2} + c\frac{N_2}{2}\left(\int_0^t g_1(s)\, ds\right)\right) (g_1 \circ \nabla u)(t) \\
& + \left(\frac{N}{2} + c\frac{N_2}{2}\left(\int_0^t g_2(s)\, ds\right)\right) (g_2 \circ \nabla u)(t)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{N}{2}\eta+C\right)\int_{\Omega}\int_{0}^{t}\int_{\tau_1}^{\tau_2}\varrho\left|\mu_1\left(\varrho\right)\right|\left|\nabla z\left(x,\rho,\varrho,t\right)\right|^2 d\varrho d\rho dx\\
&+\left(\frac{N}{2}\eta+C\right)\int_{\Omega}\int_{0}^{t}\int_{\tau_1}^{\tau_2}\varrho\left|\mu_2\left(\varrho\right)\right|\left|\nabla y\left(x,\rho,\varrho,t\right)\right|^2 d\varrho d\rho dx\\
&+\frac{N}{2}\left(\lambda+\mu\right)\left[\left\|divu\right\|^2+\left\|divv\right\|^2\right]-N\int_{\Omega}F\left(u,v\right)dx.
\end{aligned}
\tag{4.3.29}
$$

Then, for any N there exist $\kappa_1>0$ such that

$$
\mathcal{F}\leq\kappa_1 E\left(t\right).
$$

On the other hand, we can fined

$$
\begin{aligned}
\mathcal{F}\left(t\right)\geq&\left(\frac{N}{2}-\frac{N_1}{2}-\frac{N_2}{2}\right)\left[\left\|u_t\right\|^2+\left\|v_t\right\|^2\right]\\
&+\left(\frac{N}{2}\left(\mu-\int_0^t g_1\left(s\right)ds\right)-c\frac{N_1}{2}-c\frac{N_2}{2}\right)\left\|\nabla u\right\|^2\\
&+\left(\frac{N}{2}\left(\mu-\int_0^t g_2\left(s\right)ds\right)-c\frac{N_1}{2}-c\frac{N_2}{2}\right)\left\|\nabla v\right\|^2\\
&+\left(\frac{N}{2}-c\frac{N_2}{2}\left(\int_0^t g_1\left(s\right)ds\right)\right)\left(g_1\circ\nabla u\right)\left(t\right)\\
&+\left(\frac{N}{2}-c\frac{N_2}{2}\left(\int_0^t g_2\left(s\right)ds\right)\right)\left(g_2\circ\nabla u\right)\left(t\right)\\
&+\left(\frac{N}{2}\eta+c\right)\int_{\Omega}\int_{0}^{t}\int_{\tau_1}^{\tau_2}\varrho\left|\mu_1\left(\varrho\right)\right|\left|\nabla z\left(x,\rho,\varrho,t\right)\right|^2 d\varrho d\rho dx\\
&+\left(\frac{N}{2}\eta+c\right)\int_{\Omega}\int_{0}^{t}\int_{\tau_1}^{\tau_2}\varrho\left|\mu_2\left(\varrho\right)\right|\left|\nabla y\left(x,\rho,\varrho,t\right)\right|^2 d\varrho d\rho dx\\
&+\frac{N}{2}\left(\lambda+\mu\right)\left[\left\|divu\right\|^2+\left\|divv\right\|^2\right]-N\int_{\Omega}F\left(u,v\right)dx.
\end{aligned}
\tag{4.3.30}
$$

We choose N large enough so that

$$
\frac{N}{2}-\frac{N_1}{2}-\frac{N_2}{2}>0,\ \frac{N}{2}\left(\mu-\int_0^t g_i\left(s\right)ds\right)-c\frac{N_1}{2}-c\frac{N_2}{2}>0,
$$

and

$$
\frac{N}{2}-c\frac{N_2}{2}\left(\int_0^t g_i\left(s\right)ds\right)>0,\ i=1,2.
$$

Then, there exist $\kappa_2>0$ such that

$$
\mathcal{F}\left(t\right)\geq\kappa_2 E\left(t\right).
$$

Lemma 19. *The functional $\mathcal{F}(t)$ satisfies for any $t \geq t_1$,*

$$\mathcal{F}'(t) \leq -4(\mu - l_1)\|\nabla u(t)\|^2 - 4(\mu - l_2)\|\nabla v(t)\|^2 - \|u_t(t)\|^2 - \|v_t(t)\|^2$$
$$- \|divu(t)\|^2 - \|divv(t)\|^2 + \frac{1}{4}(g_1 \circ \nabla u)(t) + \frac{1}{4}(g_2 \circ \nabla v)(t) + c\int_\Omega F(u(t), v(t))\,dx$$
$$- \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_1(\varrho)| |\nabla z(x, \rho, \varrho, t)|^2 \,d\varrho d\rho dx - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| |\nabla y(x, \rho, \varrho, t)|^2 \,d\varrho d\rho dx. \tag{4.3.31}$$

Proof. Let

$$g_0 = \min\left\{\int_0^{t_1} g_1(s)\,ds, \int_0^{t_1} g_2(s)\,ds\right\}.$$

From Lemmas 14, 15 and 16, noting that $g_i' = \zeta g_i - h_i$ we have for any $t \geq t_1$,

$$\begin{aligned}
\mathcal{F}'(t) \leq & -\left(\frac{l_1}{2}N_1 - N_2\delta(1 + \Lambda_3 l_1) - N_2\delta\Lambda_4 l_1\right)\|\nabla u(t)\|^2 \\
& - \left(\frac{l_2}{2}N_1 - N_2\delta(1 + \Lambda_4 l_2) - N_2\delta\Lambda_3 l_2\right)\|\nabla v(t)\|^2 \\
& - (g_0 N_2 - \delta N_2 - N_1)\left[\|u_t(t)\|^2 + \|v_t(t)\|^2\right] \\
& + \frac{\zeta N}{2}\left[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] + N_1(p+1)\int_\Omega F(u(t), v(t))\,dx \\
& - \left[\frac{N}{2} - N_2\frac{c[C_{\zeta,1} + 1]}{\delta} - \frac{3N_1 C_{\zeta,1}}{2l_1}\right](h_1 \circ \nabla u)(t) \\
& - \left[\frac{N}{2} - N_2\frac{c[C_{\zeta,2} + 1]}{\delta} - \frac{3N_1 C_{\zeta,2}}{2l_2}\right](h_2 \circ \nabla v)(t) \\
& - [(\lambda + \mu)N_1 - \delta N_2]\left[\|divu(t)\|^2 + \|divv(t)\|^2\right] \\
& - \left[N\sigma + e^{-\tau_2} - N_1\frac{3}{2l_1}k_1 - \delta N_2 k_1\right]\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |\nabla z(x, 1, \varrho, t)|^2 \,d\varrho dx \\
& - \left[N\sigma + e^{-\tau_2} - N_1\frac{3}{2l_2}k_2 - \delta N_2 k_2\right]\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |\nabla y(x, 1, \varrho, t)|^2 \,d\varrho dx \\
& - \left[N\sigma_1 - N_1\frac{3k_1^2}{2l_1} - \delta N_2 k_1 - k_1\right]\|\nabla u_t(t)\|^2 \\
& - \left[N\sigma_2 - N_1\frac{3k_2^2}{2l_2} - \delta N_2 k_2 - k_2\right]\|\nabla v_t(t)\|^2 - I(t),
\end{aligned}$$

where

$$\sigma_1 = \left[k_1 - \left(\frac{\eta + 1}{2}\right)\left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)|\,d\varrho\right)\right], \; \sigma_2 = \left[k_2 - \left(\frac{\eta + 1}{2}\right)\left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\,d\varrho\right)\right],$$

$$\sigma = \left(\frac{\eta - 1}{2}\right).$$

Taking $\delta = \frac{1}{2N_2}$, we can get

$$\begin{aligned}
\mathcal{F}'(t) \leq & -\left(\frac{l_1}{2}N_1 - \frac{1}{2}(1+\Lambda_3 l_1) - \frac{1}{2}\Lambda_4 l_1\right)\|\nabla u(t)\|^2 \\
& -\left(\frac{l_2}{2}N_1 - \frac{1}{2}(1+\Lambda_4 l_2) - \frac{1}{2}\Lambda_3 l_2\right)\|\nabla v(t)\|^2 \\
& -\left(g_0 N_2 - \frac{1}{2} - N_1\right)\left[\|u_t(t)\|^2 + \|v_t(t)\|^2\right] \\
& + \frac{\zeta N}{2}\left[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] + N_1(p+1)\int_\Omega F(u(t), v(t))\,dx \\
& - \left[\frac{N}{2} - 2cN_2^2 - C_{\zeta,1}\left(2cN_2^2 + \frac{3N_1}{2l_1}\right)\right](h_1 \circ \nabla u)(t) \\
& - \left[\frac{N}{2} - 2cN_2^2 - C_{\zeta,2}\left(2cN_2^2 + \frac{3N_1}{2l_2}\right)\right](h_2 \circ \nabla v)(t) \\
& - \left[(\lambda+\mu)N_1 - \frac{1}{2}\right]\left[\|divu(t)\|^2 + \|divv(t)\|^2\right] \\
& - \left[N\sigma + e^{-\tau_2} - N_1\frac{3}{2l_1}k_1 - \frac{k_1}{2}\right]\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\,|\nabla z(x,1,\varrho,t)|^2\,d\varrho dx \\
& - \left[N\sigma + e^{-\tau_2} - N_1\frac{3}{2l_2}k_2 - \frac{k_2}{2}\right]\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,|\nabla y(x,1,\varrho,t)|^2\,d\varrho dx \\
& - \left[N\sigma_1 - N_1\frac{3k_1^2}{2l_1} - \frac{3k_1}{2}\right]\|\nabla u_t(t)\|^2 \\
& - \left[N\sigma_2 - N_1\frac{3k_2^2}{2l_2} - \frac{3k_2}{2}\right]\|\nabla v_t(t)\|^2 - I(t).
\end{aligned}$$

First, we take $N_1 > 0$ large such that

$$(\lambda+\mu)N_1 - \frac{1}{2} > 0,\ \left(\frac{l_1}{2}N_1 - \frac{1}{2}(1+\Lambda_3 l_1) - \frac{1}{2}\Lambda_4 l_1\right) > 4(\mu - l_1),$$

and

$$\left(\frac{l_2}{2}N_1 - \frac{1}{2}(1+\Lambda_3 l_2) - \frac{1}{2}\Lambda_4 l_2\right) > 4(\mu - l_2).$$

We choose $N_2 > 0$ large enough so that

$$g_0 N_2 - \frac{1}{2} - N_1 > 1.$$

Note that

$$0 < \frac{\zeta g_i^2(s)}{\zeta g_i(s) - g_i'(s)} < \frac{\zeta g_i^2(s)}{-g_i'(s)},\ i = 1, 2.$$

Then, for any $s \in [0, \infty)$, we get

$$\lim_{\zeta \to 0} \frac{\zeta g_i^2(s)}{\zeta g_i(s) - g_i\prime(s)} = 0,\ i = 1, 2.$$

By using the fact $\frac{\zeta g_i^2(s)}{\zeta g_i(s)-g_i'(s)} < g_i(s)$, $i = 1, 2$ and using Lebesgue-dominated convergence theorem, we can get

$$\lim_{\zeta \to 0} \zeta C_{\zeta,i} = \lim_{\zeta \to 0} \int_0^\infty \frac{\zeta g_i^2(s)}{\zeta g_i(s) - g_i'(s)} = 0,\ i = 1, 2.$$

Thus, there exist some ζ_0 $(0 < \zeta_0 < 1)$ such that if $\zeta < \zeta_0$, then

$$\zeta C_{\zeta,2} < \frac{1}{8\left[\frac{N_1}{2l_1} + 2cN_2^2\right]} \ \textit{and}\ \zeta C_{\zeta,2} < \frac{1}{8\left[\frac{N_1}{2l_2} + 2cN_2^2\right]}.$$

At last, we choose N large enough and choose ζ satisfying

$$\frac{1}{4}N - 2cN_2^2 > 0 \ \textit{and}\ \zeta = \frac{1}{2N} > \zeta_0,$$

so, we arrive at

$$\left[\frac{N}{2} - 2cN_2^2 - C_{\zeta,1}\left(2cN_2^2 + \frac{3N_1}{2l_1}\right)\right] > 0 \ \textit{and}\ \left[\frac{N}{2} - 2cN_2^2 - C_{\zeta,1}\left(2cN_2^2 + \frac{3N_1}{2l_1}\right)\right] > 0.$$

Therefor, we choose N even larger (if needed) so that

$$\left[Nn_1 + e^{-\tau_2} - N_1\frac{3}{2l_1}k_1 - \frac{k_1}{2}\right] > 0, \left[Nn_1 + e^{-\tau_2} - N_1\frac{3}{2l_2}k_2 - \frac{k_2}{2}\right] > 0,$$

$$\left[Nm - N_1\frac{3k_1^2}{2l_1} - \frac{3k_1}{2}\right] > 0 \,\textit{and}\, \left[Nm - N_1\frac{3k_2^2}{2l_2} - \frac{3k_2}{2}\right] > 0.$$

Thus, (4.3.31) is established. □

Proof of Theorem 12. Taking into account (4.3.3) and (4.2.17), we obtain that for any $t \geq t_1$,

$$\begin{aligned} &\int_0^{t_1} g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \\ &\leq -\frac{g_1(0)}{a_1} \int_0^{t_1} g_1'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \leq -cE'(t), \end{aligned} \tag{4.3.32}$$

and

$$\begin{aligned} &\int_0^{t_1} g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 \, dxds \\ &\leq -\frac{g_2(0)}{a_2} \int_0^{t_1} g_2'(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 \, dxds \leq -cE'(t). \end{aligned} \tag{4.3.33}$$

Noting (4.3.31), we shall see that there exists a constant $m > 0$ such that for all $t \geq t_1$,

$$\begin{aligned} \mathcal{F}'(t) \leq -mE(t) - cE' + c\int_{t_1}^{t} g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \\ + c\int_{t_1}^{t} g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 \, dxds. \end{aligned} \tag{4.3.34}$$

Denote $\mathcal{L}(t) = \mathcal{F}(t) + cE(t)$. It is obvious that $\mathcal{L}(t)$ is equivalent to $E(t)$. It follows from (4.3.34) that

$$\begin{aligned} \mathcal{L}'(t) \leq -mE(t) + c\int_{t_1}^{t} g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \\ + c\int_{t_1}^{t} g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 \, dxds. \end{aligned} \tag{4.3.35}$$

We consider two cas

Cas 1. $G(t)$ is linear: By multiplying (4.3.35) by $\xi(t)$ and using **(A2)** and (4.2.17), we obtain

$$\begin{aligned} \xi(t)\mathcal{L}'(t) &\leq -m\xi(t)E(t) + c\xi(t)\int_{t_1}^{t} g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \\ &+ c\xi(t)\int_{t_1}^{t} g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 \, dxds \\ &\leq -m\xi(t)E(t) + c\int_{t_1}^{t} \xi_1(t) g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \\ &+ c\int_{t_1}^{t} \xi_2(t) g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 \, dxds \\ &\leq -m\xi(t)E(t) - c\int_{t_1}^{t} g_1'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \end{aligned}$$

$$\begin{aligned} & - c\int_{t_1}^{t} g_2'(s)\int_{\Omega}|\nabla v(t)-\nabla v(t-s)|^2\,dxds \\ & \leq -m\xi(t)E(t) - cE'(t), \end{aligned} \tag{4.3.36}$$

which gives, as $\xi(t)$ is nonincreasing,

$$[\xi(t)\mathcal{L}(t)+cE(t)]' \leq \xi(t)\mathcal{L}'(t)+cE'(t) \leq -m\xi(t)E(t),\ \forall t\geq t_1. \tag{4.3.37}$$

Denote $K(t)=\xi(t)\mathcal{L}(t)+cE(t)$. we get

$$K'(t)\leq -m\xi(t)E(t).$$

Hence, using the fact that $K(t)\sim E(t)$ we easily obtain

$$E(t)\leq c_1 exp\left(-c_2\int_{t_1}^{t}\xi(s)\,ds\right).$$

Cas 2. $G(t)$ is nonlinear: First, we use Lemmas 17 and 18 to deduce that

$$J(t)=\mathcal{F}(t)+\theta_1(t)+\theta_2(t)$$

is nonnegative and it satisfies for some positive constant k and for any $t\geq t_1$,

$$\begin{aligned} J'(t) \leq & -(\mu-l_1)\|\nabla u\|^2-(\mu-l_2)\|\nabla v\|^2-\|u_t\|^2-\|v_t\|^2 \\ & -\|divu\|^2-\|divv\|^2-\frac{1}{4}(g_1\circ\nabla u)-\frac{1}{4}(g_2\circ\nabla v)+c\int_{\Omega}F(u,v)\,dx \\ & -\int_{\Omega}\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_1(\varrho)||\nabla z(x,\rho,\varrho,t)|^2\,d\varrho d\rho dx \\ & -\int_{\Omega}\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)||\nabla y(x,\rho,\varrho,t)|^2\,d\varrho d\rho dx \\ & \leq -kE(t)\leq 0. \end{aligned} \tag{4.3.38}$$

Therefore,

$$k\int_{t_1}^{t}E(s)\,ds\leq J(t_1)-J(t)\leq J(t_1),$$

this implies that

$$\int_0^{\infty}E(s)\,ds<\infty. \tag{4.3.39}$$

Now, we define $I_i(t)$ by

$$I_1(t) := q\int_{t_1}^{t}\int_{\Omega}|\nabla u(t)-\nabla u(t-s)|^2\,dxds$$

and

$$I_2(t) := q\int_{t_1}^{t}\int_{\Omega}|\nabla v(t)-\nabla v(t-s)|^2\,dxds,$$

where (4.3.39) allows for a constant $0<q<1$ chosen so that, for all $t\geq t_1$

$$I_i(t)<1,\;\; i=1,2. \tag{4.3.40}$$

We also assume, without loss of generality that $I_i(t)>0$ for all $t\geq t_1$; otherwise (4.3.35) yields an exponential decay. Also, we define $\lambda_1(t)$ and $\lambda_2(t)$ by

$$\lambda_1(t) := -\int_{t_1}^{t} g_1'(s)\int_{\Omega}|\nabla u(t)-\nabla u(t-s)|^2\,dxds,$$

$$\lambda_2(t) := -\int_{t_1}^{t} g_2'(s)\int_{\Omega}|\nabla v(t)-\nabla v(t-s)|^2\,dxds.$$

We observe that $\lambda_i(t)\leq -cE'(t)\,,\;\; i=1,2$. Noting that $G_i(t)$ is strictly convex on $(0,r]$ and $G_i(0)=0$, then

$$G_i(\nu x)\leq \nu G_i(x)\,,\;\; i=1,2,$$

provided $0\leq\nu\leq 1$ and $x\in(0,r]$. By using (**A2**), (4.3.40) and Jensen's inequality, we can obtain

$$\begin{aligned}
\lambda_1(t) &= \frac{1}{qI_1(t)}\int_{t_1}^{t} I_1(t)\left(-g_1'(s)\right)\int_{\Omega} q\,|\nabla u(t)-\nabla u(t-s)|^2\,dxds\\
&\geq \frac{1}{qI_1(t)}\int_{t_1}^{t} I_1(t)\,\xi_1(s)\,G_1(g_1(s))\int_{\Omega} q\,|\nabla u(t)-\nabla u(t-s)|^2\,dxds\\
&\geq \frac{\xi_1(t)}{qI_1(t)}\int_{t_1}^{t} G_1(I_1(t)\,g_1(s))\int_{\Omega} q\,|\nabla u(t)-\nabla u(t-s)|^2\,dxds\\
&\geq \frac{\xi_1(t)}{q}G_1\left(\frac{1}{I_1(t)}\int_{t_1}^{t} I_1(t)\,g_1(s)\int_{\Omega} q\,|\nabla u(t)-\nabla u(t-s)|^2\,dxds\right)\\
&= \frac{\xi_1(t)}{q}G_1\left(q\int_{t_1}^{t} g_1(s)\int_{\Omega}|\nabla u(t)-\nabla u(t-s)|^2\,dxds\right)\\
&= \frac{\xi_1(t)}{q}\overline{G_1}\left(q\int_{t_1}^{t} g_1(s)\int_{\Omega}|\nabla u(t)-\nabla u(t-s)|^2\,dxds\right),
\end{aligned} \tag{4.3.41}$$

where $\overline{G_1}$ is an extension of G_1 such that $\overline{G_1}$ is strictly increasing and strictly convex C^2

function on $(0,+\infty)$, see [53]. We have from (4.3.41)

$$\int_{t_1}^{t} g_1(s)\int_{\Omega}|\nabla u(t)-\nabla u(t-s)|^2\,dxds \leq \frac{1}{q}\overline{G_1}^{-1}\left(\frac{q\lambda_1(t)}{\xi_1(t)}\right). \tag{4.3.42}$$

Similarly, we have

$$\int_{t_1}^{t} g_2(s)\int_{\Omega}|\nabla v(t)-\nabla v(t-s)|^2\,dxds \leq \frac{1}{q}\overline{G_2}^{-1}\left(\frac{q\lambda_2(t)}{\xi_2(t)}\right). \tag{4.3.43}$$

We infer from (4.3.35), (4.3.42) and (4.3.43) that for any $t \geq t_1$

$$\mathcal{L}'(t) \leq -mE(t) + c\overline{G_1}^{-1}\left(\frac{q\lambda_1(t)}{\xi_1(t)}\right) + c\overline{G_2}^{-1}\left(\frac{q\lambda_2(t)}{\xi_2(t)}\right). \tag{4.3.44}$$

Let us denote

$$G_0(t) = \min\left\{\overline{G_1}^{-1}, \overline{G_2}^{-1}\right\}.$$

For $\varepsilon_0 < r$, the functional $\mathcal{K}_1(t)$ defined by:

$$\mathcal{K}_1(t) = G_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{L}(t) + E(t)$$

is equivalent to E, and by using the fact that $E' \leq 0$, $\overline{G_i}' > 0$, and $\overline{G_i}'' > 0$, we infer from (4.3.44) that

$$\begin{aligned}
\mathcal{K}_1'(t) &= \varepsilon_0\frac{E'(t)}{E(0)}G_0'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{L}(t) + G_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{L}'(t) + E'(t)\\
&\leq -mE(t)G_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + cG_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\overline{G_1}^{-1}\left(\frac{q\lambda_1(t)}{\xi_1(t)}\right)\\
&+ cG_0\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\overline{G_2}^{-1}\left(\frac{q\lambda_2(t)}{\xi_2(t)}\right).
\end{aligned} \tag{4.3.45}$$

Let $\overline{G}^*$ be the convex conjugate of $\overline{G}$ in the sense of Young (see Arnold [7]), then

$$\overline{G_i}^*(s) = s\left(\overline{G_i}'\right)^{-1}(s) - \overline{G_i}\left[\left(\overline{G_i}'\right)^{-1}(s)\right],\ i = 1,2, \tag{4.3.46}$$

and $\overline{G}^*$ satisfies the following Young's inequality

$$MD_i \leq \overline{G_i}^*(M) + \overline{G_i}(D_i),\ i = 1,2.$$

With $M = G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$ and $D_i = \overline{G_i}^{-1}\left(\frac{q\lambda_i(t)}{\xi_i(t)}\right)$ and noting $\overline{G_i}^{*}(t) \le t\left(\overline{G_i}'\right)^{-1}(t)$ and (4.3.45), we conclude

$$\begin{aligned}
\mathcal{K}_1'(t) &\le -mE(t)\, G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\overline{G_1}^{*}\left(G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) + c\left(\frac{q\lambda_1(t)}{\xi_1(t)}\right) \\
&+ c\overline{G_2}^{*}\left(G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) + c\left(\frac{q\lambda_2(t)}{\xi_2(t)}\right) \\
&\le -mE(t)\, G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cG_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\left(\overline{G_1}'\right)^{-1}\left(G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) \\
&+ c\left(\frac{q\lambda_1(t)}{\xi_1(t)}\right) + cG_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\left(\overline{G_2}'\right)^{-1}\left(G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) + c\left(\frac{q\lambda_2(t)}{\xi_2(t)}\right) \\
&\le -mE(t)\, G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cG_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\left(\overline{G_1}'\right)^{-1}\left(\overline{G_1}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) \\
&+ c\left(\frac{q\lambda_1(t)}{\xi_1(t)}\right) + cG_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\left(\overline{G_2}'\right)^{-1}\left(\overline{G_2}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) + c\left(\frac{q\lambda_2(t)}{\xi_2(t)}\right) \\
&\le -\left(mE(0) - c\varepsilon_0\right)\frac{E(t)}{E(0)} G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cq\left(\frac{\lambda_1(t)}{\xi_1(t)} + \frac{\lambda_2(t)}{\xi_2(t)}\right).
\end{aligned} \tag{4.3.47}$$

Multiplying by $\xi(t)$, we get

$$\begin{aligned}
\xi(t)\,\mathcal{K}_1'(t) &\le -\left(mE(0) - c\varepsilon_0\right)\xi(t)\frac{E(t)}{E(0)} G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cq\left(\lambda_1(t) + \lambda_2(t)\right) \\
&\le -\left(mE(0) - c\varepsilon_0\right)\xi(t)\frac{E(t)}{E(0)} G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t).
\end{aligned} \tag{4.3.48}$$

Consequently, with $\mathcal{K}_2(t) = \xi(t)\,\mathcal{K}_1(t) + cE(t)$, which satisfies, for some $\beta_1, \beta_2 > 0$,

$$\beta_1 \mathcal{K}_2(t) \le E(t) \le \beta_2 \mathcal{K}_2(t). \tag{4.3.49}$$

Choosing a suitable ε_0, we can get from (4.3.48) that there exists a constant $k > 0$,

$$\mathcal{K}_2'(t) \le -k\xi(t)\frac{E(t)}{E(0)} G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) := -k\xi(t)\, G_3\left(\varepsilon_0 \frac{E(t)}{E(0)}\right), \tag{4.3.50}$$

where $G_3(t) = tG_0(\varepsilon_0 t)$.

From $0 \le \varepsilon_0 \frac{E(t)}{E(0)} < r$, we conclude that for any $t > 0$,

$$\begin{aligned}
G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) &= \min\left\{\overline{G_1}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right), \overline{G_2}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right\} \\
&= \min\left\{G_1'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right), G_2'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right\}.
\end{aligned}$$

Denote $R(t) = \frac{\beta_1 \mathcal{K}_2(t)}{E(0)}$. Using (4.3.49), then

$$R(t) \sim E(t). \tag{4.3.51}$$

Since $G_3'(t) = G_0(\varepsilon_0 t) + \varepsilon_0 t G_0'(\varepsilon_0 t)$, then, using the strct convexity of G_0 on $(0, r]$, we arrive at $G_3'(t), G_3(t) > 0$ on $(0, 1]$. We infer From (4.3.50) that there exists a constant $b_1 > 0$ such that for all $t \le t_1$,

$$R'(t) \le -b_1 \xi(t) G_3(R(t)). \tag{4.3.52}$$

Then, by integration over (t_1, t), we have

$$\begin{aligned} \int_{t_1}^{t} \frac{-R'(s)}{G_3(R(s))} ds \ge b_1 \int_{t_1}^{t} \xi(s)\, ds \implies \int_{\varepsilon_0 R(t)}^{\varepsilon_0 R(t_1)} \frac{1}{sG_0(s)} ds \ge b_1 \int_{t_1}^{t} \xi(s)\, ds \\ \implies R(t) \le \frac{1}{\varepsilon_0} G_4^{-1}\left(b_1 \int_{t_1}^{t} \xi(s)\, ds \right), \end{aligned} \tag{4.3.53}$$

where $G_4(t) = \int_t^r \frac{1}{sG_0(s)} ds$ is strictly decreasing on $(0, r]$ and $\lim_{t \to 0} G_4(t) = +\infty$. A combination of (4.3.51) and (4.3.53), estimate (4.3.1) is established.

Conclusion and Prospects

In this thesis, we have studied the effect of the presence of delay term on the existence and uniqueness of the solution, as well as rate of decay of the energy of some problems that involve the wave equation. In the second chapter we have studied the wave equation with strong delay and distributed delay. We proved the well-posedness and an exponential decay result under suitable assumptions on the weight of the damping and the weight of the delay. In the third chapter, we have proved the global existence and an exponential decay rate for the Kirchhoff's coupled system with a distributed delay term.

The results we obtained encourage us to expand our study of the effect of delay term on the existence of solution and stability of solutions to include a broader class of other physical problems.

In the fourth chapter, we are proved a general energy decay of a coupled Lamé system with distributed time delay. This result is an extension of what Baowei Feng obtained in [29]. In order to complete this work, we will later study the existence and uniqueness of a local and global or blow-up of solution in the next work.

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Abstract

The thesis aims to provide the reader with how to use the most popular method for studying the existence and uniqueness of the solution and the general energy decay of some wave problems with strong delay and distributed delay, similar to the Kirchhoff system and the Lamé system. The first chapter deals with introducing some basic notions in bounded and unbounded operators and some main theorems in functional analysis. In second chapter, we proved the well-posedness and an exponential decay result under a suitable assumptions on the weight of the damping and the weight of the delay for a wave equation with a strong damping and a strong constant (respectively, distributed) delay. Finally "third and forth chapters", we proved the global existence of Kirchhoff's coupled system and general decay for the coupled system of Kirchhoff and system of Lamé with a distributed delay term.

Résumé

Le but de cette thèse est de fournir au lecteur comment utiliser la méthode la plus populaire pour étudier l'existence et l'unicité de la solution et la décroissance générale d'énergie de certains problèmes d'ondes à fort retard et retard distribué, similaire au système de Kirchhoff et de Lamé. Dans le premier chapitre, nous avons introduit quelques notions de base sur les opérateurs bornés et non bornés et quelques théorèmes principaux en analyse fonctionnelle. Dans le deuxième chapitre, nous avons prouvé l'existence et l'unicité avec un résultat de décroissance exponentielle sous des hypothèses appropriées sur le poids de l'amortissement et le poids du retard pour une équation d'onde avec un fort amortissement et un fort retard constant (respectivement, distribué). Finalement "troisième et quatrième chapitres", nous avons prouvé l'existence globale du système couplé de Kirchhoff et la décroissance générale pour le système couplé de Kirchhoff et de Lamé avec un terme retard distribué.

الخلاصة

الهدف من هذه الأطروحة هو تزويد القارئ بكيفية إستخدام الطريقة الأكثر شيوعًا لدراسة وجود ووحدانية الحل والانحلال العام لبعض مشاكل الموجة مع تأخير شديد وتأخر موزع ، على غرار نظام كيرشوف ونظام لامي. في الفصل الأول، قدمنا بعض المفاهيم الأساسية في المؤثرات المحدودة وغير المحدودة وبعض النظريات الرئيسية في التحليل الدالي. في الفصل الثاني، أثبتنا وجود ووحدانية الحل و نتيجة الإستقرار الأسي في ظل وجود فرضيات مناسبة على وزن التخميد ووزن التأخير لمعادلة موجية مع تخميد قوي و تأخير قوي ثابت (على التوالي ، موزع). في الفصلين الثالث والرابع ، أثبتنا الوجود العام والانحلال العام للنظام المزدوج لكيرشوف ولامي مع تأخير موزع.