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Par

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Titre

Application du calcul de Malliavin aux problèmes de contrôle singulier

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Dédicace

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Contents

Introduction

We consider optimal mixed stochastic regular-singular control problems, where the state process satisfies the following stochastic differential equation:

$$\begin{cases} dx_t = b\left(t, x_t, u_t, \omega\right) dt + \sigma\left(t, x_t, u_t, \omega\right) dB_t + \lambda\left(t, \omega\right) d\xi_t; \\ x\left(0\right) = x \in \mathbb{R}. \end{cases} \tag{1}$$

The control is a pair (u_t, ξ_t) such that u_t stands for the regular, called also the absolutely continuous part and ξ_t is the singular part.

The expected cost has the form

$$J\left(u, \xi\right) = E\left[g\left(x_T, \omega\right) + \int_0^T f\left(t, x_t, u_t, \omega\right) dt + \int_0^T h\left(t, \omega\right) d\xi_t\right], \quad (u, \xi) \in \mathcal{A}_{\mathcal{E}}. \tag{2}$$

A major approach to deal with stochastic control problems is to derive optimality necessary conditions satisfied by some optimal control, known as the stochastic maximum principle. The first fundamental result on this subject was obtained by Kushner [44], for classical regular or absolutely continuous controls. Since then, a huge literature has been produced on this subject, among them, in particular, those by Benssoussan [10], Bismut [16], Haussmann [40] and Peng [53]. One can refer to the excellent book by Yong and Zhou [53] for a complete account on the subject and the references therein.

We use Malliavin calculus techniques [49], to express the adjoint process in an explicit form. Our result extends those by Baghery and Oksendal [2], Meyer-Brandis & Øksendal.

[47] and Øksendal & Sulem [51], to mixed regular-singular control problems. See also [48] for the mean field control problems. Note that in the stochastic maximum principle, a serious drawback is the computation at least numerically of the adjoint process. This process is given by a conditional expectation and satisfies a linear backward stochastic differential equation (BSDE). Numerical and Monte Carlo methods have been developed recently to deal with BSDEs by using Malliavin calculus, see [19], [20], [24] and [35]. This could be seen as a step forward to solve numerically stochastic control problems by using these methods.

Stochastic control problems of singular type, have been studied extensively in the literature, as they model numerous situations in different areas, see [46], [50] and [51]. A typical example in mathematical finance is the so called portfolio optimization problem, under transaction costs [28] and [37]. These problems were studied through dynamic programming principle, see [41], where it was shown in particular that, the value function is continuous and is the unique viscosity solution of the HJB variational inequality. In particular the value function satisfies a variational inequality, which gives rise to a free boundary problem, and the optimal state process is a diffusion reflected at the free boundary. Bather and Chernoff [8] were the first to study such a problem. Beněs, Shepp and Witsenhaussen [14] solved a one dimensional example by observing that the value function in their example satisfies the so called the principle of smooth fit. Davis and Norman [28] solved the two dimensional problem, arising in portfolio selection models, under transaction costs. The case of diffusions with jumps has been studied in Øksendal and Sulem [50].

The first maximum principle for singular stochastic control problems was derived by Cadenillas and Haussmann [23], for systems with linear dynamics, convex cost criterion and convex state constraints. An extension to non linear systems has been developed via convex perturbations method for both absolutely continuous and singular components by Bahlali and Chala [3]. The second order stochastic maximum principle for nonlinear SDEs with a controlled diffusion matrix was obtained by Bahlali and Mezerdi [7], extending the Peng maximum principle [53], [1] to singular control problems. Similar techniques have been used by Anderson in [1] and Bahlali et al. [6], to study the stochastic maximum principle for relaxed-singular controls. The case of systems with non smooth coefficients has been treated by Bahlali et al. in [4], where the classical derivatives are replaced by the generalized ones in the definition of adjoint processes. See also the recent paper by Oksendal and Sulem [51], where Malliavin calculus techniques have been used to define the adjoint process. The relationship between the stochastic maximum principle and dynamic programming has been investigated in [5], [25]. See also [50] for some worked examples.

Let us briefly describe the contents of this thesis. In Chapter 1 we give some background on optimal control theory. In chapter 2, we present the maximum principle in singular control in which the control domain need not be convex, the control variable has two components, the first being absolutely continuous and the second singular. The coefficients of the state equation are non linear and depend explicitly on the absolutely continuous component of the control. This result was established by Seid Bahlali and Brahim Mezerdi [7]. In chapter 3 we give an introduction of Malliavin derivative, we use an approach based on chaos expansions, this approach has the advantage of being more intuitive. which turns out to be a useful framework for both Malliavin calculus, Skorohod integrals, and anticipative calculus in general. In chapter 4 we present the tow papers Brandis, Øksendal and Zhou [47] and Øksendal and Sulem [51]. Chapter 5, comprises the main result of this thesis, in this chapter we study general regular-singular stochastic control problems, in which the controller has only partial information. The control has two components, the first one is a classical regular control and the second one is a singular control. We consider systems driven by random coefficients and the running and the final costs are allowed to be random. It is clear that for such systems the dynamic programming does not hold, as the state process is no longer a Markov process. Our goal is to obtain necessary conditions for optimality satisfied by some optimal control.

Chapter 1

Introduction to stochastic controle problems

1.1 Introduction

In this chapter we give some background on optimal control theory. Optimal control theory can be described as the study of strategies to optimally influence a system x with dynamics evolving over time according to a differential equation. The influence on the system is modeled as a vector of parameters, u, called the control. It is allowed to take values in some set U, which is known as the action space. For a control to be optimal, it should minimize a cost functional (or maximize a reward functional), which depends on the whole trajectory of the system x and the control u over some time interval $[0, T]$. The infimum of the cost functional is known as the value function (as a function of the initial time and state). This minimization problem is infinite dimensional, since we are minimizing a functional over the space of functions $u(t)$, $t \in [0, T]$. Optimal control theory essentially consists of different methods of reducing the problem to a less transparent, but more manageable problem. The two main methods are dynamic programming and the maximum principle.

This chapter will be organized as follows. In section 2, we present strong and weak formulations of stochastic optimal control problems and the existence of stochastic optimal controls for both strong and weak formulation. Section 3 presents some others stochastic control problems. Section 4 and 5 is concerned to the presentation of the two important methods which are dynamic programming and the maximum principle.

1.2 Formulations of stochastic optimal controle problems

We now present two mathematical formulations (strong and weak formulations) of stochastic optimal control problems.

1.2.1 Strong formulation

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ be a given filtered probability space satisfying the usual condition, on witch we define an m-dimensional standard Brownian motion $W(.)$, consider the following controlled stochastic differential equation :

$$\begin{cases} dx(t) = b(t, x(t), u(t))\,dt + \sigma(t, x(t), u(t))\,dW_t, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where

$$b : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n,$$
$$\sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m},$$

U is a separable metric space, and $T \in (0, \infty)$ is fixed.

The function $u(.)$ is called the control representing the action of the decision-makers (controllers). At any time instant the controller has some information (as specified by the information filed $\{\mathcal{F}_t\}_{t\geq 0}$) of what has happened up to that moment, but not able to foretell what is going to happen afterwards due to the uncertainty of the system (as a consequence, for any t the controller cannot exercise his/her decision $u(t)$ befor the time t really comes).witche can be expressed in mathematical term as " $u(.)$ is $(\mathcal{F}_t)_{t\geq 0}$ −adapted". the control u is an element of the set :

$$U[0,T] = \left\{u : [0,T] \times \Omega \to U \quad / \ u(.) \text{ is } \{\mathcal{F}_t\}_{t\geq 0}\text{-adapted}\right\}. \tag{1.2}$$

We introduce the cost functional as follows :

$$J(u(.)) = E\left[\int_0^T f(t, x(t), u(t))\, dt + h(x(T))\right]. \tag{1.3}$$

Definition 1.2.1 *Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ be given satisfying the usual conditions and let $W(t)$ be a given m-dimensional standard $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion. A control $u(.)$ is called an s-admissible control, and $(x(.), u(.))$ an s-admissible pair, if*

i) $u(.) \in [0,T]$,

ii) *$x(.)$ is the unique solution of equation* (1.1),

iii) *$f(., x(.), u(.)) \in L^1_{\mathcal{F}}(0,T;\mathbb{R})$ and $h(x(T)) \in L^1_{\mathcal{F}_T}(\Omega;\mathbb{R})$.*

The set of all admissible controls is denoted by $\mathcal{U}^s_{ad}[0,T]$. We can now give the stochastic control problem under strong formulation as follow :

Problem 1.2.1 *Minimize* (1.3) *over* $\mathcal{U}^s_{ad}[0,T]$.

The objective is to find $\hat{u}(.) \in \mathcal{U}^s_{ad}[0,T]$ (if it exists), such that

$$J(\hat{u}) = \inf_{u(.)\in\mathcal{U}^s_{ad}[0,T]} J(u). \tag{1.4}$$

Any $\hat{u}(.) \in \mathcal{U}^s_{ad}[0,T]$ satisfying (1.4) is called an optimal control. The corresponding state process $\hat{x}(.)$.

1.2.2 Weak formulation

We remarque that in the strong formulation the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ on witch we define the Brownian motion W are all fixed, but it is not the cas in the weak formulation, where we consider them as a parts of the control.

Definition 1.2.2 *We call* $\pi = \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P, W(.), u(.)\right)$ *an* w*-admissible control, and* $(x(.), u(.))$ *an* w*-admissible pair, if*

i) $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ *is a filtered probability space satisfying the usual conditions;*

ii) $W(.)$ *is an* m*-dimensional standard Brownian motion defined on* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$;

iii) $u(.)$ *is an* $\{\mathcal{F}_t\}_{t\geq 0}$*-adapted process on* $(\Omega, \mathcal{F}, P)$ *taking values in* U;

iv) $x(.)$ *is the unique solution of equation* (1.1)*;*

v) $f(., x(.), u(.)) \in L^1_{\mathcal{F}}(0, T; \mathbb{R})$ *and* $h(x(T)) \in L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$.

The set of all admissible controls is denoted by $\mathcal{U}^w_{ad}[0, T]$, from now on if there is no ambiguity you can write $u(.) \in \mathcal{U}^w_{ad}[0, T]$ instead of $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P, W(.), u(.)\right) \in \mathcal{U}^w_{ad}[0, T]$. Our stochastic optimal control problem under weak formulation can be formulated as follows:

Problem 1.2.2 *Minimize* (1.3) *over* $\mathcal{U}^w_{ad}[0, T]$.

The objective is to find $\hat{\pi} \in \mathcal{U}^w_{ad}[0, T]$ such

$$J(\hat{\pi}) = \inf_{\pi \in \mathcal{U}^w_{ad}[0,T]} J(\pi). \tag{1.5}$$

1.2.3 Existence of optimal controls

In this section, we will discuss the existence of optimal controls, we use the theory that a lower semi-continuous function on a compact metric space reaches its minimum. we will first give an example from deterministic control.

Example 1.2.1 *Consider the cost functional*

$$J(u) = \int_0^T \left(x^2(t) + \left(1 - u^2(t)\right)^2 \right) dt,$$

to be minimized over the set of controls $u : [0,T] \mapsto U = [-1,1]$. *The state of the system is given by*

$$\begin{cases} dx_t = u(t)\, dt, \\ x(0) = 0. \end{cases}$$

Now, consider the following sequence of controls

$$u_n(t) = (-1)^k \quad \textit{if} \quad t \in \left[\tfrac{k}{n}, \tfrac{k+1}{n}\right), \ 0 \leq k \leq n-1.$$

Then we have $\left|x^{(n)}(t)\right| \leq \dfrac{1}{n}$ *and* $\left|J\left(u^{(n)}\right)\right| \leq \dfrac{T}{n^2}$, *which implies that* $\inf_u J(u) = 0$. *The limit of* $u^{(n)}$ *is however not in the space of strict*

controls. Instead the sequence $\delta_{u^{(n)}(t)}(du)dt$ *converges weakly to* $1/2(\delta_{-1} + \delta_1)(du)dt$. *Thus, there does not exist an optimal strict*

control in this case but only a relaxed one. But since we can construct a sequence of strict controls such that the cost functional

is arbitrarily close to its infimum, it is clear that there does exist an optimal solution, albeit in a wider sense.

Existence under strong formulation

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ be given and W is one dimensional Brownian motion. Consider the following linear controlled system :

$$\begin{cases} dx(t) = [Ax(t) + Bu(t)]\, dt + [Cx(t) + Du(t)]\, dW(t), \quad t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{1.6}$$

where A,B,C,D are matrices. The state $x(.)$ takes value in $\mathbb{R}^n$, and the control $u(.)$ is in

$$\mathcal{U}^L[0,T]=\left\{u(.)\in L^2_{\mathcal{F}}\left(0,T,\mathbb{R}^k\right)\ /\ u(t)\in U,\ \text{a.e.}t\in[0,T],\ P\text{-}a.s.\right\}, \tag{1.7}$$

$U\subset\mathbb{R}^k$, The cost functional is

$$J(u(.))=E\left[\int_0^T f(x(t),u(t))\,dt+h(x(T))\right], \tag{1.8}$$

with $f:\mathbb{R}^n\times U\to\mathbb{R}$ and $h:\mathbb{R}^n\to\mathbb{R}$.

We have the following assumptions

($\mathbf{H_1}$) $U\subset\mathbb{R}^k$ is convex and closed, and the functions f and h are convex and for some $\delta,k>0$,

$$f(x,u)\geq\delta|u|^2-k,\quad h(x)\geq-k\quad,\ \forall(x,u)\in\mathbb{R}^n\times U. \tag{1.9}$$

($\mathbf{H_2}$) $U\subset\mathbb{R}^k$ is convex and compact, and the functions f and h are convex.

the optimal control problem is as follows :

Problem 1.2.3 *Minimize* (1.8) *subject to* (1.6) *over* $\mathcal{U}^L[0,T]$.

We can now give the theorem of existence of optimal control in the linear case.

Theorem 1.2.1 *Under either* (H_1) *and* (H_2)*, if the problem is finite, then it admits an optimal control*

Proof. If we suppose that (H_1) holds. if we put

$$\alpha=\inf_{u\in\mathcal{U}^L[0,T]}J(u),$$

which is equivalent to

$$\forall\varepsilon>0,\exists u_\varepsilon\in\mathcal{U}^L[0,T]:\alpha\leq J(u_\varepsilon)<\alpha+\varepsilon,$$

which implies by putting $\varepsilon = \dfrac{1}{j}$ that :

$$\forall j \geq 1, \exists u_j \in \mathcal{U}^L [0, T] : \alpha \leq J(u_j) < \alpha + \frac{1}{j}, \tag{1.10}$$

we see that

$$\lim_{j \to \infty} J(u_j) = \alpha,$$

we call $(u_j(.), x_j(.))$ a minimizing sequence. we have then by (1.9) and (1.10)

$$E\left[\int_0^T |u_j(t)|^2 dt\right] \leq K, \forall j \geq 1, \tag{1.11}$$

for some constant $K > 0$. Then there exist a subsequence, witch is still noted by $u_\varepsilon(.)$, such that

$$u_j(.) \to \hat{u}, \text{ weakly in } L^2\left(0, T; \mathbb{R}^k\right). \tag{1.12}$$

By Mazur's theorem, we have a sequence of convex combinations

$$\tilde{u}_j(.) = \sum_{i \geq 1} a_{ij} u_{i+j}(.), \quad \text{with} \quad a_{ij} \geq 0, \ \sum_{i \geq 1} a_{ij} = 1,$$

such that

$$\tilde{u}_j(.) \to \hat{u}(.) \text{ strongly in } C_{\mathcal{F}}([0, T], \mathbb{R}^n). \tag{1.13}$$

we have $\hat{u}(.) \in \mathcal{U}^L[0, T]$ because U is convex and closed, we have also

$$\tilde{x}_j(.) \to \hat{x}(.), \quad \text{strongly in} \quad C_{\mathcal{F}}([0, T], \mathbb{R}^n).$$

It is clear that $(\hat{u}(.), \hat{x}(.))$ is admissible, by the convexity of f and h, we have

$$J(\hat{u}(.)) = \lim_{j \to \infty} J(\tilde{u}_j(.)) = \lim_{j \to \infty} \sum_{i \geq 1} a_{ij} J(u_{i+j}(.)) = \alpha, \tag{1.14}$$

hence $(\hat{x}(.), \hat{u}(.))$ is optimal. ■

Remark 1.2.1 *In the case where* (H_2) *holds, we obtain directely* (1.11). *The linearity play an essential role here, in general case, we do not have the convergence of* $u_j(.)$ *and* $x_j(.)$ *because the infinitedimensional space* $L^2_{\mathcal{F}}(\Omega; \mathbb{R}^n)$ *isn't localy compacte.*

Existence under weak formulation

We will now examine the existence of optimal control under weak formulation. Let consider the following hypotheses.

(S1) (U, d) is a compact metric space and $T > 0$.

(S2) The maps b, σ, f, and h are all continuous, and there exists a constant $L > 0$ such that for $\varphi(t, x, u) = b(t, x, u)$, $\sigma(t, x, u)$, $f(t, x, u)$, $h(x)$,

$$\begin{cases} |\varphi(t, x, u) - \varphi(t, \hat{x}, u)| \leq L|x - \hat{x}|, \\ |\varphi(t, 0, u)| \leq L, \forall t \in [0, T], \quad x, \hat{x} \in \mathbb{R}^n, \ u \in U. \end{cases}$$

(S3) For every $(t, x) \in [0, T] \times \mathbb{R}^n$, the set

$$S = \left\{ \left(b_i(t, x, u), \ \left(\sigma\sigma^T\right)^{ij}(t, x, u), \ f(t, x, u) \right) \ / \ u \in U, \ i = 1, ..., n, \ j = 1, ..., m \right\},$$

is convex in $\mathbb{R}^{n+nm+1}$.

(S3) $x(t) \in \mathbb{R}^n$.

Theorem 1.2.2 *Under the conditions* $(S1) - (S3)$*, if the problem is finite, then it admits an optimal control.*

The idea of the proof is to embed the space of admissible control in a large space with proper compactness which is the space of all (non-negative) measures on $[0, T] \times U$. For more details you can see the proof in [57] page 71.

1.3 Other stochastic control problem

1.3.1 Random horizon

In problem formulation (1.2.1), the time horizon is fixed, until a deterministic terminal time T. In some real applications, the time horizon may be random, then the control problem is to minimize :

$$J(u) = E\left[\int_0^\tau f(t, x(t), u(t))\, dt + h(x(\tau))\right], \tag{1.15}$$

over admissible control, her τ is a finite random time. In standard cases, the terminal time τ is a stopping time at which the state process exits from a certain relevant domain. For example, in a reinsurance model, the state process X is the reserve of a company that may control it by reinsuring a proportion $1-\alpha$ of premiums to another company. The terminal time τ is then the bankruptcy time of the company defined as

$$\tau = \inf\{t \geq 0 : X_t \leq 0\}.$$

More generally, given some open set Θ of $\mathbb{R}^n$,

$$\tau = \inf\{t \geq 0 : X_t \notin \Theta\} \wedge T.$$

(which depends on the control). In this case, the control problem (1.15) leads via the dynamic programming approach to a Dirichlet boundary-value problem. The problem (1.15) may be reduced to a stochastic control problem under a fixed deterministic horizon, see [5], for a recent application in portfolio optimization model. In the general random time case, the associated control problem has been relatively lightly studied in the literature, see [6] or [7] for a utility maximization problem in finance.

1.3.2 Optimal stopping

In the models presented above, the horizon of the problem is either fixed or indirectly influenced by the control. When one has the possibility to control directly the terminal time, which is then modeled by a controlled stopping time, the associated problem is an optimal stopping time problem. In the general formulation of such models, the control is mixed, composed by a pair control/stopping time (u, τ) and the functional to optimize is

$$J(u(.), \tau) = E\left[\int_0^\tau f(t, x(t), u(t))\, dt + h(x(\tau))\right]. \tag{1.16}$$

The theory of optimal stopping, thoroughly studied in the seventies, has received a renewed interest with a variety of applications in economics and finance. These applications range from asset pricing (American options) to firm investment and real options. Extensions of classical optimal stopping problems deal with multiple optimal stopping with eventual changes of regimes in the state process. They were studied e.g. in [12], [56], and applied in finance in [21], [29], [38], [22] or [52].

Example 1.3.1 *A person who owns an asset (house, stock, etc...) decides to sell. The price of the asset evolves as:*

$$dX_t = rX_t dt + \sigma X_t dB_t.$$

Suppose that there is a transaction cost $a > 0$. If the person decides to sell at date t, the profit of this transaction will be

$$e^{-\rho t}(X_t - a),$$

where $\rho > 0$ is the inflation factor. The problem is to find a stopping time which maximizes the expected benefit.

1.3.3 Partial observation control problem

It is assumed that the controller completely observes the state system. In many real applications, he is only able to observe partially the state via other variables and there is noise in the observation system. For example in financial models, one may observe the asset price but not completely its rate of return and/or its volatility, and the portfolio investment is based only on the asset price information. We are facing a partial observation control problem. This may be formulated in a general form as follows : we have a controlled signal (unobserved) process governed by :

$$dX_s = b\left(s, X_s, Y_s, u_s\right) ds + \sigma\left(s, X_s, Y_s, u_s\right) dW_s, \tag{1.17}$$

and an observation process :

$$dY_s = \eta\left(s, X_s, Y_s, u_s\right) ds + \gamma\left(s, X_s, Y_s, u_s\right) dB_s, \tag{1.18}$$

where B is another Brownian motion, eventually correlated with W. The control is adapted with respect to the filtration generated by the observation $F^Y = \left(\mathcal{F}_t^Y\right)$ and the functional to optimize is :

$$J\left(u\left(.\right)\right) = E\left[\int_0^T f\left(X\left(t\right), Y\left(t\right), u\left(t\right)\right) dt + h\left(X\left(T\right), Y\left(T\right)\right)\right]. \tag{1.19}$$

By introducing the filter measure-valued process

$$\pi_t\left(dx\right) = P\left[X\left(t\right) \in dx \mid \mathcal{F}_t^Y\right],$$

one may rewrite the functional $J(u)$ in the form :

$$J\left(u\left(.\right)\right) = E\left[\int_0^T \hat{f}\left(\pi_t, Y\left(t\right), u\left(t\right)\right) dt + \hat{h}\left(\pi_T, Y\left(T\right)\right)\right], \tag{1.20}$$

where we use the notation : $\hat{f}(\pi, y) = \int f(x, y)\,\pi(dx)$ for any finite measure π on the signal state space, and similarly for $\hat{h}$. Since by definition, the process (π_t) is $\mathcal{F}_t^Y$-adapted, the original partial observation control problem is reformulated as a complete observation control model, with the new observable state variable defined by the filter process. The additional main difficulty is that the filter process is valued in the infinite-dimensional space of probability measures: it satisfies the Zakai stochastic partial differential equation. The dynamic programming principle or maximum principle are still applicable and the associated Bellman equation or Hamiltonian system are now in infinite dimension. For a theoretical study of optimal control under partial observation under this infinite dimensional viewpoint, we mention among others the works [28], [27], [9], [11], [45] or [58].There are relatively few explicit calculations in the applications to finance of partial observation control models and this area should be developed in the future.

1.3.4 Singular and impulse control

In formulation of the problem in (1.1), the displacement of the state changes continuously in time in response to the control effort. However, in many real applications, this displacement may be discontinuous. For example, in insurance company models, the company distributes the dividends once or twice a year rather than continuously. In transaction costs models, the agent should not invest continuously in the stock due to the costs but only at discrete times A similar situation occurs in a liquidity risk model, see e.g. [24]. Let us first introduce the following function space: $D \equiv D([0,T], \mathbb{R}^n)$ the set of all functions $\xi : [0,T] \to \mathbb{R}^n$ that are right continuous with left limits (càdlàg for short). We define the total variation of ξ on $[0,T]$ by

$$\int_0^T |d\xi(s)| \equiv |\xi|_{[0,T]} = \sum_{i=1}^{i=n} \left|\xi^i\right|_{[0,T]},$$

where $|\xi^i|_{[0,T]}$ is the total variation of the ith component of ξ on $[0,T]$ in the usual sense. We define $|\xi|_t \equiv |\xi|_{[0,T]}$, $t > 0$, for simplicity. For $\xi \in D$, we define

$$\triangle\xi(s) := \xi(s) - \xi(s-),$$

and

$$S_\xi := \{ s \in [0,T] / \ \triangle \xi(s) \neq 0 \}.$$

Further, we define

$$BV_{\mathcal{F}}([0,T], \mathbb{R}^n) = \left\{ \xi \in D/ \ |\xi|_T < \infty, \xi \text{ is } (\mathcal{F}_t)_{t\geq 0}\text{-adapted} \right\}. \tag{1.21}$$

For any $\xi \in BV([0,T], \mathbb{R}^n)$ we define the pure jump part of ξ by $\xi^{jp}(t) := \sum_{0\leq s<t} \triangle\xi(s)$, and its Lebegue decomposition is :

$$\xi(.,\omega) = \xi^{ac}(.,\omega) + \xi^{sc}(.,\omega) + \xi^{jp}(.,\omega),$$

where

$$\begin{cases} \xi^{ac}(.,\omega) : \text{absolutely continuous part,} \\ \xi^{sc}(.,\omega) : \ \text{singular continuous part,} \\ \xi^{jp}(.,\omega) : \ \text{jump part,} \end{cases}$$

and

$$\begin{aligned} \xi^c(t) &= \xi(t) - \xi^{jp}(t) \\ &= \xi^{ac}(t) - \xi^{sc}(t), \end{aligned}$$

where

$$\xi^{ac}(t) := \int_0^t \dot{\xi}^c(s)\, ds.$$

The controlled state diffusion process is governed by

$$dX_s = b(s, X_s)dt + \sigma(s, X_s)dWs + d\xi_s,$$

the functional objective to optimize is in the form

$$J(\xi(.)) = E\left\{\int_0^T f(t, x(t))\,dt + \int_0^T f^a(t)\left\|\dot{\xi}^{ac}(t)\right\|_1 dt + \int_0^T f^s(t)\left|d\xi^{sc}(t)\right| dt + \sum_{t\in S_\xi[0,T]} l(t, \Delta\xi(t)) + h(x(T))\right\}. \quad (1.22)$$

Here, f, f^s, l and h are given functions, and $\left\|\dot{\xi}^{ac}(t)\right\|_1$ and $|d\xi^{sc}(t)|$ are the measures generated by the total variations of ξ^{ac} and ξ^{sc}. The optimal singular control problem is to minimize the cost functional (1.22) over $BV([0,T], \mathbb{R}^n)$.

1.3.5 Ergodic control

Some stochastic systems may exhibit over a long period a stationary behavior characterized by an invariant measure. This measure, if it does exists, is obtained by the average of the states over a long time. An ergodic control problem consists in optimizing over the long term some criterion taking into account this invariant measure. A standard formulation resulting from the criterion presented is to optimize over control u functional of the form

$$\lim \sup_{T\to+\infty} \frac{1}{T} E\left[\int_0^T f(X_t, u_t)\,dt\right], \quad (1.23)$$

or

$$\lim \sup_{T\to+\infty} \frac{1}{T} \ln E\left[\exp\left(\int_0^T f(X_t, u_t)\,dt\right)\right]. \quad (1.24)$$

This last formulation is called risk-sensitive control on an infinite horizon. Ergodic and risk-sensitive control problems were studied in [32], [13], or [32]. Risk-sensitive control problems were recently applied in a financial context in [15] and [33]

1.3.6 Stochastic target problems

Motivated by the super replication problem in finance, and in particular under gamma constraints [54], Soner and Touzi introduced a new class of stochastic control problems. The state process is described by a pair (X,Y) valued in $\mathbb{R}^n\times\mathbb{R}$, and controlled by a control process according to :

$$dX_s = b\left(s, X_s, u_s\right) ds + \sigma\left(s, X_s, u_s\right) dW_s, \tag{1.25}$$

$$dY_s = \eta\left(s, X_s, Y_s, u_s\right) + \gamma\left(s, X_s, Y_s, u_s\right) dW_s. \tag{1.26}$$

Given $(t,x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}$, $(X^{t,x},Y^{t,x,y})$ is the unique solution to of (1.25)-(1.26) with initial condition $(X_t^{t,x},Y_t^{t,x,y})=(x,y)$. The coefficients b, σ, η, γ are bounded functions and satisfy usual conditions ensuring that $(X^{t,x},Y^{t,x,y})$ is well-defined. The stochastic target problem is defined as follows. Given a real-valued measurable function g on $\mathbb{R}^n$, the value function of the control problem is defined by :

$$v\left(t,x\right)=\inf\left\{y\in\mathbb{R},\ \exists u\in\mathcal{U},\ Y_T^{t,x,y}\geq h\left(X_T^{t,x}\right)\ \text{a.s.}\right\}.$$

In finance, X is the price process, Y is the wealth process controlled by the portfolio strategy , and $v(t,x)$ is the minimum capital in order to super replicate the payoff option $h(X_T)$. The derivation of the associated dynamic programming equation is obtained in [55].

1.4 Dynamic programming

Let $T > 0$ be given and let U be a metric space. For any $(s,x) \in [0,T) \times \mathbb{R}^n$, consider the state equation

$$\begin{cases} dx(t) = b(t, x(t), u(t))\, dt + \sigma(t, x(t), u(t))\, dW_t, \ t \in [s,T], \\ x(0) = x, \end{cases} \tag{1.27}$$

along with the cost functional

$$J(s,x,u(.)) = E\left[\int_0^T f(x(t), u(t))\, dt + h(x(T))\right]. \tag{1.28}$$

Dynamic programming equation.

Fixes $s \in [0,T)$, we denote by $\mathcal{U}[s,T]$ the set of all 5-tuples $(\Omega, \mathcal{F}, P, W(.), u(.))$ satisfying the following :

a) $(\Omega, \mathcal{F}, P)$ is complete probability space.

b) $\{W_t\}_{t \geq s}$ is an m-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ over $[s,T]$ (with $W(s) = 0$ $a.s.$), and $\mathcal{F}_t^s = \sigma\{W_r : s \leq r \leq t\}$ augmented by all P-null sets in $\mathcal{F}$.

c) $u : [s,T] \times \Omega \to \mathcal{U}$ is an $\{\mathcal{F}_t^s\}_{t \geq s}$-adapted process on $(\Omega, \mathcal{F}, P)$.

d) under $u(.)$, for any $x \in \mathbb{R}^n$ equation (1.27) admits a unique solution $f(., x(.), u(.)) \in L^1_{\mathcal{F}}(0,T;\mathbb{R})$ and $h(x(T)) \in L^1_{\mathcal{F}_T}(\Omega;\mathbb{R})$ are defined on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}_{t \geq s}, P\right)$.

we define the value function

$$v(s,x) = \inf_{u(.) \in \mathcal{U}[s,T]} J(s,x,u(.)), \ (s,x) \in [0,T] \times \mathbb{R}^n.$$

We now introduce some assumptions

S_1') $(\mathcal{U}, d)$ is polish space and $T > 0$.

S_2') b, σ, f and h are uniformly continuous, and there exists a constant $L > 0$ such that for : $\varphi(t,x,u) = b(t,x,u)\,,\ \sigma(t,x,u)\,,\ f(t,x,u)\,, h(x)\,,$

$$\begin{cases} |\varphi(t,x,u) - \varphi(t,\hat{x},u)| \leq L\,|x - \hat{x}|\,, \\ |\varphi(t,0,u)| \leq L, \forall t \in [0,T]\,, \quad x, \hat{x} \in \mathbb{R}^n,\ u \in U. \end{cases}$$

Our optimal control problem can be stated as follows :
Find $\left(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \hat{W}(.)\,, \hat{u}(.)\right) \in \mathcal{U}[s,T]$ such that

$$J(s,x,\hat{u}(.)) = \inf_{u \in \mathcal{U}[s,T]} J(s,x,u(.))\,. \tag{1.29}$$

The dynamic programming principle says that if a trajectory is optimal each time, then starting from another point one can do no better than follow the optimal trajectory.

Theorem 1.4.1 *Let* (S_1')-(S_2') *hold. then for any* $(s,x) \in [0,T) \times \mathbb{R}^n$,

$$v(s,x) = \inf_{u(.) \in \mathcal{U}[s,T]} E\left\{ \int_s^{\hat{s}} f(t, x(t,s,x,u(.)\,, u(t)))\,dt + v(\hat{s}, x(\hat{s},s,x,u(.))) \right\},\ \forall 0 \leq s \leq \hat{s} \leq T. \tag{1.30}$$

We call (1.30) the dynamic programming equation. This equation is very complicated, and it seems impossible to solve such an equation directly.

1.4.1 Hamilton-Jacobi-Belman equation

Based on equation (1.30). we let $C^{1,2}([0,T] \times \mathbb{R}^n)$ be the set of all continuous functions $v : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ such that v_t, v_x and v_{xx} are all continuous in (t,x)

Proposition 1.4.1 (The HJB equation) *Suppose* $(S_1') - (S_2')$ *hold and the value function* $v \in C^{1,2}([0,T] \times \mathbb{R}^n)$. *Then* v *is a solution of the following terminal value problem* *of a (possibly degenerate) second-order partial differential equation:*

$$\begin{cases} -v_t + \sup_{u \in U} G\left(t, x, u, -v_{xx}\right) = 0, & (t, x) \in [0, T] \times \mathbb{R}^n \\ v\backslash_{t=T} = h\left(x\right), \quad x \in \mathbb{R}^n, \end{cases}$$

where

$$\begin{aligned} G\left(t, x, u, p, P\right) &:= \frac{1}{2} tr\left(P\sigma\left(t, x, u\right)\sigma\left(t, x, u\right)^T\right) \\ &+ \langle p, b\left(t, x, u\right)\rangle - f\left(t, x, u\right), \\ &\forall \left(t, x, u, p, P\right) \in [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times S^n. \end{aligned}$$

1.4.2 The classical verification approach

The classical verification approach consists in finding a smooth solution to the HJB equation, and to check that this candidate, under suitable sufficient conditions, coincides with the value function. This result is usually called a verification theorem and provides as a byproduct an optimal control. It relies mainly on Itô's formula. The assertions of a verification theorem may slightly vary from problem to problem, depending on the required sufficient technical conditions. The verification theorem is stated as follows :

Theorem 1.4.2 *Let v be a $C^{1,2}$ function on $[0, T) \times R^n$ and continuous in T, with suitable growth condition. Suppose that for all $(t, x) \in [0, T) \times R^n$, there exists $\hat{\alpha}(t, x)$ measurable, valued in $A = \mathbb{R}_+$ such that v solves the HJB equation :*

$$\begin{aligned} 0 &= w_t\left(t, x\right) - \sup_{a \in A}\left[\mathcal{L}^a w\left(t, x\right) + f\left(t, x, a\right)\right] \\ &= w_t\left(t, x\right) - \mathcal{L}^{\hat{\alpha}(t,x)} w\left(t, x\right) - f\left(t, x, \hat{\alpha}(t, x)\right), \\ &\qquad on\ [0, T) \times R^n \end{aligned}$$

together with the terminal condition $w\left(T, .\right) = g$ on $\mathbb{R}^n$ and the S.D.E. :

$$dX_s = b\left(s, X_s, \hat{\alpha}(t, x)\right) dt + \sigma\left(s, X_s, \hat{\alpha}(t, x)\right) dW_s$$

admits a unique solution, denoted $\hat{X}_s^{t,x}$, given an initial condition $X_t = x$. Then, $w = v$ and $\left\{\hat{\alpha}(s, \hat{X}_s^{t,x})\ t \le s \le T\right\}$ is an optimal control for $v(t,x)$.

A proof of this verification theorem may be found in any textbook on stochastic control, see e.g. [57], [34] or [43].

Example 1.4.1 (Merton's portfolio selection problem) *This is the situation where an investor may decide at any time over a finite horizon T to invest a proportion valued in $A = \mathbb{R}$ of his wealth X in a risky stock of constants rate of return μ and volatility σ and the rest of proportion $1 - \alpha$ in a bank account of constant interest r. His wealth controlled process is then governed by :*

$$dX_s = X_s \left(r + (\mu - r)\, \alpha_s\right) ds + X_s \sigma \alpha_s dW_s,$$

and the objective of the investor is given by the value function :

$$v\left(t,x\right) = \sup_{\alpha \in A} E\left[U\left(X_T^{t,x}\right)\right], \quad (t,x) \in [0,T] \times \mathbb{R}_+,$$

where U is a utility function, i.e. a concave and increasing function on $\mathbb{R}_+$. For the popular specific choice of the power utility function $U(x) = x^p$, with $p < 1$, it is possible to find an explicit (smooth) solution to the associated HJB equation with the terminal condition $v(T,.) = U$, namely:

$$v\left(t,x\right) = \exp\left(\rho\left(T-t\right)\right) x^p,$$

with $\rho = \dfrac{\left(\mu - r\right)^2}{2\sigma^2} \dfrac{p}{1-p} + rp$. Moreover, the optimal control is constant and given by :

$$\hat{\alpha} = \frac{\mu - r}{\sigma^2\left(1-p\right)}.$$

The key point in the explicit resolution of the Merton problem is that the value function v may be separated into a function of t and of x : $v\left(t,x\right) = \varphi\left(t\right) x^p$. With this transformation *and substitution into the HJB equation, it turns out that* φ*' is the solution of an ordinary differential equation with terminal condition* $\varphi(T) = 1$*, which is explicitly solved.*

For other applications of verification theorems to stochastic control problems in finance see [50].

1.5 The Pontriagin stochastic maximum principle

A classical approach for optimization and control problems is to derive necessary conditions satisfied by an optimal solution. The argument is to use an appropriate calculus of variations on the gain function $J(t, x, .)$ with respect to the control variable in order to derive a necessary condition of optimality. The maximum principle, initiated by Pontryagin in the 1960s, states that an optimal state trajectory must solve a Hamilton system together with a maximum condition of a function called a generalized Hamilton. In principle, solve a Hamilton should be easier than solving the original control problem. The original version of Pontryagin's maximum principle was derived for deterministic problems. As in classical calculus of variation, the basic idea of is to perturb an optimal control and to use some sort of Taylor expansion of the state trajectory and objective functional around the optimal control. By sending the perturbation to zero, one obtains some inequality, and by duality, the maximum principle is expressed in terms of an adjoint variable (Lagrange multiplier in the finite-dimensional case).

The stochastic control case was extensively studied in the 1970s by Bismut, Kushner, Bensoussan or Haussmann. However, at that time, the results were essentially obtained under the condition that there is no control on the diffusion coefficient. For example, Haussmann investigated maximum principle by Girsanov's transformation and this limitation explains why this approach does not work with control-dependent and degenerate diffusion coefficients.

The main difficulty when facing a general controlled diffusion is that the Itô integral term is not of the same order as the Lebesgue term and thus the first-order variation method fails. This difficulty was overcome by Peng , who studied the second-order term in the Taylor expansion of the perturbation method arising from the Itô integral. He then obtained a maximum principle for possibly degenerate and control-dependent diffusion, which involves in addition to the first-order adjoint variable, a second-order adjoint variable. In order to make applicable the maximum principle, one needs some explicit description of the adjoint variables. These variables obtained originally by duality in functional analysis may be represented by Riesz representation of a certain functional. By completing with martingale representation in stochastic analysis, the adjoint variables are then described by what is called today backward stochastic differential equations (BSDE).

1.5.1 Deterministic control problem

We provide a sketch of how the maximum principle for a deterministic control problem is derived. In this setting, the state of the system is given by the differential equation

$$\begin{cases} dx\left(t\right) = b\left(x\left(t\right), u\left(t\right)\right) dt, \\ x\left(0\right) = 0, \end{cases} \tag{1.31}$$

where $u(t) \in U$ for all $t \in [0, T]$, and the action space U is some subset of $\mathbb{R}$. The objective is to minimize some cost function

$$J\left(u\right) = \int_0^T h\left(x\left(t\right), u\left(t\right)\right) dt + g\left(x_T\right). \tag{1.32}$$

That is, the function h inflicts a running cost and the function g inflicts a terminal cost. We now assume that there exists a control $\hat{u}(t)$ which is optimal, i.e.

$$J\left(\hat{u}\right) = \inf_{u \in U} J\left(u\right). \tag{1.33}$$

We denote by $\hat{x}(t)$ the solution to (1.31) with the optimal control $\hat{u}(t)$. We are going to derive necessary conditions for optimality by analyzing what happens when we make a small perturbation of the optimal control. Therefore we introduce a so called spike variation, i.e. a control which is equal to $\hat{u}$ except on some small time interval :

$$u^{\theta}(t) = \begin{cases} v & \text{for} \qquad \tau - \theta \leq t \leq \tau, \\ \hat{u}(t) & \text{otherwise.} \end{cases}$$

We denote by $x^{\theta}(t)$ the solution to (1.31) with the control $u^{\theta}(t)$. We see that $x^{\theta}(t)$ and $\hat{x}(t)$ are equal up to $t = \tau - \theta$ and that

$$\begin{aligned} x^{\theta}(\tau) - \hat{x}(\tau) &= \int_{\tau-\theta}^{\tau} \left(b\left(x^{\theta}(r), v(r)\right) - b\left(\hat{x}(r), \hat{u}(r)\right)\right) dr \\ &= \left(b\left(x^{\theta}(\tau), v(\tau)\right) - b\left(\hat{x}(\tau), \hat{u}(\tau)\right)\right)\theta + o(\theta) \\ &= \left(b\left(\hat{x}_{\tau}, v_{\tau}\right) - b\left(\hat{x}_{\tau}, \hat{u}_{\tau}\right)\right)\theta + o(\theta), \end{aligned} \tag{1.34}$$

where the third equality holds since $x^{\theta}(\tau) - \hat{x}(\tau)$ is of order θ. Next, we look at the Taylor expansion of the state with respect to θ. Let

$$z(t) = \frac{\partial}{\partial \theta} x^{\theta}(t)\,|_{\theta=0}\ , \tag{1.35}$$

i.e. the Taylor expansion of $x^{\theta}(t)$ is

$$x^{\theta}(t) = \hat{x}(t) + z(t)\,\theta + o(\theta). \tag{1.36}$$

Then, by (1.34),

$$z(\tau) = b\left(\hat{x}_{\tau}, v_{\tau}\right) - b\left(\hat{x}_{\tau}, \hat{u}_{\tau}\right). \tag{1.37}$$

Moreover, we can derive the following differential equation for $z(t)$

$$\begin{aligned} dz_t &= \frac{\partial}{\partial\theta} dx^\theta(t)\,|_{\theta=0} \\ &= \frac{\partial}{\partial\theta} b\left(x^\theta(t), u^\theta(t)\right) dt\,|_{\theta=0} \\ &= b_x\left(x^\theta(t), u^\theta(t)\right) \frac{\partial}{\partial\theta} x^\theta(t)\, dt\,|_{\theta=0} \\ &= b_x\left(x^\theta(t), u^\theta(t)\right) z_t dt, \end{aligned}$$

where b_x denotes the derivative of b with respect to x. If we for the moment assume that $h = 0$, the optimality of $\hat{u}(t)$ leads to the inequality

$$\begin{aligned} 0 &< \frac{\partial}{\partial\theta} J\left(u^\theta\right)|_{\theta=0} = \frac{\partial}{\partial\theta} g\left(x^\theta(T)\right)|_{\theta=0} = g_x\left(x^\theta(T)\right) \frac{\partial}{\partial\theta} x^\theta(T)\,|_{\theta=0} \\ &= g_x\left(\hat{x}(T)\right) z(T). \end{aligned}$$

We shall use duality to obtain a more explicit necessary condition from this. To this end we introduce the adjoint equation:

$$\begin{cases} dp(t) = -b_x\left(\hat{x}(t), \hat{u}(t)\right) p(t)\, dt, \\ p(T) = g_x\left(\hat{x}(T)\right). \end{cases}$$

Then it follows that

$$d\left(p(t)\, z(t)\right) = 0,$$

i.e. $p(t)z(t) =$ constant. By the terminal condition for the adjoint equation we have

$$p(t)z(t) = g_x\left(\hat{x}(T)\right) z(T) > 0, \quad \text{for all} \quad 0 \le t \le T.$$

In particular, by (1.37)

$$p(\tau)\left(b\left(\hat{x}(\tau), v(\tau)\right) - b\left(\hat{x}(\tau), \hat{u}(\tau)\right)\right) \ge 0.$$

Since τ was chosen arbitrarily, this is equivalent to

$$p(t)\, b(\hat{x}(t), \hat{u}(t)) = \inf_v p(t)\, b(\hat{x}(t), v(t)), \quad \text{for all} \quad 0 \leq t \leq T.$$

This specifies a necessary condition for $\hat{u}(t)$ to be optimal when $h = 0$. To account for the running cost h one can construct an extra state $dx^0(t) = h(x(t), u(t))dt$, which allows us to write the cost function in terms of two terminal costs :

$$J(u) = x^0(T) + g(x(T)).$$

By repeating the calculations above for this two-dimensional system, one can derive the necessary condition :

$$H(\hat{x}(t), \hat{u}(t), p(t)) = \inf_v H(\hat{x}(t), v, p(t)), \quad \text{for all} \quad 0 \leq t \leq T, \tag{1.38}$$

where H is the so-called Hamiltonian (sometimes defined with a minus sign which turns the minimum condition above into a maximum condition):

$$H(x, u, p) = h(x, u) + pb(x, u),$$

and the adjoint equation is given by

$$\begin{cases} dp(t) = -\left(h_x(\hat{x}(t), \hat{u}(t)) + b_x(\hat{x}(t), \hat{u}(t))\, p(t)\right) dt, \\ p(T) = g_x(\hat{x}(T)). \end{cases} \tag{1.39}$$

The minimum condition (1.38) together with the adjoint equation (1.39) specifies the Hamiltonian system for our control problem.

1.5.2 The stochastic maximum principle

The earliest paper on the extension of the maximum principle to stochastic control problems is Kushner and Schweppe (1964). One major difficulty that arises in such an extension is that the adjoint equation (1.39) becomes a SDE with terminal conditions. In contrast to a deterministic differential equation, one cannot simply reverse the time since the control process, and consequently the solution to the SDE, is required to be adapted to the filtration. Bismut solved this problem by introducing conditional expectations and obtained the solution to the adjoint equation from the martingale representation theorem, see e.g. Bismut (1978) [16] and also Haussmann (1986) [39]. An extensive study of these so-called backward SDEs can be found in e.g. Ma and Yong (1999) [46].

For the case where σ isn't controlled the adjoint equation is given by

$$\begin{cases} dp\left(t\right)=-\left(h_x\left(\hat{x}\left(t\right),\hat{u}\left(t\right)\right)+b_x\left(\hat{x}\left(t\right),\hat{u}\left(t\right)\right)p\left(t\right)+\sigma_x\left(\hat{x}\left(t\right)\right)q\left(t\right)\right)dt-q\left(t\right)dW_t, \\ p\left(T\right)=g_x\left(\hat{x}\left(T\right)\right). \end{cases} \tag{1.40}$$

A solution to this kind of backward SDE is a pair $(p(t),q(t))$ which fulfills (1.40). The Hamiltonian is

$$H\left(x,u,p,q\right)=h\left(x,u\right)+pb\left(x,u\right)+q\sigma\left(x\right),$$

and the maximum principle reads

$$H\left(\hat{x}\left(t\right),\hat{u}\left(t\right),p\left(t\right),q\left(t\right)\right)=\inf_v H\left(\hat{x}\left(t\right),v,p\left(t\right),q\left(t\right)\right), \text{ for all } \quad 0\leq t\leq T,\ P\text{-}a.s.$$

For the stochastic maximum principle, there is a major difference between the cases where σ isn't controlled and σ is controlled. As for (1.1), when performing the expansion with respect to the perturbation θ (1.36), the fact that the perturbed Itô integral turns out to be of order $\sqrt{\theta}$ (rather than θ as with the ordinary Lebesgue integral) poses a problem. In fact, one needs to take into account both the first-order and second-order terms in the

Taylor expansion (1.36). This ultimately leads to a maximum principle containing two adjoint equations, both in the form of linear backward SDEs. The Hamiltonian is replaced by a extended Hamiltonian :

$$\mathcal{H}^{(\hat{x}(t),\hat{u}(t))}(t,x,v) = H\left(t,x,v,p\left(t\right),q\left(t\right) - P\left(t\right)\sigma\left(t,\hat{x}\left(t\right),\hat{u}\left(t\right)\right)\right) - \frac{1}{2}\sigma^{2}\left(t,\hat{x}\left(t\right),v\right)P\left(t\right),$$

where $(p(t),q(t))$ is the solution to the first order adjoint equation (1.40) and $(P(t),Q(t))$ is the solution to the second order adjoint equation – see Peng (1990) [49] where the first proof of this general stochastic maximum principle is given. The optimal control is in this case characterized by

$$\mathcal{H}^{(\hat{x}(t),\hat{u}(t))}\left(t,\hat{x}\left(t\right),\hat{u}\left(t\right)\right) = \inf\nolimits_{v}\mathcal{H}^{(\hat{x}(t),\hat{u}(t))}\left(t,\hat{x}\left(t\right),v\right), \quad \text{for all} \quad 0 \leq t \leq T,\ P\text{-}a.s.$$

There is also third case: if the state is given by (1.1) but the action space U is convex, it is possible to derive the maximum principle in a local form. This is accomplished by using a convex perturbation of the control instead of a spike variation, see Bensoussan (1982) [34]. The necessary condition for optimality is then the following.:

$$\frac{d}{dv}H\left(t,\hat{x}\left(t\right),\hat{u}\left(t\right),\hat{p}\left(t\right),\hat{q}\left(t\right)\right)\left(v-\hat{u}\left(t\right)\right) \geq 0, \quad \text{for all} \quad 0 \leq t \leq T,\ P\text{-}a.s.$$

Chapter 2

A general stochastic maximum principle for singular control problems

2.1 Introduction

In this chapter we will give a detailed demonstration of the maximum principle in singular control in which the control domain need not be convex, the control variable has two components, the first being absolutely continuous and the second singular. The coefficients of the state equation are non linear and depend explicitly on the absolutely continuous component of the control. This result was established by Seid Bahlali and Brahim Mezerdi [3] using a spike variation on the absolutely continuous part of the control and a convex perturbation on the singular one to establish a maximum principle. This result is a generalization of Peng's maximum principle to singular control problems.

2.1.1 Problem formulation and assumptions

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ be a probability space equipped with a filtration satisfying the usual conditions, on which a d-dimensional Brownian motion $\mathcal{B} = (\mathcal{B}_t)_{t\geq 0}$ is defined. We assume that $(\mathcal{F}_t)$ is the P-augmentation of the natural filtration of $(\mathcal{B}_t)_{t\geq 0}$.

Let T be a strictly positive real number and consider the following sets

A_1 is a non empty subset of $\mathbb{R}^k$ and $A_2 = ([0,\infty))^m$.

U_1 is the class of measurable, adapted processes $u : [0,T] \times \Omega \to A_1$.

U_2 is the class of measurable, adapted processes $\xi : [0,T] \times \Omega \to A_2$ such that ξ is no decreasing, left-continuous with right limits and $\xi_0 = 0$.

Definition 2.1.1 *An admissible control is an $\mathcal{F}_t$-adapted process $(u,\xi) \in U_1 \times U_2$ such that*

$$E\left[\sup_{t\in[0,T]} |u(t)|^2 + |\xi_T|^2\right] < \infty.$$

We denote by $\mathcal{U}$ the set of all admissible controls.

For any $(u,\xi) \in \mathcal{U}$, we consider the following stochastic equation :

$$\begin{cases} dx(t) &= b(t, x(t), u(t))\, dt + \sigma(t, x(t), u(t))\, d\mathcal{B}_t + G_t d\xi_t, \\ x(0) &= x_0, \end{cases} \tag{2.1}$$

where

$$b : [0,T] \times \mathbb{R}^n \times A_1 \to \mathbb{R}^n,$$

$$\sigma : [0,T] \times \mathbb{R}^n \times A_1 \to \mathcal{M}_{n\times d}(\mathbb{R}),$$

$$G : [0,T] \to \mathcal{M}_{n\times m}(\mathbb{R}).$$

The expected cost has the form

$$J(u,\xi) = E\left[g(x_T) + \int_0^T h(t, x(t), u(t))\, dt + \int_0^T h(t)\, d\xi_t\right], \tag{2.2}$$

where

$$g : \mathbb{R}^n \to \mathbb{R},$$
$$h : [0,T] \times \mathbb{R}^n \times A_1 \to \mathbb{R},$$
$$k : [0,T] \to ([0,\infty))^m .$$

The control problem is to minimize the functional $J(.)$ over $\mathcal{U}$. If $\left(\hat{u},\hat{\xi}\right) \in \mathcal{U}$ is an optimal solution, that is

$$J\left(\hat{u},\hat{\xi}\right) = \inf_{(u,\xi)\in\mathcal{U}} J(u,\xi),$$

we may ask, how we can characterize it, in other words what conditions must $\left(\hat{u},\hat{\xi}\right)$ necessarily satisfy?

We have the following assumptions :

1. b, σ, g, h are twice continuously differentiable with respect to x
2. The derivatives $b_x, b_{xx}, \sigma_x, \sigma_{xx}, G_x, G_{xx}, h_x, h_{xx}$ are continuous in (x,u) and uniformly bounded.
3. b, σ are bounded by $C(1+|x|+|u|)$.
4. G and k are continuous and G is bounded.

Under the above hypothesis, for every $(u,\xi) \in \mathcal{U}$ equation (2.1) has a unique strong solution given by

$$x^{(u,\xi)}(t) = x_0 + \int_0^t b\left(s, x^{(u,\xi)}(s), u(s)\right) ds + \int_0^t \sigma\left(s, x^{(u,\xi)}(s), u(s)\right) d\mathcal{B}_s + \int_0^t G_s d\xi_s,$$

and the cost functional J is well defined from $\mathcal{U}$ into $\mathbb{R}$.

2.1.2 Preliminary Results

We assume the existence of an optimal control $\left(\hat{u},\hat{\xi}\right)$ minimizing the cost J over $\mathcal{U}$ and $\hat{x}(t)$ denotes the optimal trajectory, that is, the solution of (2.1) corresponding to $\left(\hat{u},\hat{\xi}\right)$. Let us introduce the following perturbation of the optimal control $\left(\hat{u},\hat{\xi}\right)$:

$$\left(u^{\theta}(t),\xi^{\theta}(t)\right)=\begin{cases}\left(v,\hat{\xi}(t)+\theta\left(\eta(t)-\hat{\xi}(t)\right)\right) & \text{if } t\in[\tau,\tau+\theta],\\ \left(\hat{u}(t),\hat{\xi}(t)+\theta\left(\eta(t)-\hat{\xi}(t)\right)\right) & \text{otherwise,}\end{cases} \tag{2.3}$$

where $0\leq\tau<T$ is fixed, $\theta>0$ is sufficiently small, v is a -measurable random variable and η is an increasing process with $\eta_0=0$.
Since $\left(\hat{u},\hat{\xi}\right)$ is optimal, we have

$$J\left(u^{\theta},\xi^{\theta}\right)-J\left(\hat{u},\hat{\xi}\right)\geq 0.$$

Let

$$J_1=J\left(u^{\theta},\xi^{\theta}\right)-J\left(u^{\theta},\hat{\xi}\right), \tag{2.4}$$

$$J_2=J\left(u^{\theta},\hat{\xi}\right)-J\left(\hat{u},\hat{\xi}\right). \tag{2.5}$$

The variational inequality will be derived from the fact that

$$\lim_{\theta\to 0}\frac{1}{\theta}J_1+\lim_{\theta\to 0}\frac{1}{\theta}J_2\geq 0, \tag{2.6}$$

let x_t^{θ}, $x_t^{\left(u^{\theta},\hat{\xi}\right)}$ be the trajectories associated respectively with $\left(u^{\theta},\xi^{\theta}\right)$ and $\left(u^{\theta},\hat{\xi}\right)$.
For simplicity of notation, we denote

$$f(t)=f(t,\hat{x},\hat{u}),$$

$$f^{\theta}(t)=f\left(t,\hat{x},u^{\theta}\right),$$

where f stands for one of the functions $b, b_x, b_{xx}, \sigma, \sigma_x, \sigma_{xx}, h, h_x, h_{xx}$.

To obtain the variational inequality we need the following technical lemmas.

Lemma 2.1.1 *Under assumptions 1, we have*

$$\lim_{\theta \to 0} E \left| \frac{x^\theta (t) - x^{\left(u^\theta, \hat{\xi}\right)} (t)}{\theta} - z (t) \right|^2 = 0, \tag{2.7}$$

where z is the solution of the linear stochastic differential equation

$$z (t) = \int_0^t b_x (s) z (s) ds + \int_0^t \sigma_x (s) z (s) d\mathcal{B}_s + \int_0^t G (s) d \left(\eta - \hat{\xi} \right)_s .$$

Proof. We first proof that

$$\lim_{\theta \to 0} E \left[\sup_{0 \le t \le T} \left| x^\theta (t) - x^{\left(u^\theta, \hat{\xi}\right)} (t) \right|^2 \right] = 0, \tag{2.8}$$

$$\lim_{\theta \to 0} E \left[\sup_{0 \le t \le T} \left| x^{\left(u^\theta, \hat{\xi}\right)} (t) - \hat{x} (t) \right|^2 \right] = 0, \tag{2.9}$$

$$E \left[|z_t|^2 \right] < \infty, \tag{2.10}$$

where $x^\theta (t)$ and $x^{\left(u^\theta, \hat{\xi}\right)} (t)$ are the solutions of the equation (2.1) associated respectively to the controls $\left(u^\theta, \xi^\theta\right)$ and $\left(u^\theta, \hat{\xi}\right)$,we have then

$$\begin{aligned} x^\theta (t) &= x_0 + \int_0^t b \left(s, x^\theta (s), u^\theta (s)\right) ds + \int_0^t \sigma \left(s, x^\theta (s), u^\theta (s)\right) d\mathcal{B}_s \\ &\quad + \int_0^t G_s d\xi_s^\theta, \\ x^{\left(u^\theta, \hat{\xi}\right)} (t) &= x_0 + \int_0^t b \left(s, x^{\left(u^\theta, \hat{\xi}\right)} (s), u^\theta (s)\right) ds + \int_0^t \sigma \left(s, x^{\left(u^\theta, \hat{\xi}\right)} (s), u^\theta (s)\right) d\mathcal{B}_s \\ &\quad + \int_0^t G_s d\hat{\xi}_s, \end{aligned}$$

using the two equations above and applying the expectation we have

$$E\left[\sup_{0\leq t\leq T}\left|x^{\theta}(t)-x^{\left(u^{\theta},\hat{\xi}\right)}(t)\right|^{2}\right] \leq 3TE\left[\int_{0}^{T}\left|b\left(s,x^{\theta}(s),u^{\theta}(s)\right)-b\left(s,x^{\left(u^{\theta},\hat{\xi}\right)}(s),u^{\theta}(s)\right)\right|^{2}ds\right]+$$

$$3E\left[\sup_{0\leq t\leq T}\left(\int_{0}^{t}\left|\sigma\left(s,x^{\theta}(s),u^{\theta}(s)\right)-\sigma\left(s,x^{\left(u^{\theta},\hat{\xi}\right)}(s),u^{\theta}(s)\right)\right|d\mathcal{B}_{s}\right)^{2}\right]$$

$$+3\theta^{2}E\left[\sup_{0\leq t\leq T}\left(\int_{0}^{t}G_{s}d\left(\eta-\hat{\xi}\right)_{s}\right)^{2}\right],$$

using Burkholder Davis Gundy inequality, we get

$$E\left[\sup_{0\leq t\leq T}\left|x^{\theta}(t)-x^{\left(u^{\theta},\hat{\xi}\right)}(t)\right|^{2}\right] \leq 3TE\left[\int_{0}^{T}\left|b\left(s,x^{\theta}(s),u^{\theta}(s)\right)-b\left(s,x^{\left(u^{\theta},\hat{\xi}\right)}(s),u^{\theta}(s)\right)\right|^{2}ds\right]$$

$$+3cE\left[\int_{0}^{T}\left|\sigma\left(s,x^{\theta}(s),u^{\theta}(s)\right)-\sigma\left(s,x^{\left(u^{\theta},\hat{\xi}\right)}(s),u^{\theta}(s)\right)\right|^{2}ds\right]$$

$$+\theta^{2}K.$$

Under assumption (1), we have b and σ are Lipschitz then the inequality above becomes

$$\left|b\left(s,x^{\theta}(s),u^{\theta}(s)\right)-b\left(s,x^{\left(u^{\theta},\hat{\xi}\right)}(s),u^{\theta}(s)\right)\right|\leq k\left|x^{\theta}(s)-x^{\left(u^{\theta},\hat{\xi}\right)}(s)\right|,$$

$$\left|\sigma\left(s,x^{\theta}(s),u^{\theta}(s)\right)-\sigma\left(s,x^{\left(u^{\theta},\hat{\xi}\right)}(s),u^{\theta}(s)\right)\right|\leq k\left|x^{\theta}(s)-x^{\left(u^{\theta},\hat{\xi}\right)}(s)\right|,$$

$$E\left[\sup_{0\leq t\leq T}\left|x^{\theta}(t)-x^{\left(u^{\theta},\hat{\xi}\right)}(t)\right|^{2}\right] \leq (3Tk^{2}+3ck^{2})\int_{0}^{T}E\left[\left|x^{\theta}(s)-x^{\left(u^{\theta},\hat{\xi}\right)}(s)\right|^{2}\right]ds$$

$$+K\theta^{2}.$$

In the other hand we have by using the isomitry property of the stochastic integral and the fact that b and σ are Lipschitz

$$\begin{aligned}
E\left[\left|x^{\theta}(t)-x^{\left(u^{\theta},\hat{\xi}\right)}(t)\right|^{2}\right] &\leq 3TE\left[\int_{0}^{t}\left|b\left(s,x^{\theta}(s),u^{\theta}(s)\right)-b\left(s,x^{\left(u^{\theta},\hat{\xi}\right)}(s),u^{\theta}(s)\right)\right|^{2}ds\right] \\
&+3TE\left[\int_{0}^{t}\left|\sigma\left(s,x^{\theta}(s),u^{\theta}(s)\right)-\sigma\left(s,x^{\left(u^{\theta},\hat{\xi}\right)}(s),u^{\theta}(s)\right)\right|^{2}ds\right] \\
&+K\theta^{2} \\
&\leq K'E\left[\int_{0}^{t}\left|x^{\theta}(t)-x^{\left(u^{\theta},\hat{\xi}\right)}(t)\right|^{2}ds\right]+\theta^{2}K,\ \text{where } K'=6Tk^{2}.
\end{aligned}$$

Using Gronwall lemma, we have

$$E\left(\left|x^{\theta}(t)-x^{\left(u^{\theta},\hat{\xi}\right)}(t)\right|^{2}\right)\leq c\theta^{2}\quad\text{where}\quad c=K\exp\left(K'T\right).$$

Finally we have

$$E\left[\sup_{0\leq t\leq T}\left|x^{\theta}(t)-x^{\left(u^{\theta},\hat{\xi}\right)}(t)\right|^{2}\right]\leq C\theta^{2}.$$

We use the same arguments to prove that

$$\begin{aligned}
&\lim_{\theta\to 0}E\left[\sup_{0\leq t\leq T}\left|x^{\left(u^{\theta},\hat{\xi}\right)}(t)-\hat{x}(t)\right|^{2}\right]=0, \\
&E\left[\left|z_{t}\right|^{2}\right]<\infty.
\end{aligned}$$

Let

$$y^{\theta}(t)=\frac{x^{\theta}(t)-x^{\left(u^{\theta},\hat{\xi}\right)}(t)}{\theta}-z(t).$$

If we use for $f=b$ or $f=\sigma$ the following equality :

$$\begin{aligned}
f\left(s,x^{\theta},u^{\theta}\right)-f\left(s,x^{\left(u^{\theta},\hat{\xi}\right)},u^{\theta}\right) &=\int_{0}^{1}\frac{d}{d\lambda}\left(f\left(s,x^{\left(u^{\theta},\hat{\xi}\right)}+\lambda\left(x^{\theta}-x^{\left(u^{\theta},\hat{\xi}\right)}\right),u^{\theta}\right)\right)d\lambda \\
&=\int_{0}^{1}f_{x}\left(s,x^{\left(u^{\theta},\hat{\xi}\right)}+\lambda\left(x^{\theta}-x^{\left(u^{\theta},\hat{\xi}\right)}\right),u^{\theta}\right)\left(x^{\theta}-x^{\left(u^{\theta},\hat{\xi}\right)}\right)d\lambda,
\end{aligned} \tag{2.11}$$

and the inequality : $(a+b+c)^2 \leq 3a^2+3b^2+3c^2$, it hold that

$$\begin{aligned} E\left|y^{\theta}(t)\right|^2 \leq 3T\int_0^t E\left|\int_0^1 b_x\left(s, x_s^{\left(u^{\theta},\hat{\xi}\right)}+\lambda\left(x_s^{\theta}-x_s^{\left(u^{\theta},\hat{\xi}\right)}\right), u_s^{\theta}\right) y^{\theta}(s)\, d\lambda\right|^2 ds \\ +3\int_0^t E\left|\int_0^1 \sigma_x\left(s, x_s^{\left(u^{\theta},\hat{\xi}\right)}+\lambda\left(x_s^{\theta}-x_s^{\left(u^{\theta},\hat{\xi}\right)}\right), u_s^{\theta}\right) y^{\theta}(s)\, d\lambda\right|^2 ds \\ +3E\left|\rho^{\theta}(t)\right|^2, \end{aligned}$$

where

$$\begin{aligned} \rho^{\theta}(t) = \int_0^t\int_0^1 \left[b_x\left(s, x_s^{\left(u^{\theta},\hat{\xi}\right)}+\lambda\left(x_s^{\theta}-x_s^{\left(u^{\theta},\hat{\xi}\right)}\right), u_s^{\theta}\right)-b_x\left(s,\hat{x}_s,\hat{u}_s\right)\right] z_s d\lambda ds \\ +\int_0^t\int_0^1 \left[\sigma_x\left(s, x_s^{\left(u^{\theta},\hat{\xi}\right)}+\lambda\left(x_s^{\theta}-x_s^{\left(u^{\theta},\hat{\xi}\right)}\right), u_s^{\theta}\right)-\sigma_x\left(s,\hat{x}_s,\hat{u}_s\right)\right] z_s d\lambda d\mathcal{B}_s. \end{aligned}$$

Since b_x and σ_x are bounded, we have

$$E\left|y^{\theta}(t)\right|^2 \leq c\int_0^t E\left|y^{\theta}(s)\right|^2 ds + 3E\left|\rho^{\theta}(t)\right|^2.$$

b_x, σ_x being continuous and bounded, then using (2.8), (2.9), (2.10) and the dominated convergence theorem, we get

$$\lim_{\theta\to 0} E\left|\rho^{\theta}(t)\right|^2 = 0.$$

We conclude by using Gronwall's lemma. ■

Lemma 2.1.2 *Under assumption* $1-4$*, the following estimate holds*

$$E\left[\sup_{t\in[0,T]}\left|x_t^{\left(u^{\theta},\hat{\xi}\right)}-\hat{x}_t-x_1(t)-x_2(t)\right|^2\right] \leq C\theta^2, \tag{2.12}$$

where x_1, x_2 are solutions of

$$x_1(t) = \int_0^t \left[b_x(s) x_1(s) + b^\theta(s) - b(s) \right] ds + \int_0^t \left[\sigma_x(s) x_1(s) + \sigma^\theta(s) - \sigma(s) \right] d\mathcal{B}_s,$$

$$\begin{aligned} x_2(t) &= \int_0^t \left[b_x^\theta(s) - b_x(s) \right] x_1(s) \, ds \\ &+ \int_0^t \left[b_x(s) x_2(s) + \frac{1}{2} b_{xx}(s) x_1(s) x_1(s) \right] ds \\ &+ \int_0^t \left[\sigma_x^\theta(s) - \sigma_x(s) \right] x_1(s) \, d\mathcal{B}_s \\ &+ \int_0^t \left[\sigma_x(s) x_2(s) + \frac{1}{2} \sigma_{xx}(s) x_1(s) x_1(s) \right] d\mathcal{B}_s. \end{aligned}$$

The above equations are called the first and the second-order variational equations.

Proof. We put

$$\tilde{x}(t) = \hat{x}(t) - \int_0^t G(s) \, d\hat{\xi}_s,$$

$$\tilde{x}_t^{\left(u^\theta, \hat{\xi}\right)} = \hat{x}_t^{\left(u^\theta, \hat{\xi}\right)} - \int_0^t G(s) \, d\hat{\xi}_s.$$

It is clear that

$$x_t^{\left(u^\theta, \hat{\xi}\right)} - \hat{x}_t - x_1(t) - x_2(t) = \tilde{x}_t^{\left(u^\theta, \hat{\xi}\right)} - \tilde{x}_t - x_1(t) - x_2(t).$$

By using the same proof as in [53], lemma 1 page 968, we show that

$$E \left[\sup_{t \in [0,T]} \left| \tilde{x}_t^{\left(u^\theta, \hat{\xi}\right)} - \tilde{x}_t - x_1(t) - x_2(t) \right|^2 \right] \leq C \theta^2,$$

which prove the lemma ■

Lemma 2.1.3 *Under assumptions of lemma* (2.2.1), *we have*

$$\lim_{\theta \to 0} \frac{J_1}{\theta} = E\left[z\left(t\right) g_x\left(\hat{x}\left(T\right)\right)\right] + E\int_0^T z\left(t\right) h_x\left(t\right) dt + E\int_0^T k\left(t\right) d\left(\eta - \hat{\xi}\right)_t . \tag{2.13}$$

Proof. From (2.4) we have by using equality (2.12)

$$\begin{aligned} \frac{J_1}{\theta} &= E\int_0^T \int_0^1 \left(\frac{x_s^\theta - x_s^{\left(u^\theta, \hat{\xi}\right)}}{\theta}\right) h_x\left(s, x_s^{\left(u^\theta, \hat{\xi}\right)} + \lambda\left(x_s^\theta - x_s^{\left(u^\theta, \hat{\xi}\right)}\right), u_s^\theta\right) d\lambda ds \\ &+ \int_0^1 \left(\frac{x_T^\theta - x_T^{\left(u^\theta, \hat{\xi}\right)}}{\theta}\right) g_x\left(x_T^{\left(u^\theta, \hat{\xi}\right)} + \lambda\left(x_T^\theta - x_T^{\left(u^\theta, \hat{\xi}\right)}\right)\right) d\lambda \\ &+ E\int_0^T k\left(t\right) d\left(\eta - \hat{\xi}\right)_t . \end{aligned}$$

Since g_x and h_x are continuous and bounded, then from (2.3), (2.7), (2.9) and by letting θ going to zero we conclude. ■

Lemma 2.1.4 *Under assumptions of lemma* (2.2.2) *we have*

$$\begin{aligned} J_2 &\leq E\left[g_x\left(\hat{x}_T\right)\left(x_1\left(T\right) + x_2\left(T\right)\right) + \int_0^T h_x\left(t\right)\left(x_1\left(t\right) + x_2\left(t\right)\right) dt\right] \\ &+ \frac{1}{2} E\left[g_{xx}\left(x_T\right) x_1\left(T\right) x_1\left(T\right)\right] + \int_0^T h_{xx}\left(t\right) x_1\left(t\right) x_1\left(t\right) dt \\ &+ E\int_0^T \left[h^\theta\left(t\right) - h\left(t\right)\right] dt + o\left(\theta\right). \end{aligned} \tag{2.14}$$

Proof. From (2.4) and the estimate (2.12) we have

$$\begin{aligned}
J_2 &= E\left[g\left(x_T^{(u^\theta,\hat{\xi})}\right) - g\left(\hat{x}_T\right)\right] \\
&+ E\left[\int_0^T \left(h\left(t, x_t^{(u^\theta,\hat{\xi})}, u_t^\theta\right) - h\left(t, \hat{x}_t, \hat{u}_t\right)\right) dt\right] \\
&\leq E\left[g\left(\hat{x}_T + x_1(T) + x_2(T)\right) - g\left(\hat{x}_T\right)\right] \\
&+ E\left[\int_0^T \left(h\left(t, \hat{x}_t + x_1(t) + x_2(t), \hat{u}_t\right) - h\left(t, \hat{x}_t, \hat{u}_t\right)\right) dt\right] \\
&+ E\left[\int_0^T \left(h\left(t, \hat{x}_t + x_1(t) + x_2(t), u_t^\theta\right) - h\left(t, \hat{x}_t + x_1(t) + x_2(t), \hat{u}_t\right)\right) dt\right] \\
&+ o(\theta) \\
&= E\left[g_x\left(\hat{x}_T\right)\left(x_1(T) + x_2(T)\right)\right] + \frac{1}{2} E\left[g_{xx}\left(\hat{x}_T\right) x_1(T) x_1(T)\right] \\
&+ E\left[\int_0^T h_x(t)\left(x_1(t) + x_2(t)\right) dt\right] \\
&+ E\left[\int_0^T \frac{1}{2} h_{xx}(t)\left(x_1(t) + x_2(t)\right)\left(x_1(t) + x_2(t)\right)\right] dt \\
&+ E\left[\int_0^T h^\theta(t) - h(t)\, dt\right] \\
&+ E\left[\int_0^T \left(h_x^\theta(t) - h_x(t)\right)\left(x_1(t) + x_2(t)\right) dt\right] \\
&+ \frac{1}{2} E\left[\int_0^T \left(h_{xx}^\theta(t) - h_{xx}(t)\right)\left(x_1(t) + x_2(t)\right)\left(x_1(t) + x_2(t)\right) dt\right] \\
&+ o(\theta). \qquad (2.15)
\end{aligned}$$

From the construction of $u^\theta(.)$, it is easy to verify by Gronwall's inequality and the moment inequality (see Ikeda and Watanabe [41]) that

$$\sup_{0 \leq t \leq T} E\left(\left|x_1(t)\right|^2\right) \leq c\theta,$$

$$\sup_{0 \leq t \leq T} E\left(\left|x_2(t)\right|^2\right) \leq c\theta^2,$$

now we use this inequalities and the inequality (2.15) to obtain the result ■

2.1.3 Variational inequalities and adjoint processes

We now use (2.13) and (2.14) to derive the variational inequalities.

The first-order expansion

The linear terms in (2.13) and (2.14) may be treated in the following way. Let Φ_1 be the fundamental solution of the linear equation

$$\begin{aligned} d\Phi_1(t) &= b_x(t)\,\Phi_1(t)\,dt + \sigma_x(t)\,\Phi_1(t)\,d\mathcal{B}_t, \\ \Phi_1(0) &= I_d. \end{aligned} \tag{2.16}$$

This equation is linear with bounded coefficients, then it admits a unique strong solution. This solution is invertible and its inverse $\Psi_1(t)$ is the unique solution of the following equation

$$d\Psi_1(t) = \left[\sigma_x(t)\,\Psi_1(t)\,\sigma_x^*(t) - b_x(t)\,\Psi_1(t)\right]dt - \sigma_x(t)\,\Psi_1(t)\,d\mathcal{B}_t. \tag{2.17}$$

Moreover Φ_1 and Ψ_1 satisfy

$$E\left[\sup_{t\in[0,T]}\left|\Phi_1(t)\right|^2\right] + E\left[\sup_{t\in[0,T]}\left|\Psi_1(t)\right|^2\right] < \infty. \tag{2.18}$$

We introduce the following processes

$$\alpha_1(t) = \Psi_1(t)\left[x_1(t) + x_2(t)\right], \tag{2.19}$$

$$\beta_1(t) = \Psi_1(t)\,z(t), \tag{2.20}$$

$$X_1 = \Phi_1^*(T)\, g_x(\hat{x}(T)) + \int_0^T \Phi_1^*(t)\, h_x(t)\, dt \tag{2.21}$$

$$Y_1(t) = E[X_1 \mid \mathcal{F}_t] - \int_0^t \Phi_1^*(s)\, h_x(s)\, ds. \tag{2.22}$$

we have by replacing X_1 by its value in (2.22)

$$Y_1(t) = E\left[\Phi_1^*(T)\, g_x(\hat{x}(T)) + \int_t^T \Phi_1^*(s)\, h_x(s)\, ds \mid \mathcal{F}_t\right], \tag{2.23}$$

we have then

$$E[\alpha_1(T)\, Y_1(T)] = E[g_x(\hat{x}(T))\,(x_1(T) + x_2(T))], \tag{2.24}$$

$$E[\beta_1(T)\, Y_1(T)] = E[g_x(\hat{x}(T))\, Z(t)]. \tag{2.25}$$

Since g_x and h_x are bounded, then from (2.18), X_1 is square integrable. Hence $(E[X_1 \mid \mathcal{F}_t])_{t\geqslant 0}$ is a square integrable martingale with respect to the natural filtration of the Brownian motion $(B_t)_{t\geqslant 0}$.Then from Ito's representation theorem we have

$$Y_1(t) = E[X_1] + \int_0^t Q_1(s)\, dB_S - \int_0^t \Phi_1^*(s)\, h_x(s)\, ds,$$

where $Q_1(s)$ is an adapted process such that $E\left(\int_0^T |Q_1(s)|^2\, ds\right) < \infty.$
By applying the Ito's formula to $\beta_1 Y_1$, we obtain

$$\begin{aligned} d(\beta_1(.)\, Y_1(.))_t &= \beta_1(t)\, dY_1(t) + Y_1(t)\, d\beta_1(t) + d\langle \beta_1(.), Y_1(.)\rangle_t \\ &= \beta_1(t)\, Q_1(t)\, dB_t - Z(t)\, h_x(t)\, dt + Y_1(t)\, d(\Psi_1(t)\, Z(t)) \\ &\quad + d\langle \Psi_1(.)\, Z(.), Y_1(.)\rangle_t, \end{aligned}$$

in the other hand we have

$$\begin{aligned} d\left(\Psi_1(.)\, Z(.)\right)_t &= \Psi_1(t)\, dZ(t) + Z(t)\, d\Psi_1(t) + d\left\langle \Psi_1(.)\,, Z(.)\right\rangle_t \\ &= \Psi_1 G(t)\, d\left(\eta - \hat{\xi}\right)_t, \end{aligned}$$

then we have

$$d\left(\beta_1(.)\, Y_1(.)\right)_t = \beta_1 Q_1 dB_t - Z(t)\, h_x(t)\, dt + Y_1(t)\, \Psi_1 G(t)\, d\left(\eta - \hat{\xi}\right)_t,$$

by integrating from 0 to T and taking expectation we obtain

$$\begin{aligned} E\left(\beta_1(T)\, Y_1(T)\right) = &-E\left[\int_0^T Z(t)\, h_x(t)\, dt\right] \\ &+ E\left[\int_0^T G(t)^* \,\Psi_1^* Y_1(t)\, d\left(\eta - \hat{\xi}\right)_t\right]. \end{aligned}$$

By using (2.25),we can rewrite (2.13)

$$\lim_{\theta\to 0}\frac{J_1}{\theta} = E\int_0^T \left(k(t) + G(t)^* \,p_1(t)\right) d\left(\eta - \hat{\xi}\right)_t, \tag{2.26}$$

where p_1 is adapted process defined in (2.28).

Now by applying the Itô's formula to $\alpha_1 Y_1$, to obtain

$$\begin{aligned} d\left(\alpha_1(.)\, Y_1(.)\right)_t &= \alpha_1(t)\, dY_1(t) + Y_1(t)\, d\alpha_1(t) + d\left\langle \alpha_1(.)\,, Y_1(.)\right\rangle_t \\ &= \alpha_1(t)\, Q(t)\, dB_t - \left[x_1(t) + x_2(t)\right] h_x(t)\, dt + Y_1(t)\, d\left(\Psi_1 x_1\right)_t \\ &+ Y_1(t)\, d\left(\Psi_1 x_2\right)_t + d\left\langle \alpha_1(.)\,, Y_1(.)\right\rangle_t, \end{aligned}$$

then by completing the calculus as we done before, and using (2.24), we can rewrite (2.14) as

$$\begin{aligned}
J_2 &\leq E\int_0^T \{H(t,\hat{x}(t),u_\theta(t),p_1(t),q_1(t)) - H(t,\hat{x}(t),\hat{u}(t),p_1(t),q_1(t))\}\,dt \\
&+\frac{1}{2}E\int_0^T x_1^*(t)\,H_{xx}(t,\hat{x}(t),\hat{u}(t),p_1(t),q_1(t))\,x_1(t)\,dt \\
&+\frac{1}{2}E\left[g_{xx}(x_T)\,x_1(T)\,x_1(T)\right] + o(t), \qquad (2.27)
\end{aligned}$$

where p_1 and q_1 are adapted processes given by

$$p_1(t) = \Psi_1^*(t)\,Y_1(t);\ \ p_1 \in \mathcal{L}^2([0,T];\mathbb{R}^n) \qquad (2.28)$$

$$q_1(t) = \Psi_1^*(t)\,Q_1(t) - \sigma_x^*(t)\,p_1(t);\ \ q_1 \in \mathcal{L}^2\left([0,T];\mathbb{R}^{n\times d}\right), \qquad (2.29)$$

and the Hamiltonian H is defined from $[0,T]\times\mathbb{R}^n\times A_1\times\mathbb{R}^n\times\mathcal{M}_{n\times d}(\mathbb{R})$ into $\mathbb{R}$ by

$$H(t,x(t),u(t),p_1(t),q_1(t)) = h(t) + p(t)\,b(t) + \sum_{i=1}^{d}\sigma_i(t)\,q_i(t),$$

where σ_i and q_i denote respectively the i^{th} columns of matrices σ and q.

The process p_1 is called the first order adjoint process and from (2.28), it is given explicitly by

$$p_1(t) = E\left[\Psi_1^*(t)\,\Phi_1^*(T)\,g_x(\hat{x}(T)) + \Psi_1^*(t)\int_t^T \Phi_1^*(s)\,h_x(s)\,ds \mid \mathcal{F}_t\right],$$

where $\Phi_1(t)$ and $\Psi_1(t)$ are respectively the solutions of (2.16) and (2.17).

The second-order expansion

We now treat the quadratic terms of (2.14) by the same method. Let $Z = x_1 x_1^*$, by Itô's formula we obtain

$$\begin{aligned} dZ_t &= x_1(t)\, dx_1^*(t) + x_1^*(t)\, dx_1(t) + d\langle x_1, x_1^* \rangle_t \\ &= \left[Z_t b_x^*(t) + b_x(t) Z_t + \sigma_x(t) Z_t \sigma_x^*(t) + A_\theta(t) \right] dt \\ &+ \left[Z(t) \sigma_x^*(t) + \sigma_x(t) Z(t) + B_\theta(t) \right] dB_t, \end{aligned} \tag{2.30}$$

where A_θ and B_θ are given by

$$\begin{aligned} A_\theta(t) &= x_1(t) \left[b^\theta(t) - b(t) \right]^* + \left[b^\theta(t) - b(t) \right] x_1^* \\ &+ \sigma_x(t)\, x_1(t) \left[\sigma^\theta(t) - \sigma(t) \right]^* + \left[\sigma^\theta(t) - \sigma(t) \right] x_1^*(t)\, \sigma_x^\theta(t) \\ &+ \left[\sigma^\theta(t) - \sigma(t) \right] \left[\sigma^\theta(t) - \sigma(t) \right]^*, \end{aligned}$$

$$B_\theta(t) = x_1(t) \left[\sigma^\theta(t) - \sigma(t) \right]^* + \left[\sigma^\theta(t) - \sigma(t) \right] x_1^*(t).$$

We consider now the following symmetric matrix-valued linear equation

$$\begin{cases} d\Phi_2(t) & = \left[\Phi_2(t)\, b_x^*(t) + b_x(t)\, \Phi_2(t) + \sigma_x(t)\, \Phi_2(t)\, \sigma_x^*(t) \right] dt \\ & \quad + \left[\Phi_2(t)\, \sigma_x^*(t) + \sigma_x(t)\, \Phi_2(t) \right] dB_t. \end{cases} \tag{2.31}$$

This equation is linear with bounded coefficients, hence it admits a unique strong solution. $\Phi_2(t)$ is invertible and its inverse Ψ_2 is is the solution of the following equation

$$\begin{cases} d\Psi_2(t) & = [\sigma_x(t) + \sigma_x^*(t)] \Psi_2(t) [\sigma_x(t) + \sigma_x^*(t)]^* dt \\ & - [\Psi_2(t) b_x^*(t) + b_x(t) \Psi_2(t) \sigma_x^*(t)] dt \\ & + [\Psi_2(t) \sigma_x^*(t) + \sigma_x(t) \Psi_2(t)] dB_t, \\ \Psi_2(0) & = I_d. \end{cases} \tag{2.32}$$

Moreover, Φ_2 and Ψ_2 satisfy

$$E\left[\sup_{t\in[0,T]} |\Phi_2(t)|^2\right] + E\left[\sup_{t\in[0,T]} |\Psi_2(t)|^2\right] < \infty, \tag{2.33}$$

we put

$$\alpha_2(t) = \Psi_2(t) Z(t), \tag{2.34}$$

$$X_2 = \Phi_2^*(T) g_{xx}(\hat{x}(T)) + \int_0^T \Phi_2^*(t) H_{xx}(\hat{x}(t), \hat{u}(t), p_1(t), q_1(t)) dt, \tag{2.35}$$

$$Y_2(t) = E(X_2 \mid \mathcal{F}_t) - \int_0^t \Phi_2^*(s) H_{xx}(\hat{x}(s), \hat{u}(s), p_1(s), q_1(s)) ds, \tag{2.36}$$

we remark that

$$E(\alpha_2(T) Y_2(T)) = E(x_1^*(T) g_{xx}(\hat{x}(T) x_1(T))). \tag{2.37}$$

Since g_{xx} and H_{xx} are bounded, then from (2.33), $(E(X_2 \mid \mathcal{F}_t))_{t\geq 0}$ is square integrable martingale with respect to the natural filtration of the Brownian motion $B(t)$. Then from Ito's representation theorem we have

$$Y_2(t) = E(X_2) + \int_0^t Q_2(s) dB_s - \int_0^t \Phi_2^*(s) H_{xx}(\hat{x}(s), \hat{u}(s), p_1(s), q_1(s)) ds, \tag{2.38}$$

where $Q_2(s)$ is an adapted process such that $E\left(\int_0^T |\Psi_2(t)|^2 dt\right) < \infty$.

By applying Ito's formula to $\alpha_2(t) Y_2(t)$ along with (2.37) and using the definition of u^θ, we can derive (2.27) as follows

$$\begin{aligned}
\lim_{\theta\to 0}\frac{J_2}{\theta} &\leq E\left\{H\left[\tau,\hat{x}(\tau),v,p_1(\tau),q_1(\tau)-p_2(\tau)\sigma(\tau,\hat{x}(\tau),\hat{u}(\tau))\right]\right\} \\
&+\frac{1}{2}E\left\{Tr\left[\sigma\sigma^*(\tau,\hat{x}(\tau),v)\right]p_2(\tau)\right\} \\
&-E\left\{H\left[\tau,\hat{x}(\tau),\hat{u}(\tau),p_1(\tau),q_1(\tau)-p_2(\tau)\sigma(\tau,\hat{x}(\tau),\hat{u}(\tau))\right]\right\} \\
&+\frac{1}{2}E\left\{Tr\left[\sigma\sigma^*(\tau,\hat{x}(\tau),)\,\hat{u}(\tau)\right]p_2(\tau)\right\},
\end{aligned} \tag{2.39}$$

where p_2 is an adapted process given by

$$p_2(t)=\Psi_2^*(t)\,Y_2(t)\ \ ;\ \ p_2\in\mathcal{L}^2\left([0,T];\mathbb{R}^{n\times n}\right). \tag{2.40}$$

The process p_2 is called the second order adjoint process and from (2.35), (2.36), (2.40) it is given explicitly by

$$\begin{aligned}
p_2(t) &= E\left[\Psi_2^*(t)\,\Phi_2^*(t)\,g_{xx}(x(T))\mid\mathcal{F}_t\right] \\
&+E\left[\Psi_2^*(t)\int_t^T\Phi_2^*(t)\,H_{xx}(\hat{x}(t),\hat{u}(t),p_1(s),q_1(s))\,ds\mid\mathcal{F}_t\right],
\end{aligned}$$

where Φ_2 and Ψ_2 are respectively the solutions of (2.31) and (2.32).

2.1.4 Adjoint equations and the maximum principle

By applying Ito's formula to the adjoint processes p_1 in (2.28) and p_2 in (2.40), we obtain the first and second order adjoint equations which are linear backward stochastic differential equations, given by

$$\begin{cases}
-dp_1(t)=H_x(\hat{x}(t),\hat{u}(t),p_1(s),q_1(s))\,dt-q_1dB_t, \\
p_1(T)=g_x(\hat{x}(T)),
\end{cases} \tag{2.41}$$

$$\begin{cases} -dp_2(t) & = \left[b_x^*(t)\, p_2(t) + p_2(t)\, b_x(t) + \sigma_x^*(t)\, p_2(t)\, \sigma_x^*(t)\right] dt \\ & + \left[\sigma_x^*(t)\, q_2(t) + q_2(t)\, \sigma_x(t)\right], \\ p_2(T) & = g_{xx}(\hat{x}(T)), \end{cases} \tag{2.42}$$

where $q_1(t)$ is given by (2.29) and $q_2(t)$ by

$$q_2(t) = \left(q_2^1(t), ..., q_2^d(t)\right) \;\; ; \; q_2 \in \left(\mathcal{L}^2\left([0,T]\,;\mathbb{R}^{n\times n}\right)\right)^d$$
$$q_2^i(t) = \Psi_2^*(t)\, Q_2^i(t) + p_2(t)\, \sigma_x^i(t) + \sigma_x^{i*}(t)\, p_2(t) \;\; ; \; i = 1, ..., d,$$

and $Q_1(t), Q_2(t)$ satisfy respectively

$$\int_0^t Q_1(s)\, dB_s = E\left[\Phi_1^*(T)\, g_x(\hat{x}(T)) + \int_0^T \Phi_1^*(t)\, h_x(t)\, dt \mid \mathcal{F}_t\right] - E\left[\Phi_1^*(T)\, g_x(\hat{x}(T)) + \int_0^T \Phi_1^*(t)\, h_x(t)\, dt\right],$$

$$\int_0^t Q_2(s)\, dB_s = E\left[\Phi_2^*(T)\, g_{xx}(\hat{x}(T)) + \int_0^T \Phi_2^*(t)\, H_{xx}\left[\hat{x}(t), \hat{u}(t), p_1(t), q_1(t)\right] dt \mid \mathcal{F}_t\right] - E\left[\Phi_2^*(T)\, g_{xx}(\hat{x}(T)) + \int_0^T \Phi_2^*(t)\, H_{xx}\left[\hat{x}(t), \hat{u}(t), p_1(t), q_1(t)\right] dt\right].$$

We can now give the important result of this chapter.

Theorem 2.1.1 (The Stochastic maximum principle) *Let $\left(\hat{u}, \hat{\xi}\right)$ be an optimal control minimizing the cost J over $\mathcal{U}$ and $\hat{x}$ denotes the corresponding optimal trajectory. Then there are two unique couples of adapted processes*

$$(p_1, q_1) \in \mathcal{L}^2([0,T]\,;\mathbb{R}^n) \times \mathcal{L}^2\left([0,T]\,;\mathbb{R}^{n\times d}\right),$$

$$(p_2, q_2) \in \mathcal{L}^2\left([0,T]\,;\mathbb{R}^{n\times n}\right) \times \left(\mathcal{L}^2\left([0,T]\,;\mathbb{R}^{n\times n}\right)\right)^d,$$

which are respectively solutions of backward stochastic differential equations (2.41) *and* (2.42) *such that*

$$\begin{aligned}
&H\left[\tau,\hat{x}\left(\tau\right),\hat{u}\left(\tau\right),p_1\left(\tau\right),q_1\left(\tau\right)-p_2\left(\tau\right)\sigma\left(\tau,\hat{x}\left(\tau\right),\hat{u}\left(\tau\right)\right)\right]\\
&+\frac{1}{2}Tr\left[\sigma\sigma^*\left(\tau,\hat{x}\left(\tau\right),\hat{u}\left(\tau\right)\right)\right]p_2\left(\tau\right)\\
&\leq H\left[\tau,\hat{x}\left(\tau\right),v,p_1\left(\tau\right),q_1\left(\tau\right)-p_2\left(\tau\right)\sigma\left(\tau,\hat{x}\left(\tau\right),\hat{u}\left(\tau\right)\right)\right]\\
&+\frac{1}{2}Tr\left[\sigma\sigma^*\left(\tau,\hat{x}\left(\tau\right),v\right)\right]p_2\left(\tau\right),\\
&\forall v\in A_1;\ \ a.e,\ \ a.s \qquad (2.43)
\end{aligned}$$

$$P\left\{\forall t\in[0,T],\ \forall i;\ \ \left(k_t^i+G_i^*\left(t\right)p_1\left(t\right)\right)\geqslant 0\right\}=1, \qquad (2.44)$$

$$P\left\{\sum_{i=1}^{d}1_{\left\{k_t^i+G_i^*(t)p_1(t)\geqslant 0\right\}}d\hat{\xi}_t^i=0\right\}=1. \qquad (2.45)$$

Proof. From (2.6), (2.26), (2.39) we have for every $\mathcal{F}_t$-measurable random variable v, and every increasing process η with $\eta_0=0$

$$\begin{aligned}
0&\leq E\left\{H\left[\tau,\hat{x}\left(\tau\right),v,p_1\left(\tau\right),q_1\left(\tau\right)-p_2\left(\tau\right)\sigma\left(\tau,\hat{x}\left(\tau\right),\hat{u}\left(\tau\right)\right)\right]\right\}\\
&+\frac{1}{2}E\left\{Tr\left[\sigma\sigma^*\left(\tau,\hat{x}\left(\tau\right),v\right)\right]p_2\left(\tau\right)\right\}\\
&-E\left\{H\left[\tau,\hat{x}\left(\tau\right),\hat{u}\left(\tau\right),p_1\left(\tau\right),q_1\left(\tau\right)-p_2\left(\tau\right)\sigma\left(\tau,\hat{x}\left(\tau\right),\hat{u}\left(\tau\right)\right)\right]\right\}\\
&-\frac{1}{2}E\left\{Tr\left[\sigma\sigma^*\left(\tau,\hat{x}\left(\tau\right),\hat{u}\left(\tau\right)\right)\right]p_2\left(\tau\right)\right\}\\
&+E\int_0^T\left[k\left(t\right)+G^*\left(t\right)p_1\left(t\right)\right]d\left(\eta-\hat{\xi}\right)_t,
\end{aligned}$$

if we put $\eta=\hat{\xi}$ we obtain (2.43). On the other hand, if we choose $v=\hat{u}$ and using the same proof of in theorem 4.2 in [3], we deduce (2.44) and(2.45). ■

Chapter 3

Introduction to Malliavin calculus

3.1 Introduction

The mathematical theory now known as Malliavin calculus was first introduced by Paul Malliavin in 1978, as an infinite-dimensional integration by parts technique. The purpose of this calculus was to prove results about the smoothness of densities of solutions of stochastic differential equations driven by Brownian motion. For several years this was the only known application.

In 1984, Ocone obtained an explicit interpretation of the Clark representation formula in terms of the Malliavin derivative (Clark-Ocone formula). In 1991 Ocone and Karatzas applied this result to finance: They proved that the Clark-Ocone formula can be used to obtain explicit formulae for replicating portfolios of contingent claims in complete markets.

Since then Malliavin calculus has been applied in various domains within finance and outside of it. In the meanwhile the very potentials in applications created the need for an extension of the calculus to other types of noise than Brownian motion. The most part of this chapter is taken from [29].

3.2 Elements of Malliavin calculus for Brownian motion

We choose to introduce the operators Malliavin derivative and Skorohod integral via chaos expansions. Other, basically equivalent, approach is to use directional derivatives on the Wiener space, see e.g. Da Prato(2007), Malliavin (1997), Nualart (2006), Sanz-Solé (2005).

Let $B(t) = B(\omega, t)$, $\omega \in \Omega$, $t \in [0,T]$ $(t > 0)$, be a Brownian motion on the complete probability space $(\Omega, \mathcal{F}, P)$ such that $B(0) = 0$ $P-a.s$. For any t, let $\mathcal{F}_t$ be the σ-algebra generated by $B(s)$, $0 < s < t$, augmented by all the P-zero measure events. The resulting (continuous) filtration is denoted :

$$\mathcal{F} = \{\mathcal{F}_t,\ t > 0\}.$$

3.2.1 Iterated Itô integrals

Let f be a deterministic function defined on

$$S_n = \{(t_1, ..., t_n) \in [0,T]^n :\ 0 < t_1 < ... < t_n < T\} \quad (n > 1),$$

such that

$$\|f\|_{L^2(S_n)} := \int_{S_n} f^2(t_1, ..., t_n)\, dt_1 ... dt_n < \infty. \tag{3.1}$$

Definition 3.2.1 *The n-fold iterated Itô integrals are given by:*

$$J_n(f) := \int_0^T \int_0^{t_n} ... \int_0^{t_2} f(t_1, ..., t_n)\, dB(t_1)\, dB(t_2) ... dB(t_n). \tag{3.2}$$

We set $J_n(f) = f$ for $f \in \mathbb{R}$.

Directly from the properties of Itô integrals we have :

- $J_n(f) \in L^2(P)$, by the Itô isometry $\|J_n(f)\|^2_{L^2(P)} = \|f\|^2_{L^2(S_n)}$.
- If $g \in L^2(S_m)$ and $f \in L^2(S_n)$ $(m < n)$, then $E[J_m(g) J_n(f)] = 0$.

Let $f \in \tilde{L}^2([0,T]^n)$, i.e. f is a symmetric square integrable functions.

Definition 3.2.2 *We also called n-fold iterated Itô integral the random variable :*

$$I_n(f) := \int_{[0,T]^n} f(t_1,...,t_n)\, dB(t_1)\, dB(t_2) ...dB(t_n) := n! J_n(f). \tag{3.3}$$

About symmetric functions:

- The function $f : [0,T]^n \to \mathbb{R}$ is symmetric if $f(t_{\sigma_1},...,t_{\sigma_n}) = f(t_1,...,t_n)$ for all permutations σ of $(1,...,n)$.
- if f is a real function on $[0,T]^n$, then the symmetrization $\tilde{f}$ of f is

$$\tilde{f}(t_1,...,t_n) := \frac{1}{n!}\sum_{\sigma} f(t_{\sigma_1},...,t_{\sigma_n}), \tag{3.4}$$

  where the sum is taken over all permutations σ of $(1,...,n)$. Naturally $\tilde{f} = f$ if and only if f is symmetric.
- If $f \in \tilde{L}^2([0,T]^n)$, then $\|f\|^2_{L^2([0,T]^n)} = n! \|f\|^2_{L^2(S_n)}$.

3.2.2 Iterated Itô integrals and Hermite polynomials

The Hermite polynomials $h_n(x)$, $x \in \mathbb{R}$, $n = 0,1,2,...$ are defined by

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}\left(e^{-\frac{1}{2}x^2}\right), \quad n = 0,1,2,... \tag{3.5}$$

Recall that the family of Hermite polynomials constitute an orthogonal basis for $L^2(\mathbb{R}, \mu(dx))$ if $\mu(dx) = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}x^2} dx$ (see e.g. Schoutens (2000)).

Note That

$$n!\int_0^T\int_0^{t_n}...\int_0^{t_2} g(t_1)...g(t_n)\,dB(t_1)\,dB(t_2)...dB(t_n) = \|g\|^n h_n\left(\frac{\theta}{\|g\|}\right), \tag{3.6}$$

where $\|g\| = \|g\|_{L^2([0,T])}$ and $\theta = \int_0^T g(t)\,dB_t$.

Example 3.2.1 *Let $g = 1$ and $n = 3$, then we get*

$$\begin{aligned} 6\int_0^T\int_0^{t_3}\int_0^{t_2} 1dB(t_1)\,dB(t_2)\,dB(t_3) &= T^{\frac{3}{2}} h_3\left(\frac{B(T)}{T^{\frac{1}{2}}}\right) \\ &= B^3(T) - 3TB(T). \end{aligned}$$

In fact the first Hermite polynomials are:

$$\begin{aligned} h_0(x) &= 1, \\ h_1(x) &= x, \\ h_2(x) &= x^2 - 1, \\ h_3(x) &= x^3 - 3x, \\ h_4(x) &= x^4 - 6x^2 + 3, ... \end{aligned}$$

The computation of the iterated Itô integrals is based on :

Proposition 3.2.1 *If ξ_1, ξ_2, ... are orthonormal functions in $L^2([0,T])$, we have that*

$$I_n\left(\xi_1^{\otimes\alpha_1}\hat{\otimes}...\hat{\otimes}\,\xi_m^{\otimes\alpha_m}\right) = \prod_{k=1}^{k=m} h_{\alpha_k}\left(\int_0^T \xi_k(t)\,dB(t)\right), \tag{3.7}$$

with $\alpha_1 + ... + \alpha_m = n$ and $\alpha_k \in \{0,1,2,...\}$ for all k.

Recall that the tensor product $f \otimes g$ of two functions f, g is defined by

$$f \otimes g(x_1, x_2) = f(x_1) f(x_2),$$

and the symmetrized tensor product $f \hat{\otimes} g$ is the symmetrization of $f \otimes g$.

3.2.3 Wiener-Itô chaos expansions

Theorem 3.2.1 *Let ξ be an $\mathcal{F}_{T}-$measurable random variable in $L^2(P)$. Then there exists a unique sequence $\{f_n\}_{n=0}^{\infty}$ of functions $f_n \in \tilde{L}^2([0,T]^n)$ such that*

$$\xi = \sum_{n=0}^{\infty} I_n(f_n), \tag{3.8}$$

where the convergence is in $L^2(P)$. Moreover, since

$$\|I_n(f_n)\|^2_{L^2(P)} = n! \, \|f_n\|_{L^2([0,T]^n)},$$

we have the isometry

$$\|\xi\|^2_{L^2(P)} = \sum_{n=0}^{\infty} n! \, \|f_n\|_{L^2([0,T]^n)}.$$

Example 3.2.2 *The chaos expansion of* $\xi = \exp\left\{B(T) - \frac{1}{2}T\right\}$ *is given by* $\xi = \sum_{n=0}^{\infty} \frac{t^{\frac{n}{2}}}{n!} h_n\left(\frac{B(t)}{\sqrt{t}}\right)$.

3.2.4 Skorohod integral

Let $u(\omega,t)$, $\omega \in \Omega$, $t \in [0,T]$, be a measurable stochastic process such that, for all $t \in [0,T]$, $u(t)$ is a $\mathcal{F}_T$-measurable random variable and $E[u^2(t)] < \infty$.
Then for each $t \in [0,T]$, we can apply the Wiener-Itô chaos expansion to the random variable $u(t) := u(\omega,t)$, $\omega \in \Omega$

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}) \quad f_{n,t} \in \tilde{L}^2([0,T]^n).$$

The functions $f_{n,t}$, $n = 1,2,...$, depend on $t \in [0,T]$ as parameter. We can define $f_n(t_1,...,t_n,t_{n+1}) := f_{n,t}(t_1,...,t_n)$ as a function of $n+1$ variables.

Its symmetrization $\tilde{f}_n$ is then given by

$$\tilde{f}_n\left(t_1,...,t_n,t_{n+1}\right) = \frac{1}{n+1}\left[f_n\left(t_1,...,t_n,t_{n+1}\right)\right.$$
$$\left.+ f_n\left(t_2,...,t_{n+1},t_1\right) + ... + f_n\left(t_1,...,t_{n-1},t_n\right)\right].$$

Definition 3.2.3 *Let $u\left(t\right),\ t\in[0,T]$, be a measurable stochastic process such that, for all $t\in[0,T]$, $u\left(t\right)$ is a $\mathcal{F}_T$-measurable random variable and $E\left[u^2\left(t\right)\right]<\infty$.*

Let its Wiener-Itô chaos expansion be

$$u\left(t\right) = \sum_{n=0}^{\infty} I_n\left(f_{n,t}\right) = \sum_{n=0}^{\infty} I_n\left(f_n\left(.,t\right)\right), \quad \left(f_n\left(.,t\right)\in\tilde{L}^2\left([0,T]^n\right)\right).$$

Then we define the Skorohod integral of u by

$$\delta\left(u\right) := -\int_0^T u\left(t\right)\delta B_t := \sum_{n=0}^{\infty} I_{n+1}\left(\tilde{f}_n\right), \tag{3.9}$$

when it converge in $L^2\left(P\right)$ (here $\tilde{f}_n$ is the symmetrization of $f_n\left(.,t\right)$.

Moreover,

$$\left\|\delta\left(u\right)\right\|^2_{L^2(P)} = \sum_{n=0}^{\infty}\left(n+1\right)!\left\|\tilde{f}_n\right\|^2_{L^2([0,T]^n)} < \infty.$$

Some basic properties of the Skorohod integral

- The Skorohod integral is a linear operator
- $E\left(\delta\left(u\right)\right) = 0$
- In general, if G is an $\mathcal{F}_T$ -measurable random variable such that; $Gu\in Dom(\delta)$; we have that

$$\int_0^T Gu\left(t\right)\delta B_t \neq G\int_0^T u\left(t\right)\delta B_t.$$

Example 3.2.3 *Let us compute* $\int_0^T B(T)\,\delta B_t$. *The Wiener-Itô chaos expansion of the integrand is given by*

$$u(t) = B(T) = \int_0^T 1dB_t = I_1(1), \quad t \in [0,T],$$

i.e for all t, $f_{0,t} = 0$, $f_{1,t} = 1$, *and* $f_{n,t} = 0$ *for all* $n > 2$. *Hence*

$$\delta(u) = I_2\left(\tilde{f}_1\right) = I_2(1) = 2\int_0^T \int_0^{t_2} dB(t_1)\,dB(t_2) = B(T)^2 = T.$$

Note that, even if the integrand does not depend on t, we have

$$\int_0^T B(T)\,\delta B_t \neq B(T) \int_0^T \delta B_t.$$

Theorem 3.2.2 (Skorohod integral as extension of the Itô integral) *Let* $u(t)$, $t \in [0,T]$, *be a measurable* $\mathcal{F}-$*adapted stochastic process such that,* $E[u^2(t)] < \infty$. *Then* u *is both Itô and Skorohod integrable and*

$$\int_0^T u(t)\,\delta B_t = \int_0^T u(t)\,dB_t.$$

3.3 Malliavin derivative

There are many ways of introducing the Malliavin derivative. The original construction was given on the Wiener space $\Omega = C_0([0,T])$ consisting of all continuous functions $\omega : [0,T] \to \mathbb{R}$ with $\omega_0 = 0$. In this section, we mainly use an approach based on chaos expansions.

Definition 3.3.1 *Let $F \in L^2(P)$ be $\mathcal{F}_T$-measurable with chaos expansion*

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $f_n(t) \in \tilde{L}^2([0,T]^n)$, $n = 1, 2, 3...$

i) *We say that $F \in \mathfrak{D}_{1,2}^{(B)}$*

$$\|F\|^2_{\mathfrak{D}_{1,2}^{(B)}} := \sum_{n=1}^{\infty} nn! \|f_n\|^2_{L^2(\lambda^n)} < \infty. \tag{3.10}$$

ii) *For any $F \in \mathfrak{D}_{1,2}^{(B)}$ we define the Malliavin derivative D_tF of F at time t, as the expansion*

$$D_tF := \sum_{n=1}^{\infty} nI_{n-1}(f_n(.,t)), \tag{3.11}$$

where $I_{n-1}(f_n(.,t))$ is the $(n-1)-$fold iterated integral of $f_n(t_1,...,t_{n-1},t)$ with respect to the first $n-1$ variables $t_1,...,t_{n-1}$ and $t_n = t$ left as parameter.

Note that $\|D_{.}F\|_{L^2(P\times\lambda)} = \|F\|^2_{\mathfrak{D}_{1,2}} < \infty$, thus the derivative D_tF is well-defined as an element of $L^2(P \times \lambda)$.

Theorem 3.3.1 (Closability) *Suppose $F \in L^2(P)$ and $F_k \in \mathfrak{D}_{1,2}^{(B)}$, $k = 1, 2, ...,$*

1. *$F_k \to F$, $k \to \infty$, in $L^2(P)$,*

2. *$\{D_tF_k\}_{k=1}^{\infty}$ converges in $L^2(P \times \lambda)$.*

Then $F \in \mathfrak{D}_{1,2}$ and $D_tF_k \to D_tF$, $k \to \infty$, in $L^2(P \times \lambda)$.

Proof. Let $F = \sum_{n=0}^{\infty} I_n(f_n)$ and $F_k = \sum_{n=0}^{\infty} I_n\left(f_n^k\right)$, $k = 1, 2, ...,$ then by (1)

$$f_n^k \to f_n, \ k \to \infty, \text{ in } L^2(\lambda^n).$$

■

Theorem 3.3.2 *for all n By (2) we have*

$$\sum_{n=1}^{\infty} nn! \left\| f_n^k - f_n^j \right\|_{L^2(\lambda^n)}^2 = \| D_t F_k - D_t F_j \|_{L^2(P\times\lambda)}^2 \to 0, \quad j,k \to 0.$$

Hence by the Fatou lemma

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} nn! \left\| f_n^k - f_n \right\|_{L^2(\lambda^n)}^2 \leq \lim_{k\to\infty} \underline{\lim}_{j\to\infty} \sum_{n=1}^{\infty} nn! \left\| f_n^k - f_n^j \right\|_{L^2(\lambda^n)}^2 = 0.$$

This implies that $F \in \mathfrak{D}_{1,2}$ and

$$D_t F_k \to D_t F, \ k \to \infty, \ in \ L^2 (P \times \lambda).$$

3.3.1 Fundamental rules of calculus

We present here a collection of results that constitute the rules of calculus of the Malliavin derivatives

Let $F = \int_0^T f(s)\, dB_s$, where $f \in L^2([0,T])$. Then

- $D_t F = f(t)$;
- $D_t (F)^n = nF^{n-1} D_t F = nF^{n-1} f(t)$.

Consider the case when $F = \sum_{n=0}^{\infty} I_n(f_n)$, and $f_n = f^{\otimes n}$ for som $f \in L^2([0,T])$, that is $f_n(t_1, ..., t_n) = f(t) ... f(t)$. Then we have by 3.6 we have

$$I_n(f_n) = \|f\|^n h_n \left(\frac{\theta}{\|f\|} \right), \tag{3.12}$$

where $\|f\| = \|f\|_{L^2[0,T]}$, $\theta = \int_0^T f(s)\, dB_s$ where h_n is the Hermite polynomial of order n.

Then by (3.11) we have

$$\begin{aligned} D_t I_n(f_n) &= n I_{n-1}(f_n(.,t)) \\ &= n I_{n-1}\left(f^{\otimes n-1}\right) f(t) \\ &= n \|f\|^{n-1} h_{n-1}\left(\frac{\theta}{\|f\|}\right) f(t). \end{aligned} \tag{3.13}$$

A basic property of the Hermite polynomials is that

$$h_n'(x) = n h_{n-1}(x), \tag{3.14}$$

combining this with (3.12) and (3.13), we get

$$D_t h_n\left(\frac{\theta}{\|f\|}\right) = h_n'\left(\frac{\theta}{\|f\|}\right)\frac{f(t)}{\|f\|}.$$

In particular, choosing $n = 1$, we get

$$D_t\left(\int_0^T f(s)\,dB_s\right) = f(t).$$

Similarly, by induction, for $n = 2, 3, ...$, we have

$$D_t\left(\int_0^T f(s)\,dB_s\right)^n = n\left(\int_0^T f(s)\,dB_s\right)^{n-1} f(t).$$

Theorem 3.3.3 (Chain rule.) *Let $F \in \mathfrak{D}_{1,2}$ and φ be a continuously differentiable function with bounded derivative. Then $\varphi(F) \in \mathfrak{D}_{1,2}$ and*

$$D_t\varphi(F) = \varphi'(F) D_t F. \tag{3.15}$$

(The chain rule can be extended to the case Lipschitz).

Malliavin Derivative and Conditional Expectation

We now present some preliminary results on conditional expectations.

Definition 3.3.2 *Let G be a Borel set on $[0,T]$. We define $\mathcal{F}_G$ to the completed $\sigma-$algebra generated by all random variables of the form :*

$$F = \int_0^T \chi_A(t)\, dB_t,$$

for all Borel sets $A \subset G$.

Lemma 3.3.1 *For any $g \in L^2[0,T]$, we have*

$$E\left[\int_0^T \mathsf{g}(t)\, dB_t \mid \mathcal{F}_G\right] = \int_0^T \chi_G(t)\, \mathsf{g}(t)\, dB_t. \tag{3.16}$$

Proof. By definition of conditional expectation, it is sufficient to verify that the random variable

$$\int_0^T \chi_G(t)\, \mathsf{g}(t)\, dB_t, \text{ is } \mathcal{F}_G\text{-mesurable,} \tag{3.17}$$

and that

$$E\left[F\int_0^T \mathsf{g}(t)\, dB_t\right] = E\left[F\int_0^T \chi_G(t)\, \mathsf{g}(t)\, dB_t\right], \tag{3.18}$$

for all bounded $\mathcal{F}_G$-measurable random variables F. To prove (3.17) we may assume that g is continuous, because the continuous functions are dense in $L^2([0,T])$. If g is continuous, then

$$\int_0^T \chi_G(t)\, \mathsf{g}(t)\, dB_t = \lim_{\Delta_{t_i}\to 0} g(t_i) \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} \chi_G(t)\, dB_t$$

where the limit is in $L^2(P)$ for the vanishing mesh Δ_{t_i} of the partitions $0 < t_1 < t_2 < \ldots < t_n = T$. Since each term in the sum is $\mathcal{F}_G$-measurable, the sum is also $\mathcal{F}_G$-measurable Then by taking a subsequence converging P-*a.s.* we conclude that the limit represents an $\mathcal{F}_G$-measurable random variable.

To prove (3.18) we may assume $F = \int_0^T \chi_A(t)\, dB_t$ for some $A \subset G$. Then by the Itô isometry we have

$$E\left[F \int_0^T \mathsf{g}(t)\, dB_t\right] = E\left[\int_0^T \chi_A(t)\, \mathsf{g}(t)\, dt\right],$$

and also

$$E\left[F \int_0^T \chi_G(t)\, \mathsf{g}(t)\, dB_t\right] = E\left[\int_0^T \chi_G(t)\, \chi_A(t)\, \mathsf{g}(t)\, dt\right] = E\left[\int_0^T \chi_A(t)\, \mathsf{g}(t)\, dt\right],$$

Then the proof can be completed by a density argument. ■

Lemma 3.3.2 *Let $G \subset [0,T]$ be a Borel set and $v = v(t)$, $t \in [0,T]$ be a stochastic process such that*

i) *for all t, $v(t)$ is measurable with respect to $\mathcal{F}_t \cap \mathcal{F}_G$,*

ii) $E\left(\int_0^T v^2(t)\, dt\right) < \infty.$

Then $\int_G v(t)\, dB_t$ is $\mathcal{F}_G$-measurable.

Lemma 3.3.3 *Let $u = u(t)$, $t \in [0,T]$, bean F-adapted stochastic process in $L^2(P \times \lambda)$. Then*

$$E\left[\int_0^T u(t)\, dB_t \mid \mathcal{F}_G\right] = \int_G E\left(u(t) \mid \mathcal{F}_G\right) dB_t.$$

Proposition 3.3.1 *Let $f_n \in \tilde{L}^2\left([0,T]^n\right)$, $n = 1, 2, ...$ Then*

$$E\left(I_n(f_n) \mid \mathcal{F}_G\right) = I_n\left(f_n \chi_G^{\otimes n}\right),$$

where $f_n \chi_G^{\otimes n}(t_1, t_2, ..., t_n) = f_n(t_1, t_2, ..., t_n)\, \chi_G(t_1) ... \chi_G(t_n)$.

Proposition 3.3.2 *If $F \in \mathcal{D}_{1,2}$, then $E[F \mid \mathcal{F}_G] \in \mathcal{D}_{1,2}$ and*

$$D_t E[F \mid \mathcal{F}_G] = E[D_t F \mid F_G]\, \chi_G(t).$$

Proof. First assume that $F = I_n(f_n)$ for some $f_n \in \tilde{L}^2([0,T]^n)$. By proposition (4.3.1), we have

$$\begin{aligned} D_t E[F \mid \mathcal{F}_G] &= D_t E[I_n(f_n) \mid \mathcal{F}_G] \\ &= D_t I_n\left(f_n \chi_G^{\otimes n}\right) \\ &= n I_{n-1}\left[f_n(.,t)\,\chi_G^{\otimes n-1}(.)\,\chi_G(t)\right] \\ &= n I_{n-1}\left[f_n(.,t)\,\chi_G^{\otimes n-1}(.)\right]\chi_G(t) \\ &= E[D_t F \mid F_G]\,\chi_G(t). \end{aligned} \tag{3.19}$$

Next, let $F = \sum_{n=0}^{\infty} I_n(f_n)$ belong to $\mathcal{D}_{1,2}$. Let $F_k = \sum_{n=0}^{k} I_n(f_n)$. Then

$$F_k \to F \text{ in } L^2(\Omega) \text{ and } D_t F_k \to D_t F \text{ in } L^2(P \times \lambda) \text{ as } k \to \infty.$$

By (3.19) we have

$$D_t E[F_k \mid \mathcal{F}_G] = E[D_t F_k \mid F_G]\,\chi_G(t),$$

for all k, and taking the limit with convergence in $L^2(P \times \lambda)$ of this, as $k \to \infty$ we obtain the result. ■

Corollary 3.3.1 *Let $u = u(s)$, $s \in [0,T]$, be an F-adapted stochastic process and assume that $u(s) \in D_{1,2}$ for all s. Then*

i) $D_t u(s)$, $s \in [0,T]$, *F-adapted for all t;*

ii) $D_t u(s) = 0$, *for $t > s$.*

Proof. By Proposition (4.3.2) we have that

$$\begin{aligned} D_t u(s) = D_t E(u(s) \mid \mathcal{F}_s) &= E(D_t u(s) \mid \mathcal{F}_s)\,\chi_{[0,s]}(t) \\ &= E(D_t u(s) \mid \mathcal{F}_s)\,\chi_{[t,T]}(s), \end{aligned}$$

from which (i) and (ii) follow immediately. ■

3.3.2 Malliavin Derivative and Skorohod Integral

The Skorohod integral is the adjoint operator to the Malliavin derivative

The following result shows that the Malliavin derivative is the adjoint operator of the Skorohod integral.

Theorem 3.3.4 (Duality formula) *Let $F \in D_{1,2}$ be $\mathcal{F}_T$-measurable and let u be a Skorohod integrable stochastic process. Then*

$$E\left[F\int_0^T u(t)\,\delta B(t)\right] = E\left[\int_0^T u(t)\,D_t F\,dt\right]. \tag{3.20}$$

Proof. Let $F = \sum\limits_{n=0}^{\infty} I_n(f_n)$ and, for all t, $u(t) = \sum\limits_{k=0}^{\infty} I_k(g_k(.,t))$ be the chaos expansions of F and $u(t)$, respectively. Then

$$\begin{aligned}
E\left[F\int_0^T u(t)\,\delta B(t)\right] &= E\left[\sum_{n=0}^{\infty} I_n(f_n)\int_0^T \sum_{k=0}^{\infty} I_k(g_k(.,t))\,\delta B(t)\right] \\
&= E\left[\sum_{n=0}^{\infty} I_n(f_n)\sum_{k=0}^{\infty} I_{k+1}(\tilde{g}_k)\right] \\
&= E\left[\sum_{k=0}^{\infty} I_{k+1}(f_{k+1})\,I_{k+1}(\tilde{g}_k)\right] \\
&= \sum_{k=0}^{\infty}(k+1)!\int_{[0,T]^{k+1}} f_{k+1}(x)\,\tilde{g}_k(x)\,dx \\
&= \sum_{k=0}^{\infty}(k+1)!\,(f_{k+1},\tilde{g}_k)_{L^2\left([0,T]^{k+1}\right)},
\end{aligned} \tag{3.21}$$

where $\tilde{g}_k$ is the symmetrization of $g_k(x_1, ..., x_n, t)$ as a function of $n+1$ variables. On the other side we have

$$
\begin{aligned}
E\left[\int_0^T u(t) D_t F dt\right] &= E\left[\int_0^T \left(\sum_{k=0}^{\infty} I_k(g_k(.,t))\right)\left(\sum_{n=1}^{\infty} n I_{n-1}(f_n(.,t))\right) dt\right] \\
&= \int_0^T \sum_{k=0}^{\infty} E\left[(k+1)(I_k(g_k(.,t))) I_k(f_{k+1}(.,t))\right] dt \\
&= \int_0^T \sum_{k=0}^{\infty} (k+1) k! \left(f_{k+1}, g_k\right)_{L^2\left([0,T]^k\right)}. \qquad (3.22)
\end{aligned}
$$

Now

$$
\begin{aligned}
\left(f_{k+1}, \tilde{g}_k\right)_{L^2\left([0,T]^{k+1}\right)} &= \int_0^T \left(f_{k+1}(.,t), \tilde{g}_k(.,t)\right)_{L^2\left([0,T]^k\right)} dt \\
&= \frac{1}{k+1} \sum_{j=1}^{k+1} \int_0^T \left(f_{k+1}(.,t), \tilde{g}_k(.,t)\right)_{L^2\left([0,T]^k\right)} dt_j \\
&= \int_0^T \left(f_{k+1}(.,t), g_k(.,t)\right)_{L^2\left([0,T]^k\right)} dt \\
&= \left(f_{k+1}, g_k\right)_{L^2\left([0,T]^{k+1}\right)}. \qquad (3.23)
\end{aligned}
$$

Therefore, by (3.21) combined with (3.18) and (3.19) the result follows ■

An Integration by Parts Formula and Closability of the Skorohod Integral

Theorem 3.3.5 (Integration by parts) *Let $u(t)$, $t \in [0,T]$ be a Skorohod integrable stochastic process and $F \in D_{1,2}$ such that the product $Fu(t)$, $t \in [0,T]$, is Skorohod integrable. Then*

$$
F \int_0^T u(t) \delta B(t) = \int_0^T F u(t) \delta B(t) + \int_0^T u(t) D_t F dt. \qquad (3.24)
$$

The duality formula is at the core of the proof of the integration by parts formula for the Skorohod integral and Malliavin derivative.

Theorem 3.3.6 (Closability of the Skorohod integral) *Suppose that* $u_n(t)$, $t \in [0,T]$, $n = 1,2,...,$ *is a sequence of Skorohod integrable stochastic processes and that the corresponding sequence of Skorohod integrals*

$$\delta(u_n) := \int_0^T u_n(t)\,\delta B_t, \;\; n = 1,2,... \tag{3.25}$$

converge in $L^2(P)$. *Moreover, suppose that*

$$\lim_{n\to\infty} u_n = 0 \;\; in \;\; L^2(P \times \lambda).$$

Then

$$\lim_{n\to\infty} \delta(u_n) = 0 \;\; in \;\; L^2(P).$$

A Fundamental Theorem of Calculus

The next result gives a useful connection between differentiation and Skorohod integration.

Theorem 3.3.7 (The fundamental theorem of calculus.) *Let* $u = u(s)$, $s \in [0,T]$ *be a stochastic process such that*

$$E\left(\int_0^T u^2(s)\,ds\right) < \infty,$$

and assume that, for all $s \in [0,T]$, $u(s) \in D_{1,2}$ *and that, for all* $t \in [0,T]$, $D_t u \in Dom(\delta)$. *Assume also that*

$$E\left(\int_0^T \left(\delta(D_t u)^2\right) dt\right) < \infty.$$

Then $\int_0^T u(s)\,\delta B_s$ *is well-defined and belongs to* $D_{1,2}$ *and*

$$D_t\left(\int_0^T u(s)\,\delta B_s\right) = \int_0^T D_t u(s)\,\delta B_s + u(t). \tag{3.26}$$

Corollary 3.3.2 *Let u be as in Theorem 4.3.7 and assume in addition that $u(s)$, $s \in [0,T]$, is $\mathcal{F}$-adapted. Then*

$$D_t \left(\int_0^T u(s)\, dB_s \right) = \int_0^T D_t u(s)\, dB_s + u(t) . \tag{3.27}$$

3.4 Clark-Ocone formula

The Clark-Ocone formula is a representation theorem for square integrable random variables in terms of Itô stochastic integrals in which the integrand is explicitly characterized in terms of the Malliavin derivative

Theorem 3.4.1 (Clark-Ocone formula) *Let $F \in D_{1,2}$ be $\mathcal{F}_T$-measurable. Then*

$$F = E(F) + \int_0^T E(D_t F \mid \mathcal{F}_t)\, dB_t. \tag{3.28}$$

Remark 3.4.1 *The formula can only be applied to random variables in $D_{1,2}$. Extensions beyond this domain to the whole $L^2(P)$ are possible in the white noise framework. Other Itô integral representations exist where the integrand is given in terms of the non-anticipating derivative. This operator is defined on the whole $L^2(P)$ See e.g. Di Nunno (2002, 2007). Some rules of calculus are given for this operator, however much has still to be discovered.*

Proof. Write $F = \sum\limits_{n=0}^{\infty} I_n(f_n)$ with $f_n \in \tilde{L}^2([0,T]^n)$, $n = 1, 2, ...$ Hence,

$$\begin{aligned}
\int_0^T E(D_t F \mid \mathcal{F}_t)\, dB_t &= \int_0^T E\left(\sum_{n=1}^{\infty} n I_{n-1}(f_n(.,t)) \mid \mathcal{F}_t\right) dB_t \\
&= \int_0^T \sum_{n=1}^{\infty} n E\left(I_{n-1}(f_n(.,t)) \mid \mathcal{F}_t\right) dB_t \\
&= \int_0^T \sum_{n=1}^{\infty} n I_{n-1}\left[(f_n(.,t)) . \chi_{[0,t]}^{\otimes n-1}(.)\right] dB_t \\
&= \int_0^T \sum_{n=1}^{\infty} n(n-1)! J_{n-1}\left[(f_n(.,t)) . \chi_{[0,t]}^{\otimes n-1}(.)\right] dB_t \\
&= \sum_{n=1}^{\infty} n! J_n(f_n) = \sum_{n=1}^{\infty} I_n(f_n) \\
&= F - I_0(f_0) = F - E(f).
\end{aligned}$$

■

3.4.1 A generalized Clark-Ocone formula

Suppose that $\tilde{B}_t = B_t + \int_0^T \theta_s ds$, where $\theta = \{\theta_t, t \in [0,T]\}$ is an adapted measurable process such that $\int_0^T \theta_t^2 dt < \infty$ almost surely. Suppose that $E[Z_T] = 1$, where the process Z_t is given by

$$Z_t = \exp\left(-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right).$$

Then by the Girsanov Theorem, the process $\tilde{B} = \left\{\tilde{B}_t,\ t \in [0,T]\right\}$ is a Brownian motion under the probability Q on $\mathcal{F}_T$ given by $\frac{dQ}{dP} = Z_t$.

The Clark-Ocone formula can be generalized in order to represent an $\mathcal{F}_T-$measurable random variable F as stochastic integral with respect to the process $\tilde{B}$. Notice that, in general, we have $\mathcal{F}_T^{\tilde{\mathcal{B}}} \subset \mathcal{F}_T$ $\left(\text{where } \left\{\mathcal{F}_t^{\tilde{\mathcal{B}}},\ 0 \le t \le T\right\} \text{ denotes the family of } \sigma\text{-fields generated by } \tilde{B}\right)$ and usually $\mathcal{F}_T^{\tilde{\mathcal{B}}} \neq \mathcal{F}_T$. Thus, an $\mathcal{F}_T$-measurable random variable F may not be $\mathcal{F}_T^{\tilde{\mathcal{B}}}$-measurable and we cannot obtain a representation of F as an integral with respect to $\tilde{B}$

simply by applying the Clark-Ocone formula to the Brownian motion $\tilde{B}$ on the probability space $\left(\Omega, \mathcal{F}_T^{\tilde{\mathcal{B}}}, Q\right)$.

Theorem 3.4.2 (Clark-Ocone formula under change of measure) *Let F be an $\mathcal{F}_T$-measurable random variable such that $F \in \mathfrak{D}^{1,2}$ and let $\theta \in L^{1,2}$. Assume*

1. $E\left(Z_T^2 F^2\right) + E\left(Z_T^2 \int_0^T \left(D_t F\right)^2 dt\right) < \infty,$

2. $E\left(Z_T^2 F^2 \int_0^T \left(\theta_t + \int_t^T D_t \theta_s dB_s + \int_t^T \theta_s D_t \theta_s ds\right)^2 dt\right) < \infty.$

Then

$$F = E_Q\left(F\right) + \int_0^T E_Q\left(D_t F - F \int_t^T D_t \theta_s d\tilde{B}_s \mid \mathcal{F}_t\right) d\tilde{B}_s. \tag{3.29}$$

The proof of this theorem can be found in [49] page 337.

Chapter 4

A Malliavin calculus in stochastic control problems

In this chapiter we will present the tow results one is established by Brandis, Øksendal and Zhou [47] and the other is estabilished by Øksendal and Sulem [51] witch treat singular control problem, both of them established stochastic maximum principle, where they considers controlled Itô-L'evy process where the information available to the controller is possibly less than the overall information. All the system coefficients and the objective performance functional are allowed to be random, possibly non-Markovian. Malliavin calculus is employed to derive a maximum principle for the optimal control of such a system.

4.1 A stochastic maximum principle via Malliavin calculus

4.1.1 Formulation of the problem

Suppose the state process $X(t) = X^u(t,u)$; $t \geqslant 0$, $\omega \in \Omega$, s a controlled Itô-L'evy process in $\mathbb{R}$ of the form of the form

$$\begin{cases} dx_t = b(t, x(t), u(t), \omega)\, dt + \sigma(t, x(t), u(t), \omega)\, dB_t \\ + \int\limits_{\mathbb{R}_0} \theta(t, x(t^-), u(t^-), z, \omega)\, \tilde{N}(dt, dz); \\ x(0) = x \in \mathbb{R}. \end{cases} \tag{4.1}$$

Here $\mathbb{R}_0 = \mathbb{R} - \{0\}$, $B(t) = B(t,\omega)$, and $\eta(t) = \eta(t,\omega)$, given by

$$\eta(t) = \int_0^t \int\limits_{\mathbb{R}_0} z\tilde{N}(ds, dz);\ t \geqslant 0,\ \omega \in \Omega,$$

are a 1-dimensional Brownian motion and an independent pure jump L'evy martingale, respectively, on a given filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P\right)$. Thus

$$\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)\, dt,$$

is the *compensated jump measure* of $\eta(.)$, where $N(dt, dz)$ is the *jump measure* and $\nu(dz)$ is the *L'evy measure* of the *L'evy process* $\eta(.)$ The process $u(t)$ is our control process, assumed to be $\mathcal{F}_t$-adapted and have values in a given open convex set $U \subset \mathbb{R}$. The coefficients

$$b : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R},$$

$$\sigma : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R},$$

$$\theta : [0,T] \times \mathbb{R} \times U \times \mathbb{R}_0 \times \Omega \to \mathbb{R},$$

are given $\mathcal{F}_t-$predictable processes. for more information about stochastic control of Itô diffusions and jump diffusions one can see [32]. Let $T > 0$ be a given constant. For simplicity, we assume that

$$\int_{\mathbb{R}_0} z^2 \nu (dz) < \infty.$$

Suppose in addition that we are given a sub-filtration $\varepsilon_t \subset \mathcal{F}_t$, $t \in [0,T]$, representing the information available to the controller at time t and satisfying the usual conditions, meaning that the controller gets a delayed information compared to $\mathcal{F}_t$.

Let $\mathcal{A} = \mathcal{A}_\varepsilon$ denote a given family of controls, contained in the set of ε_t-adapted càdlàg controls $u(.)$ such that (4.1) has a unique strong solution up to time T. Suppose we are given a performance functional of the form

$$J(u) = E\left[\int_0^T f(t, x(t), u(t), \omega)\, dt + g(x(T), \omega)\right];\ u \in \mathcal{A}_\varepsilon,$$

where $E = E_p$ denotes expectation with respect to P and $f : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$ and $g : \mathbb{R} \times \Omega \to \mathbb{R}$ are given $\mathcal{F}_t$-adapted processes with

$$E\left[\int_0^T |f(t, x(t), u(t))|\, dt + |g(x(T))|\right] < \infty \ \text{ for all } u \in \mathcal{A}_\varepsilon.$$

The partial information control problem we consider is the following:

Problem 4.1.1 *Find $\Phi_\varepsilon \in \mathbb{R}$ and $u^* \in \mathcal{A}_\varepsilon$, (if it exists) such that*

$$\Phi_\varepsilon = \sup_{u \in \mathcal{A}_\varepsilon} (J(u)) = J(u^*).$$

Remark 4.1.1 *This problem is not of Markovian type, because b, σ, θ, f and g are allowed to be stochastic processes and also because our controls must be ε_t-adapted, and hence cannot be solved by dynamic programming. We instead investigate the maximum principle, and derive an explicit form for the adjoint process.*

We have the following assumptions

Assumption

1) The functions b, σ, f and g are all continuously differentiable (C^1) with respect to $x \in \mathbb{R}$ and $u \in U$ for each $t \in [0,T]$ and $a.a.\omega \in \Omega$.

2) For all $t, r \in (0,T)$ $t \leq r$, and all bounded ε_t-measurable random variables $\alpha = \alpha(\omega)$ the control $\beta_\alpha(s) = \alpha(\omega)\chi_{[t,r]}(s)$; $s \in [0,T]$ belongs to $\mathcal{A}_\varepsilon$.

3) For all $u, \beta \in \mathcal{A}_\varepsilon$ with β bounded, there exists $\delta > 0$ such that $u + y\beta \in \mathcal{A}_\varepsilon$ for all $y \in (-\delta, \delta)$, and such that the family and such that the family

$$\left\{ \frac{\partial f}{\partial x}\left(t, x^{u+y\beta}(t), u(t) + y\beta(t)\right) \frac{d}{dy} x^{u+y\beta}(t) + \frac{\partial f}{\partial u}\left(t, x^{u+y\beta}(t), u(t) + y\beta(t)\right) \beta(t) \right\}_{y \in (-\delta,\delta)},$$

is $\lambda \times P$-uniformly integrable and the family

$$\left\{ g'\left(x^{u+y\beta}(T)\right) \frac{d}{dy} x^{u+y\beta}(T) \right\}_{y \in (-\delta,\delta)},$$

is P-uniformly integrable.

4) For all $u, \beta \in \mathcal{A}_\varepsilon$ with β bounded the process $y(t) = y^\beta(t) = \frac{d}{dy} x^{u+y\beta}(t)\,|_{y=0}$ exists and satisfies the equation

$$\begin{aligned} dy(t) = {} & \left[y\left(t^-\right) \frac{\partial b}{\partial x}(t, x(t), u(t))\,dt + \frac{\partial \sigma}{\partial x}(t, x(t), u(t))\,dB(t) \right. \\ & \left. + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}\left(t, x\left(t^-\right), u\left(t^-\right), z\right) \tilde{N}(dt, dz) \right] \\ & + \beta\left(t^-\right) \left[\frac{\partial b}{\partial u}(t, x(t), u(t))\,dt + \frac{\partial \sigma}{\partial u}(t, x(t), u(t))\,dB(t) \right. \\ & \left. + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial u}\left(t, x\left(t^-\right), u\left(t^-\right), z\right) \tilde{N}(dt, dz) \right]; \\ y(0) = {} & 0. \end{aligned}$$

5) For all $u \in \mathcal{A}_\varepsilon$, the following processes

$$\begin{aligned} K(t) &:= g'(x(T)) + \int_t^T \frac{\partial}{\partial x} f(s, x(s), u(s))\,ds, \\ D_t K(t) &:= D_t g'(x(T)) + \int_t^T D_t \frac{\partial}{\partial x} f(s, x(s), u(s))\,ds, \\ D_{t,z} K(t) &:= D_{t,z} g'(x(T)) + \int_t^T D_{t,z} \frac{\partial}{\partial x} f(s, x(s), u(s))\,ds, \\ H_0(s, x, u) &= k(s)\, b(s, x, u) + D_s K(s)\, \sigma(s, x, u) \\ &\quad + \int_{\mathbb{R}_0} D_{s,z} K(s)\, \theta(s, x, u, z)\, \nu(dz), \\ G(t, s) &:= \exp\left(\int_t^s \left\{ \frac{\partial}{\partial x} b(r, x(r), u(r), \omega) - \frac{1}{2} \left(\frac{\partial \sigma}{\partial x} \right)^2 (r, x(r), u(r), \omega) \right\} dB(r) \right. \\ &\quad + \int_t^s \frac{\partial \sigma}{\partial x}(r, x(r), u(r), \omega)\, dB(r) \\ &\quad + \int_t^s \int_{\mathbb{R}_0} \left\{ \ln\left(1 + \frac{\partial \theta}{\partial x}(r, x(r), u(r), z, \omega) \right) - \frac{\partial \theta}{\partial x}(r, x(r), u(r), z, \omega) \right\} \nu(dz)\, dr \\ &\quad + \int_t^s \int_{\mathbb{R}_0} \ln\left(1 + \frac{\partial \theta}{\partial x}\left(r, x\left(r^-\right), u\left(r^-\right), z, \omega\right) \right) \tilde{N}(dr, dz), \qquad (4.2) \end{aligned}$$

$$p(t) := K(t) + \int_t^T \frac{\partial H_0}{\partial x}(s, x(s), u(s))\, G(t,s)\, ds, \tag{4.3}$$

$$q(t) := D_t p(t), \text{ and} \tag{4.4}$$

$$r(t,z) := D_{t,z} p(t), \tag{4.5}$$

all exist for $0 \le t \le s \le T$, $z \in \mathbb{R}_0$

We now define the Hamiltonian of this general problem

Definition 4.1.1 (The general stochastic Hamiltonian) *The general stochastic Hamiltonian is the process*

$$H : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R},$$

defined by

$$H(t,x,u,\omega) = f(t,x,u,\omega) + p(t)\, b(t,x,u,\omega) + q(t)\, \sigma(t,x,u,\omega) + \int_{\mathbb{R}_0} r(t,z)\, \theta(t,x,u,z,\omega)\, \nu(dz). \tag{4.6}$$

4.1.2 The stochastic maximum principle

We can now formulate the stochastic maximum principle:

Theorem 4.1.1 (Maximum Principle) **1.** *Suppose $u \in \mathcal{A}_\varepsilon$ is a critical point for $J(u)$, in the sense that*

$$\frac{d}{dy} J(\hat{u} + y\beta)\,|_{y=0} = 0 \text{ for all bounded } \beta \in \mathcal{A}_\varepsilon. \tag{4.7}$$

Then

$$E\left[\frac{\partial \hat{H}}{\partial u}(t, \hat{x}(t), \hat{u}(t)) \mid \varepsilon_t\right] = 0 \text{ for a.a. } t,\ \omega, \tag{4.8}$$

where

$$\hat{x}(t) = x^{\hat{u}}(t),$$

$$\hat{H}(t,\hat{x}(t),u) = f(t,\hat{x}(t),u) + \hat{p}b(t,\hat{x}(t),u) + \hat{q}(t)\sigma(t,\hat{x}(t),u)$$
$$\int_{\mathbb{R}_0} \hat{r}(t,z)\theta(t,\hat{x}(t),u,z)\nu(dz),$$

with

$$\hat{p}(t) = \hat{k}(t) + \int_t^T \frac{\partial H_0}{\partial x}(s,\hat{x}(s),\hat{u}(s))\hat{G}(t,s)\,ds,$$

and

$$\begin{aligned}\hat{G}(t,s) &:= \exp\left(\int_t^s \left\{\frac{\partial}{\partial x}b(r,\hat{x}(r),u(r),\omega) - \frac{1}{2}\left(\frac{\partial\sigma}{\partial x}\right)^2(r,\hat{x}(r),u(r),\omega)\right\}dB(r)\right.\\ &+ \int_t^s \frac{\partial\sigma}{\partial x}(r,\hat{x}(r),u(r),\omega)\,dB(r)\\ &+ \int_t^s\int_{\mathbb{R}_0}\left\{\ln\left(1+\frac{\partial\theta}{\partial x}(r,\hat{x}(r),u(r),z,\omega)\right) - \frac{\partial\theta}{\partial x}(r,\hat{x}(r),u(r),z,\omega)\right\}\nu(dz)\,dr\\ &+ \int_t^s\int_{\mathbb{R}_0}\ln\left(1+\frac{\partial\theta}{\partial x}\left(r,\hat{x}\left(r^-\right),u\left(r^-\right),z,\omega\right)\right)\tilde{N}(dr,dz),\end{aligned}$$

$$\hat{K}(t) = g'(\hat{x}(T)) + \int_t^T \frac{\partial}{\partial x}f(s,\hat{x}(s),u(s))\,ds,$$

2. *Conversely, suppose there exists* $\hat{u} \in \mathcal{A}_\varepsilon$ *such that* (4.8) *holds. Then* $\hat{u}$ *satisfies* (4.7).

Proof.

1. We suppose that u is a critical point for $J(u)$, we have by assumption 3

$$\begin{aligned}0 &= \frac{d}{dy}J(u+y\beta)\,|_{y=0}\\ &= E\left[\int_0^T \left\{\frac{\partial f}{\partial x}(t,x(t),u(t))\,y(t) + \frac{\partial f}{\partial u}(t,x(t),u(t))\,\beta(t)\right\}dt\right.\\ &+ g'(x(T))\,y(T),\end{aligned}$$

where

$$\begin{aligned} y(t) &= \frac{d}{dy} x^{(u+y\beta)}(t)\,|_{y=0} \\ &= \int_0^t \left\{ \frac{\partial b}{\partial x}(s, x(s), u(s))\, y(s) + \frac{\partial b}{\partial u}(s, x(s), u(s))\, \beta(s) \right\} ds \\ &+ \int_0^t \left\{ \frac{\partial \sigma}{\partial x}(s, x(s), u(s))\, y(s) + \frac{\partial \sigma}{\partial u}(s, x(s), u(s))\, \beta(s) \right\} dB_s \\ &+ \int_0^t \int_{\mathbb{R}_0} \left\{ \frac{\partial \theta}{\partial x}(s, x(s), u(s), z)\, y(s) + \frac{\partial \theta}{\partial u}(s, x(s), u(s), z)\, \beta(s) \right\} \tilde{N}(ds, dz). \end{aligned} \tag{4.9}$$

From now on we use the short hand notation

$$\frac{\partial f}{\partial x}(t, x(t), u(t)) = \frac{\partial f}{\partial x}(t), \;\; \frac{\partial f}{\partial u}(t, x(t), u(t)) = \frac{\partial f}{\partial u}(t),$$

and similarly for $\frac{\partial b}{\partial x}$, $\frac{\partial b}{\partial u}$, $\frac{\partial \sigma}{\partial x}$, $\frac{\partial b}{\partial u}$, $\frac{\partial \theta}{\partial x}$, and $\frac{\partial \theta}{\partial u}$. By replacing $y(T)$ by its value, and using the duality formulas, we get

$$\begin{aligned} &E\left[g'(x(T))\, y(T)\right] \\ &= E\left[g'(x(T)) \left(\int_0^T \left\{ \frac{\partial b}{\partial x}(s)\, y(s) + \frac{\partial b}{\partial u}(s)\, \beta(s) \right\} ds \right.\right. \\ &+ \int_0^T \left\{ \frac{\partial \sigma}{\partial x}(s)\, y(s) + \frac{\partial \sigma}{\partial u}(s)\, \beta(s) \right\} dB_s \\ &\left.\left. + \int_0^T \int_{\mathbb{R}_0} \left\{ \frac{\partial \theta}{\partial x}(s)\, y(s) + \frac{\partial \theta}{\partial u}(s)\, \beta(s) \right\} \tilde{N}(ds, dz) \right) \right] \\ &= E\left[\int_0^T \left\{ g'(x(T)) \frac{\partial b}{\partial x}(s)\, y(s) + g'(x(T)) \frac{\partial b}{\partial u}(s)\, \beta(s) \right.\right. \\ &+ \int_0^T D_t\left(g'(x(T))\right) \frac{\partial \sigma}{\partial x}(s)\, y(s) + D_t\left(g'(x(T))\right) \frac{\partial \sigma}{\partial u}(s)\, \beta(s) \\ &\left.\left. + \int_{\mathbb{R}_0} \left[D_{t,z}\left(g'(x(T))\right) \frac{\partial \theta}{\partial x}(s)\, y(s) + D_{t,z}\left(g'(x(T))\right) \frac{\partial \theta}{\partial u}(s)\, \beta(s) \right] \nu(dz) \right\} ds \right]. \end{aligned} \tag{4.10}$$

Similarly we have,

$$
\begin{aligned}
&E\left[\int_0^T \frac{\partial f}{\partial x}(t)\, y(t)\, dt\right] \\
&= E\left[\int_0^T \frac{\partial f}{\partial x}(t)\left(\int_0^t \left\{\frac{\partial b}{\partial x}(s)\, y(s) + \frac{\partial b}{\partial u}(s)\, \beta(s)\right\} ds\right.\right. \\
&+ \int_0^t \left\{\frac{\partial \sigma}{\partial x}(s)\, y(s) + \frac{\partial \sigma}{\partial u}(s)\, \beta(s)\right\} dB_s \\
&\left.\left.+ \int_0^t \int_{\mathbb{R}_0} \left\{\frac{\partial \theta}{\partial x}(s)\, y(s) + \frac{\partial \theta}{\partial u}(s)\, \beta(s)\right\} \tilde{N}(ds, dz)\right)\right] \\
&= E\left[\int_0^T \left(\int_0^t \left\{\frac{\partial f}{\partial x}(t)\frac{\partial b}{\partial x}(s)\, y(s) + \frac{\partial f}{\partial x}(t)\frac{\partial b}{\partial u}(s)\, \beta(s)\right.\right.\right. \\
&+ D_t\left(\frac{\partial f}{\partial x}(t)\right)\frac{\partial \sigma}{\partial x}(s)\, y(s) + D_t\left(\frac{\partial f}{\partial x}(t)\right)\frac{\partial \sigma}{\partial u}(s)\, \beta(s) \\
&\left.\left.\left.+ \int_{\mathbb{R}_0}\left[D_{t,z}\left(\frac{\partial f}{\partial x}(t)\right)\frac{\partial \theta}{\partial x}(s)\, y(s) + D_{t,z}\left(\frac{\partial f}{\partial x}(t)\right)\frac{\partial \theta}{\partial u}(s)\, \beta(s)\right]\nu(dz)\right\} ds\right) dt\right]
\end{aligned}
$$

then by using the Fubini theorem,

$$
\begin{aligned}
&E\left[\int_0^T \frac{\partial f}{\partial x}(t)\, y(t)\, dt\right] \\
&= E\left[\int_0^T \left\{\int_s^T \frac{\partial f}{\partial x}(t)\, dt\right\}\left[\frac{\partial b}{\partial x}(s)\, y(s) + \frac{\partial b}{\partial u}(s)\, \beta(s)\right]\right. \\
&+ \left\{\int_s^T D_s\left(\frac{\partial f}{\partial x}(t)\right) dt\right\}\left[\frac{\partial \sigma}{\partial x}(s)\, y(s) + \frac{\partial \sigma}{\partial u}(s)\, \beta(s)\right] \\
&\left.\left\{\int_s^T D_{s,z}\left(\frac{\partial f}{\partial x}(t)\right) dt\right\}\int_{\mathbb{R}_0}\left[\frac{\partial \theta}{\partial x}(s)\, y(s) + \frac{\partial \theta}{\partial u}(s)\, \beta(s)\right]\nu(dz)\right\} ds\Bigg].
\end{aligned}
$$

Changing the notation $s \to t$, this becomes

$$
\begin{aligned}
E\left[\int_0^T \frac{\partial f}{\partial x}(t)\, y(t)\, dt\right] &= E\left[\int_0^T \left\{\int_t^T \frac{\partial f}{\partial x}(s)\, ds\right\}\left[\frac{\partial b}{\partial x}(t)\, y(t) + \frac{\partial b}{\partial u}(t)\, \beta(t)\right]\right. \\
&+ \left\{\int_t^T D_t\left(\frac{\partial f}{\partial x}(s)\right) ds\right\}\left[\frac{\partial \sigma}{\partial x}(t)\, y(t) + \frac{\partial \sigma}{\partial u}(t)\, \beta(t)\right] \\
&\left.+ \left\{\int_t^T D_{t,z}\left(\frac{\partial f}{\partial x}(s)\right) ds\right\}\int_{\mathbb{R}_0}\left[\frac{\partial \theta}{\partial x}(t)\, y(t) + \frac{\partial \theta}{\partial u}(t)\, \beta(t)\right]\nu(dz)\right\} dt\Bigg].
\end{aligned}
\tag{4.11}
$$

Recall

$$K(t) := g'(x(T)) + \int_t^T \frac{\partial f}{\partial x}(s)\,ds. \tag{4.12}$$

By combining (4.10)-(4.12), we get

$$E\left[\int_0^T \left\{K(t)\left(\frac{\partial b}{\partial x}(t)\,y(t) + \frac{\partial b}{\partial u}(t)\,\beta(t)\right)\right.\right.$$
$$+ D_t K(t)\left(\frac{\partial \sigma}{\partial x}(t)\,y(t) + \frac{\partial \sigma}{\partial u}(t)\,\beta(t)\right)$$
$$\left.\left. + \int_{\mathbb{R}_0} D_{t,z}K(t)\left(\frac{\partial \theta}{\partial x}(t)\,y(t) + \frac{\partial \theta}{\partial u}(t)\,\beta(t)\right)\nu(dz) + \frac{\partial f}{\partial u}(t)\,\beta(t)\right\} dt\right]$$
$$= 0. \tag{4.13}$$

Now we apply the above to $\beta = \beta_\alpha \in \mathcal{A}_\varepsilon$ of the form $\beta_\alpha(s) = \alpha\chi_{[t,t+h]}(s)$, for some $t, h \in (0,T)$, $t+h \leq T$, where $\alpha = \alpha(\omega)$ is bounded and ε_t−measurable. We have, by 4.9 for $0 \leq s \leq t$

$$y^{\beta_\alpha}(s) = \int_0^s \left\{\frac{\partial b}{\partial x}(r)\,y^{\beta_\alpha}(r)\right\} dr$$
$$+ \int_0^s \frac{\partial \sigma}{\partial x}(r)\,y^{\beta_\alpha}(r)\,dB_r$$
$$+ \int_0^s \int_{\mathbb{R}_0} \left[\frac{\partial \theta}{\partial x}(r)\,y^{\beta_\alpha}(r)\right] N(dr,dz).$$

Then $y^{\beta_\alpha}(s) = 0$, for $0 \leq s \leq t$ and hence (4.12) becomes

$$A_1 + A_2 = 0. \tag{4.14}$$

where

$$A_1 = E\left[\int_t^T \left\{K(s)\frac{\partial b}{\partial x}(s) + D_s K(s)\frac{\partial \sigma}{\partial x}(s) + \int_{\mathbb{R}_0} D_{s,z}K(s)\frac{\partial \theta}{\partial x}(t)\,\nu(dz)\right\} y^{\beta_\alpha}(s)\,ds\right]$$

$$A_2 = E\left[\int_t^{t+h} \left\{K(s)\frac{\partial b}{\partial u}(s) + D_s K(s)\frac{\partial \sigma}{\partial u}(s) + \int_{\mathbb{R}_0} D_{s,z}K(s)\frac{\partial \theta}{\partial u}(t)\,\nu(dz)\right\} \alpha\,ds\right].$$

In the other hand for $s \geqslant t+h$ we have by (4.9) with $y(s) = y^{\beta_\alpha}(s)$

$$dy(s) = y(s^-)\left\{\frac{\partial b}{\partial x}(s)\,ds + \frac{\partial \sigma}{\partial x}(s) + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(t)\,\tilde{N}(dr,dz)\right\}, \tag{4.15}$$

with initial condition $y(t+h)$ in time $t+h$. This equation can be solved explicitly and we get

$$y(s) = y(t+h)\,G(t+h,s)\,;\;\; s \geqslant t+h, \tag{4.16}$$

where, in general, for $s \geqslant t$,

$$\begin{aligned} G(t,s) := \exp\Bigg(&\int_t^s \left\{\frac{\partial b}{\partial x}(r) - \frac{1}{2}\left(\frac{\partial \sigma}{\partial x}\right)^2(r)\right\} dB(r) \\ &+ \int_t^s \frac{\partial \sigma}{\partial x}(r)\,dB(r) \\ &+ \int_t^s \int_{\mathbb{R}_0} \left\{\ln\left(1 + \frac{\partial \theta}{\partial x}(r)\right) - \frac{\partial \theta}{\partial x}(r)\right\} \nu(dz)\,dr \\ &+ \int_t^s \int_{\mathbb{R}_0} \ln\left(1 + \frac{\partial \theta}{\partial x}(r)\right) \tilde{N}(dr,dz), \end{aligned} \tag{4.17}$$

$y(s)$ given in (4.16) is the solution of (4.17), it can be verified by applying the Itô formula to $y(s)$ given in (4.16).We define

$$H_0(s,x,u) = K(s)\,b(s,x,u) + D_s K(s)\,\sigma(s,x,u) + \int_{\mathbb{R}_0} D_{s,z} K(s)\,\theta(s,x,u,z)\,\nu(dz)$$

Then

$$A_1 = E\left[\int_t^T \frac{\partial H_0}{\partial x}(s)\,y(s)\,ds\right].$$

Differentiating with respect to h at $h=0$ we get

$$\frac{d}{dh} A_1 \mid_{h=0} = \frac{d}{dh} E\left[\int_t^{t+h} \frac{\partial H_0}{\partial x}(s)\,y(s)\,ds\right] \mid_{h=0} + \frac{d}{dh} E\left[\int_{t+h}^T \frac{\partial H_0}{\partial x}(s)\,y(s)\,ds\right] \mid_{h=0}.$$

Since $y(t) = 0$ and since $\dfrac{\partial H_0}{\partial x(s)}$ is càdlàg we see that

$$\frac{d}{dh}E\left[\int_t^{t+h}\frac{\partial H_0}{\partial x}(s)\,y(s)\,ds\right]_{h=0} = 0. \tag{4.18}$$

Therefore, using (4.16) and $y(t) = 0$,

$$\begin{aligned}
\frac{d}{dh}A_{1|_{h=0}} &= \frac{d}{dh}E\left[\int_{t+h}^{T}\frac{\partial H_0}{\partial x}(s)\,y(t+h)\,G(t+h,s)\,ds\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left\{E\left[\int_{t+h}^{T}\frac{\partial H_0}{\partial x}(s)\,y(t+h)\,G(t+h,s)\,ds\right]\right\} \\
&= \int_t^T \lim_{h\to 0}\frac{1}{h}E\left[\frac{\partial H_0}{\partial x}(s)\,y(t+h)\,G(t+h,s)\right]ds \\
&= \int_t^T \frac{d}{dh}E\left[\frac{\partial H_0}{\partial x}(s)\,y(t+h)\,G(t+h,s)\right]_{h=0}ds \\
&= \int_t^T \frac{d}{dh}E\left[\frac{\partial H_0}{\partial x}(s)\,y(t+h)\,G(t,s)\right]_{h=0}ds,
\end{aligned} \tag{4.19}$$

by (4.9)

$$\begin{aligned}
y(t+h) = {}& \alpha\int_t^{t+h}\left\{\frac{\partial b}{\partial u}(r)\,dr + \frac{\partial \sigma}{\partial u}(r)\,dB_r + \int_{\mathbb{R}_0}\frac{\partial \theta}{\partial u}(r)\,\tilde{N}(dr,dz)\right\} \\
&+ \int_t^{t+h} y\left(r^-\right)\left\{\frac{\partial b}{\partial x}(r)\,dr + \frac{\partial \sigma}{\partial x}(r)\,dB_r + \int_{\mathbb{R}_0}\frac{\partial \theta}{\partial x}(r)\,\tilde{N}(dr,dz)\right\}.
\end{aligned} \tag{4.20}$$

Therefore, by (4.19) and (4.20)

$$\frac{d}{dh}A_{1|_{h=0}} = \Lambda_1 + \Lambda_2,$$

$$\begin{aligned}
\Lambda_1 = {}& \int_t^T \frac{d}{dh}E\left[\frac{\partial H_0}{\partial x}(s)\,G(t,s)\,\alpha\int_t^{t+h}\left\{\frac{\partial b}{\partial u}(r)\,dr + \frac{\partial \sigma}{\partial u}(r)\,dB_r\right.\right. \\
&\left.\left.+\int_{\mathbb{R}_0}\frac{\partial \theta}{\partial u}(r)\,\tilde{N}(dr,dz)\right\}dr\right]_{h=0}ds,
\end{aligned} \tag{4.21}$$

and

$$\Lambda_2 = \int_t^T \frac{d}{dh} E\left[\frac{\partial H_0}{\partial x}(s)\, G(t,s) \int_t^{t+h} y\left(r^-\right) \left\{\frac{\partial b}{\partial x}(r)\, dr + \frac{\partial \sigma}{\partial x}(r)\, dB_r \right.\right.$$
$$\left.\left. + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(r)\, \tilde{N}(dr,dz)\right\} dr\right]_{h=0} ds. \tag{4.22}$$

By the duality formulas, we get

$$\Lambda_1 = \int_t^T \frac{d}{dh} E\left[\alpha \int_t^{t+h} \left\{\frac{\partial b}{\partial u}(r)\, F(t,s)\, dr + \frac{\partial \sigma}{\partial u}(r)\, D_r F(t,s)\, dB_r \right.\right.$$
$$\left.\left. + \int_{\mathbb{R}_0} D_{r,z} \frac{\partial \theta}{\partial u}(r)\, \tilde{N}(dr,dz)\right\} dr\right]_{h=0} ds,$$
$$= \int_t^T E\left[\alpha \left\{\frac{\partial b}{\partial u}(t)\, F(t,s)\, dt + \frac{\partial \sigma}{\partial u}(t)\, D_r F(t,s)\, dB_t + \int_{\mathbb{R}_0} D_{t,z} F(t,s)\, \frac{\partial \theta}{\partial u}(t)\, \nu(dz)\right\}\right] ds, \tag{4.23}$$

such that

$$F(t,s) = \frac{\partial H_0}{\partial x}(s)\, G(t,s).$$

Since $y(t) = 0$ we see that $\Lambda_2 = 0$. We conclude that

$$\frac{d}{dh} A_{1|_{h=0}} = \Lambda_1$$

Moreover, we see directly that

$$\frac{d}{dh} A_{2|_{h=0}} = E\left[\alpha \left\{K(t)\, \frac{\partial b}{\partial u}(t)\, dt + \frac{\partial \sigma}{\partial u}(t)\, D_r K(t)\, dB_t \right.\right.$$
$$\left.\left. + \int_{\mathbb{R}_0} D_{t,z} K(t)\, \frac{\partial \theta}{\partial u}(t)\, \nu(dz) + \frac{\partial f}{\partial u}(t)\right\}\right].$$

Therefore, differentiating (4.14) with respect to h at $h = 0$ gives the equation

$$E\left[\alpha\left\{\left(K(t)+\int_t^T F(t,s)\,ds\right)\frac{\partial b}{\partial u}(t)+D_t\left(K(t)+\int_t^T F(t,s)\,ds\right)\frac{\partial\sigma}{\partial u}(t)\right.\right.$$
$$\left.\left.+\int_{\mathbb{R}_0}D_{t,z}\left(K(t)+\int_t^T F(t,s)\,ds\right)\frac{\partial\theta}{\partial u}(t)\left(\nu(dz)+\frac{\partial f}{\partial u}(t)\right)\right\}\right]$$
$$=0. \tag{4.24}$$

if we put

$$p(t)=K(t)+\int_t^T F(t,s)\,ds=K(t)+\int_t^T\frac{\partial H_0}{\partial x}(s)\,G(t,s)\,ds,$$

then (4.24) can be written

$$E\left[\frac{\partial}{\partial u}\left\{f(t,x_t,u)+p(t)\,b(t,x_t,u)+D_tp(t)\,\sigma(t,x_t,u)\right.\right.$$
$$\left.\left.+\int_{\mathbb{R}_0}D_{t,z}p(t)\,\theta(t,x_t,u)\,\nu(dz)\right\}_{u=u(t)}\alpha\right]$$
$$=0.$$

Since this holds for all bounded ε_t-measurable random variable , we conclude that

$$E\left(\frac{\partial H}{\partial u}(t,x_t,u)_{u=u(t)}\mid\varepsilon_t\right)=0,\ \text{for } a.a.\ t,\ \omega.$$

witch complete the proof of the first part.

ii) Conversely, suppose (4.8) holds for some $\hat{u}\in\mathcal{A}_\varepsilon$. Then by reversing the above argument we get that (4.14) holds for all $\beta_\alpha\in\mathcal{A}_\varepsilon$ of the form $\beta_\alpha(s,\omega)=\alpha\chi_{(t,t+h]}(s)$ for some $t,h\in[0,T]$ with $t+h\leq T$ and some bounded ε_t-measurable α. Hence (4.14) holds for all linear combinations of such β_α.. Since all bounded $\beta\in\mathcal{A}_\varepsilon$ can be approximated pointwise boundedly in (t,ω) by such linear combinations, it follows that (4.14) holds for all bounded $\beta\in\mathcal{A}_\varepsilon$. Hence, by reversing the remaining part of the argument above, we conclude that (4.14) holds.

■

4.1.3 Application

We now give an example of application

Example 4.1.1 (Optimal portfolio) *Suppose we have a financial market with the following two investment possibilities:*

i) *A risk free asset, where the unit price* $S_0(t)$ *at time* t *is given by*

$$dS_0(t) = \rho_t S_0(t)\,dt;\;\; S_0(0) = 1;\;\; t \in [0,T]. \tag{4.25}$$

ii) *A risky asset, where the unit price* $S_1(t)$ *at time* t *is given by*

$$dS_1(t) = S_1(t^-)\left[\alpha_t dt + \beta_t dB_t + \int_{\mathbb{R}_0} \zeta(t,z)\,\tilde{N}(dt,dz);\right] \quad t \in [0,T], \tag{4.26}$$
$$S_1(0) > 0.$$

Here $\rho_t, \alpha_t, \beta_t$ *and* $\zeta(t,z)$ *are bounded* $\mathcal{F}_t$*-predictable processes,* $t \in [0,T]$, $z \in \mathbb{R}_0$ *and* $T > 0$ *is a given constant. We also assume that*

$$\zeta(t,z) \geqslant -1 \quad a.s. \text{ for } a.a.\ t, z,$$

and

$$E\left[\int_0^T \int_{\mathbb{R}_0} |\log(1+\zeta(t,z))|^2\,\nu(dz)\,dt\right] < \infty.$$

A portfolio in this market is an ε_t*-predictable process* $u(t)$ *representing the amount invested in the risky asset at time* t*. When the portfolio* $u(.)$ *is chosen, the corresponding wealth* *process $x(t) = x^u(t)$ satisfies the equation*

$$dx(t) = [\rho_t x(t) + (\alpha_t - \rho_t) u(t)] dt + \beta_t u(t) dB_t + \int_{\mathbb{R}_0} \zeta(t,z) \tilde{N}(dt,dz);$$
$$x(0) = x > 0. \tag{4.27}$$

The partial information optimal portfolio problem is to find the portfolio $u \in \mathcal{A}_\varepsilon$ which maximizes

$$J(u) = E[U(x^u(T,\omega))],$$

where $U(x) = U(x,\omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ is a given $\mathcal{F}_t$-measurable random variable for each x and $x \to U(x,\omega)$ is a utility function for each ω. We assume that $x \to U(x)$ is C^1 and $U'(x)$ is strictly decreasing.

With the notation of the previous section we see that in this case we have

$$f(t,x,u) = 0, \;\; g(x,\omega) = U(x,\omega),$$

$$b(t,x,u) = \rho_t x_t + (\alpha_t - \rho_t) u, \quad \sigma(t,x,u) = \beta_t u,$$
$$\theta(t,x,u,z) = \zeta(t,z) u,$$

thus

$$K(t) = U'(x(T)) = K,$$

and

$$H_0(t,x,u) = K(\rho_t x_t + (\alpha_t - \rho_t) u) + D_t K \beta_t u + \int_{\mathbb{R}_0} D_{t,z} K \zeta(t,z) \tilde{N}(dt,dz),$$

and

$$G(t,s) = \exp\left(\int_t^s \rho_r dr\right).$$

Thus

$$p(t) = U'(x(T)) + \int_t^T K\rho_s \exp\left(\int_t^s \rho_r dr\right) ds,$$

and the Hamiltonian becomes

$$\begin{aligned} H(t,x,u) &= p(t)\left[\rho_t x_t + (\alpha_t - \rho_t) u\right] + D_t p(t) \beta_t u \\ &+ \int_{\mathbb{R}_0} D_{t,z} p(t) \zeta(t,z) u \nu(dz). \end{aligned}$$

By the maximum principle (4.8) we get the following condition for an optimal control

Theorem 4.1.2 *Suppose that $\hat{u}$ is an optimal control corresponding $\hat{x}(t)$, $\hat{p}(t)$ then we have*

$$E\left[p(t)(\alpha_t - \rho_t) + D_t p(t)\beta_t + \int_{\mathbb{R}_0} D_{t,z} p(t) \zeta(t,z) \nu(dz) \mid \varepsilon_t\right] = 0. \tag{4.28}$$

we complete by considering a solution in the special case when

$$\nu = \rho_t = 0, \;\; |\beta_t| \geqslant \delta > 0 \text{ and } \varepsilon_t = \mathcal{F}_t,\, 0 \le t \le T,$$

where $\delta > 0.$ is a given constant. Then (4.28) simplifies to

$$\alpha_t E\left[K \mid \mathcal{F}_t\right] + \beta_t E\left[D_t K \mid \mathcal{F}_t\right] = 0. \tag{4.29}$$

By the Clark-Ocone theorem (3.28) we have

$$K = E[K] + \int_0^T E\left[D_t K \mid \mathcal{F}_t\right] dB_t, \tag{4.30}$$

then

$$E[K \mid \mathcal{F}_t] = E[K] + \int_0^t E[D_t K \mid \mathcal{F}_t] \, dB_t. \tag{4.31}$$

Define

$$M_t := E[K \mid \mathcal{F}_t] = E[U'(\hat{x}(T)) \mid \mathcal{F}_t].$$

Then by substituting (3.28) into (4.31) we get

$$M_t = E[K] - \int_0^t \frac{\alpha_s}{\beta_s} M_s dB_s,$$

or

$$dM_t = -\frac{\alpha_t}{\beta_t} M_t dB_t,$$

witch has the solution

$$M_t = E[U'(\hat{x}(T))] \exp\left(-\int_0^t \frac{\alpha_s}{\beta_s} dB_s - \frac{1}{2}\int_0^t \left(\frac{\alpha_s}{\beta_s}\right)^2 ds \right), \tag{4.32}$$

such that $M_0 = E[U'(\hat{x}(T))]$, we have $U'(\hat{x}(T)) = M_T = K$. Given K the corresponding optimal portfolio $\hat{u}$ is given as the solution of the backward stochastic differential equation

$$\begin{cases} d\hat{x}_t = \alpha_t \hat{u}(t) \, dt + \beta_t \hat{u}(t) \, dB_t; \quad t < T, \\ \hat{x}(T) = (U')^{-1}(K), \end{cases} \tag{4.33}$$

witch can be written

$$\begin{cases} d\hat{x}_t = \beta_t \hat{u}(t) \, d\tilde{B}_t; \quad t < T, \\ \hat{x}(T) = (U')^{-1}(K), \end{cases} \tag{4.34}$$

where

$$d\tilde{B}_t = \frac{\alpha_t}{\beta_t} dt + dB_t,$$

witch is a Brownian motion with respect to the probability measure Q defined by

$$dQ = N_T dP \text{ on } \mathcal{F}_T,$$

where

$$N_t = \exp\left(-\int_0^t \frac{\alpha_s}{\beta_s} dB_s - \frac{1}{2}\int_0^t \left(\frac{\alpha_s}{\beta_s}\right)^2 ds\right).$$

By the theorem(3.4.2) of Clark-Ocone under change of measure we have

$$\hat{x}(t) = E_Q[\hat{x}(T)] + \int_0^T E_Q\left[\left(D_t\hat{x}(T) - \hat{x}(T)\int_t^T D_t\left(\frac{\alpha_s}{\beta_s}\right) d\tilde{B}_t \mid \mathcal{F}_t\right) d\tilde{B}_t\right]. \tag{4.35}$$

Comparing (4.34) and (4.35) we get

$$\hat{u}(t) = \frac{1}{\beta_t} E_Q\left[\left(D_t\hat{x}(T) - \hat{x}(T)\int_t^T D_t\left(\frac{\alpha_s}{\beta_s}\right) d\tilde{B}_t \mid \mathcal{F}_t\right)\right].$$

Using Bayes'rule we conclude

Theorem 4.1.3 *Suppose $\hat{u} \in \mathcal{A}_{\mathcal{F}}$ is an optimal portfolio for the problem*

$$\sup_{u\in\mathcal{A}_{\mathcal{F}}} \left(E\left[U\left(x^u(t), \omega\right)\right]\right),$$

with

$$dx_t^u = \alpha_t u(t)\, dt + \beta_t u(t)\, dB_t.$$

Then

$$\hat{u}(t) = \frac{1}{\beta_t N_t} E\left[N_T\left(D_t\hat{x}(T) - \hat{x}(T)\int_t^T D_t\left(\frac{\alpha_s}{\beta_s}\right) d\tilde{B}_t \mid \mathcal{F}_t\right)\right],$$

and

$$\hat{x}(T) = (U')^{-1}(M_T),$$

where M_t is given by (4.32)*.*

4.2 Singular stochastic maximum principle

4.2.1 Formulation of the singular control problem

Consider a controlled singular jump diffusion $x(t) = x^{\xi}(t)$ of the form $x(0^-) = x \in \mathbb{R}$

$$\begin{aligned} dx_t &= b(t, x(t), \omega)\, dt + \sigma(t, x(t), \omega)\, dB_t + \textstyle\int_{\mathbb{R}_0} \theta(t, x(t^-), z, \omega)\, \tilde{N}(dt, dz) \\ &+ \lambda(t, \omega)\, d\xi_t;\ t \in [0, T], \end{aligned}$$

defined on a probability space $\left(\Omega, \mathcal{F}, (\mathcal{F})_{t \geqslant 0}, P\right)$, where $t \to b(t, x)$, $t \to \sigma(t, x)$ and $t \to \theta(t, x, z)$ are given $\mathcal{F}_t$-predictable processes for each $x \in \mathbb{R}$, $z \in \mathbb{R}_0 \equiv \mathbb{R} - \{0\}$ We assume that b, σ and θ are C^1 with respect to x and that there exists $\varepsilon > 0$ such that

$$\frac{\partial \theta}{\partial x}(t, x, z, \omega) \geqslant -1 + \varepsilon \quad a.s \text{ for } (t, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}_0.$$

$\lambda(t)$ is ε_t-adapted and continuous. The process $\xi(t) = \xi(t, \omega)$, is our control process, assumed to be càdlàg and non-decreasing for each ω, with $\xi(0^-) = 0$. We require that the control $\xi(t)$ is ε_t-adapted. The set of such controls is denoted by $\mathcal{A}_{\varepsilon}$. Let $t \to f(t, x)$ and $t \to h(t, x)$ be given $\mathcal{F}_t$-predictable processes and $g(x)$ an $\mathcal{F}_T$-measurable random variable for each x. De ne the performance functional

$$J(\xi) = E\left[\int_0^T f(t, x, \omega)\, dt + g(x(T), \omega) + \int_0^T h\left(t, x\left(t^-\right), \omega\right) d\xi(t)\right]. \tag{4.36}$$

Problem 4.2.1 *We want to find an optimal control $\xi^* \in \mathcal{A}_{\varepsilon}$ such that*

$$\Phi := \sup_{\xi \in \mathcal{A}_{\varepsilon}} J(\xi) = J(\xi^*). \tag{4.37}$$

For $\xi \in \mathcal{A}_{\varepsilon}$, we let $\nu(\xi)$ denote the set of ε_t-adapted processes ζ of finite variation such *that there exists* $\delta = \delta(\xi) > 0$ *such that*

$$\xi + y\zeta \in \mathcal{A}_\varepsilon \text{ for all } y \in [0, \delta].$$

We define the derivative process, for $\xi \in \mathcal{A}_\varepsilon$ *and* $\zeta \in \nu(\xi)$

$$Y(t) := \lim_{y \to 0^+} \frac{1}{y} \left(X^{\xi + y\zeta}(t) - X^{\xi}(t) \right), \ t \in [0, T]. \tag{4.38}$$

We have

$$dY(t) = Y\left(t^-\right) \left[\frac{\partial b}{\partial x}(t)\, dt + \frac{\partial \sigma}{\partial x}(t)\, dB_t + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(t, z)\, \tilde{N}(dt, dz) \right] + \lambda(t)\, d\zeta_t. \tag{4.39}$$

Note that

$$Y(0) = \lim_{y \to 0^+} \frac{1}{y} \left(X^{\xi + y\zeta}(0) - X^{\xi}(0) \right) = \frac{d}{dy} x \,|_{y=0} = 0.$$

Then we have

$$\lim_{y \to 0^+} \frac{1}{y} \left(J(\xi + y\zeta) - J(\xi) \right) = E\left[\int_0^T \frac{\partial f}{\partial x}(t, x(t))\, Y(t)\, dt + g'(x(T)\, Y(T)) \right.$$
$$\left. + \int_0^T \frac{\partial h}{\partial x}\left(t, x\left(t^-\right)\right) Y\left(t^-\right) d\xi_t + \int_0^T h\left(t, x\left(t^-\right)\right) d\zeta_t \right].$$

The solution of equation

Lemma 4.2.1 *The solution of equation* (4.39) *is*

$$Y(t) = Z(t) \left[\int_0^t Z^{-1}\left(s^-\right) \lambda(s)\, d\zeta + \sum_{0 < s \leq t} Z^{-1}\left(s^-\right) \lambda(s)\, \alpha(s)\, \Delta(\zeta(s)) \right], \ t \in [0, T],$$

with $\Delta(\zeta(s)) = \zeta(s) - \zeta(s^-)$, *where*

$$\alpha(s) = \frac{-\int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(s, z)\, N(\{s\}, dz)}{1 + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(s, z)\, N(\{s\}, dz)} \quad ; s \in [0, T],$$

and $Z(t)$ *is the solution of the "homogeneous" version of* (4.39) *i.e.* $Z(0) = 1$ *and*

$$Z(t) = Z(t^-)\left[\frac{\partial b}{\partial x}(t)\,dt + \frac{\partial \sigma}{\partial x}(t)\,dB_t + \int_{\mathbb{R}_0}\frac{\partial \theta}{\partial x}(t,z)\,\tilde{N}(dt,dz)\right], \tag{4.40}$$

Remark 4.2.1 *Note that*

$$\int_{\mathbb{R}_0}\frac{\partial \theta}{\partial x}(s,z)\,N(\{s\},dz) = \begin{cases} \dfrac{\partial \theta}{\partial x}(s,z) & \textit{if } \eta \textit{ has jump of size } z \textit{ at } s, \\ 0 & \textit{otherwise.} \end{cases}$$

We set

$$G(t,s) = \frac{Z(s)}{Z(t)} \quad \text{for} \quad t < s.$$

4.2.2 A Malliavin-calculus based necessary maximum principle

To establish the maximum principle of the problem 4.4.1 we need the following lemma

Lemma 4.2.2 *Suppose* $\xi \in \mathcal{A}_\varepsilon$ *and* $\zeta \in \nu(\varepsilon)$. *Then*

$$\begin{aligned}&\lim_{y\to 0^+}\frac{1}{y}\left(J(\xi + y\zeta) - J(\xi)\right)\\ &= E\left[\int_0^T [\lambda(t)\tilde{p}(t) + h(t)]\,d\zeta^c(t) + \sum_{0<t\le T}\{\lambda(t)(\tilde{p}(t) + S(t)\alpha(t) + h(t))\}\triangle\zeta(t)\right],\end{aligned} \tag{4.41}$$

where $\zeta^c(t)$ *denotes the continuous part of* $\zeta(t)$ *and*

$$S(t) = \int_t^T G(t,s)\frac{\partial H_0}{\partial x}(s)\,ds, \tag{4.42}$$

$$\tilde{p}(t) = R(t) + \int_t^T G(t,s)\frac{\partial H_0}{\partial x}(s)\,ds = R(t) + S(t), \tag{4.43}$$

$$R(t) = g'(x(T)) + \int_t^T \frac{\partial f}{\partial x}(s)\,ds + \int_{t^+}^T \frac{\partial h}{\partial x}(s)\,d\xi_s, \tag{4.44}$$

$$H_0(s,x) = R(s)\,b(s,x) + D_sR(s)\,\sigma(s,x) + \int_{\mathbb{R}_0} D_{s,z}R(s)\,\theta(s,x,z)\,\nu(dz). \tag{4.45}$$

We can now prove the main result of this section

Theorem 4.2.1 (Necessary maximum principle) *Set*

$$U(t) = \lambda(t)\,\tilde{p}(t) + h(t), \tag{4.46}$$

$$V(t) = \lambda(t)\left(\tilde{p}(t) + S(t)\,\alpha(t)\right) + h(t);\ t\in[0,T]. \tag{4.47}$$

1. *Suppose* $\xi \in \mathcal{A}_\varepsilon$ *is optimal for problem. Then a.a.*$t\in[0,T]$ *we have*

$$E[U(t)\mid\varepsilon_t] \le 0 \text{ and } E[U(t)\mid\varepsilon_t]\,d\xi^c(t) = 0, \tag{4.48}$$

and for all $t\in[0,T]$ *we have*

$$E[V(t)\mid\varepsilon_t] \le 0 \text{ and } E[V(t)\mid\varepsilon_t]\,\Delta\xi(t) = 0. \tag{4.49}$$

2. *Conversely, suppose that* (4.48) *and* (4.49) *hold for some* $\xi\in\mathcal{A}_\varepsilon$. *Then* ξ *is a directional sub-stationary point for* $J(\xi)$, *in the sense that*

$$\lim_{y\to 0^+}\frac{1}{y}\left(J(\xi+y\zeta) - J(\xi)\right) \le 0 \text{ for all } \zeta\in\nu(\xi). \tag{4.50}$$

Proof.

1. Suppose ξ is optimal for the problem 4.4.1 Then

$$\lim_{y\to 0^+} \frac{1}{y}\left(J\left(\xi + y\zeta\right) - J\left(\xi\right)\right) \leq 0 \text{ for all } \zeta \in \nu\left(\xi\right).$$

Hence by lemma (4.4.2)

$$E\left[\int_0^T U\left(t\right) d\zeta^c\left(t\right) + \sum_{0<t\leq T} V\left(t\right)\Delta\zeta\left(t\right)\right] \leq 0 \text{ for all } \zeta \in \nu\left(\xi\right). \tag{4.51}$$

In particular, this holds if we fix $t \in [0, T]$ and choose ζ such that

$$d\zeta_s = a\left(\omega\right)\delta_t\left(s\right); \;\; s \in [0, T],$$

where $a\left(\omega\right) \geqslant 0$ is ε_t-measurable and bounded and $\delta_t\left(.\right)$ is the unit point mass at t. Then (4.51) gets the form:

$$E\left[V\left(t\right)a\right] \leq 0. \tag{4.52}$$

Since this holds for all bounded ε_t-measurable $a \geqslant 0$, we conclude that

$$E\left[V\left(t\right) \mid \varepsilon_t\right] \leq 0. \tag{4.53}$$

Next, choose $\zeta\left(t\right) = -\xi^d\left(t\right)$, the purely discontinuous part of ξ. So by (4.51) we get

$$E\left[\sum_{0<t\leq T} V\left(t\right)\left(-\Delta\xi\left(t\right)\right)\right] \leqslant 0,$$

or

$$E\left[\sum_{0<t\leq T} V\left(t\right)\Delta\xi\left(t\right)\right] \geqslant 0, \tag{4.54}$$

on the other hand choosing $\zeta(t) = -\xi^d(t)$ in (4.51) gives

$$E\left[\sum_{0<t\leq T} V(t)\,\Delta\xi(t)\right] \leqslant 0. \tag{4.55}$$

Combining (4.55) and (4.54) we obtain

$$E\left[\sum_{0<t\leq T} E[V(t) \mid \varepsilon_t]\,\Delta\xi(t)\right] = E\left[\sum_{0<t\leq T} V(t)\,\Delta\xi(t)\right] = 0.$$

Since $E[V(t) \mid \varepsilon_t] \leq 0$ and $\Delta\xi(t) \geqslant 0$, this implies that

$$E[V(t) \mid \varepsilon_t]\,\Delta\xi(t) = 0,\ \forall t \in [0,T].$$

To prove (4.49) we proceed similarly, by choosing first $d\zeta(t) = a(t)\,dt\ t \in [0,T]$, and next $\zeta(t) = \xi^c(t)$.

2. Suppose (4.48) and (4.49) hold for some $\xi \in \mathcal{A}_\varepsilon$. Choose $\zeta \in \nu(\xi)$. Then $\xi + y\zeta \in \mathcal{A}_\varepsilon$ and hence $d\xi + yd\zeta \geqslant 0$ for all $y \in [0,\delta]$ for some $\delta > 0$. Therefore,

$$\begin{aligned}
& yE\left[\int_0^T U(t)\,d\zeta^c + \sum_{0<t\leq T} V(t)\,\Delta\zeta(t)\right] \\
&= yE\left[\int_0^T E[U(t) \mid \varepsilon_t]\,d\zeta^c + \sum_{0<t\leq T} E[V(t)/\varepsilon_t]\,\Delta\zeta(t)\right] \\
&= E\left[\int_0^T E[U(t) \mid \varepsilon_t]\,d\xi^c + \sum_{0<t\leq T} E[V(t) \mid \varepsilon_t]\,\Delta\xi(t)\right] \\
&+ yE\left[\int_0^T E[U(t) \mid \varepsilon_t]\,d\zeta^c + \sum_{0<t\leq T} E[V(t) \mid \varepsilon_t]\,\Delta\zeta(t)\right] \\
&= E\left[\int_0^T E[U(t) \mid \varepsilon_t]\,d(\xi^c + y\zeta^c) + \sum_{0<t\leq T} E[V(t) \mid \varepsilon_t]\,\Delta(\xi(t) + y\zeta(t))\right] \leqslant 0
\end{aligned}$$

Hence the conclusion follows from Lemma (4.4.2).

■

Chapter 5

A stochastic maximum principle for mixed regular-singular control problems via Malliavin calculus

In this chapter, we study general regular-singular stochastic control problems, in which the controller has only partial information. The control has two components, the first one is a classical regular control and the second one is a singular control. We consider systems driven by random coefficients and the running and the final costs are allowed to be random. It is clear that for such systems the dynamic programming does not hold, as the state process is no longer a Markov process. Our goal is to obtain necessary conditions for optimality satisfied by some optimal control.

5.1 Formulation of the problem

Suppose the state process $x_t = x_t^{(u,\xi)}$; $t \geq 0$, satisfies the following stochastic differential equation:

$$\begin{cases} dx_t = b(t, x_t, u_t)\, dt + \sigma(t, x_t, u_t)\, dB_t + \lambda_t d\xi_t; \\ x_0 = x \in \mathbb{R}. \end{cases} \tag{5.1}$$

Here (B_t) is 1-dimensional Brownian motion, defined on a filtred probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$, satisfying the usual conditions. Assume that $(\mathcal{F}_t)$ is the natural filtration of (B_t) .The coefficients

$$\begin{aligned} b &: [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}, \\ \sigma &: [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}, \\ \lambda &: [0,T] \times \Omega \to \mathbb{R}, \end{aligned}$$

are given $\mathcal{F}_t-$predictable processes.

Suppose in addition that we are given a subfiltration $\mathcal{E}_t \subset \mathcal{F}_t$, $t \in [0,T]$, representing the information available to the controller at time t and satisfying the usual conditions.

- Let T be a strictly positive real number and consider the following sets.
- $\mathcal{U}_1^{\mathcal{E}}$ is the class of measurable, $\mathcal{E}_t$-adapted processes $u : [0,T] \times \Omega \to U$, where U is some Borel subset of $\mathbb{R}^k$.
- $\mathcal{U}_2^{\mathcal{E}}$ is the class of measurable, $\mathcal{E}_t$-adapted processes $\xi : [0,T] \times \Omega \to [0,\infty)$ such that ξ is nondecreasing, right-continuous with left hand limits and $\xi_0 = 0$.

Definition 5.1.1 *An admissible control is an $\mathcal{E}_t$-adapted process $(u,\xi) \in \mathcal{U}_1^{\mathcal{E}} \times \mathcal{U}_2^{\mathcal{E}}$ such that*

$$E\left[\int_0^T |u_t|^2\, dt + |\xi_T|^2\right] < \infty.$$

We denote by $\mathcal{A}_{\mathcal{E}}$ the set of all admissible controls.

The expected reward to be maximized has the form

$$J(u,\xi) = E\left[g(x_T) + \int_0^T f(t, x_t, u_t)\, dt + \int_0^T h(t)\, d\xi_t\right], \quad (u,\xi) \in \mathcal{A}_{\mathcal{E}}, \tag{5.2}$$

where

$$f : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R},$$
$$g : \mathbb{R} \times \Omega \to \mathbb{R},$$
$$h : [0,T] \times \Omega \to \mathbb{R},$$

are given $\mathcal{F}_t$-adapted processes.

The goal of the controller is to maximize the functional $J(u, \xi)$ over $\mathcal{A}_{\mathcal{E}}$. An admissible control $\left(\hat{u}, \hat{\xi}\right) \in \mathcal{A}_{\mathcal{E}}$ is optimal if:

$$J\left(\hat{u}, \hat{\xi}\right) = \sup_{(u,\xi)\in\mathcal{A}_{\mathcal{E}}} J(u, \xi). \tag{5.3}$$

Our objective is to derive necessary conditions satisfied by $\left(\hat{u}, \hat{\xi}\right)$.

Note that since we allow b, σ, h, f and g to be random coefficients and also because our controls must be $\mathcal{E}_t$-adapted, this problem is no longer of Markovian type and hence cannot be solved by dynamic programming. Our attention will be focused on the stochastic maximum principle, for which an explicit form for the adjoint process is obtained. Malliavin calculus techniques will be used to get an explicit form of the adjoint process.

Assumptions

The following assumptions will be in force throughout this paper.

$(\mathbf{H_1})$ b, σ, g, f are adapted processes such that there exists a positive constant C satisfying:

$$|b(t,x,u)| + |\sigma(t,x,u)| + |f(t,x,u)| + |g(x)| \le C(1 + |x| + |u|).$$

$(\mathbf{H_2})$ b, σ, g, f are continuously differentiable with respect to $x \in \mathbb{R}$ and $u \in U$ for each $t \in [0,T]$, and a.s. $\omega \in \Omega$, with bounded derivatives.

$(\mathbf{H_3})$ λ, h are bounded continuous processes.

$(\mathbf{H_4})$For all bounded $\mathcal{F}_t-$measurable random variables $\alpha = \alpha(\omega)$ the process $v_s^{\alpha} = \alpha(\omega) 1_{(t,r]}(s)$; $s \in [0,T]$ belongs to $\mathcal{U}_1^{\varepsilon}$.

$(\mathbf{H_5})$For u, $v \in \mathcal{U}_1^{\mathcal{E}}$ with v bounded, there exists $\delta > 0$ such that

$$u^{\theta} = u + \theta v \in \mathcal{U}_1^{\mathcal{E}} \text{ for all } \theta \in [-\delta, \delta].$$

Under the above assumptions, for every $(u, \xi) \in \mathcal{A}_{\mathcal{E}}$, equation (5.1) admits a unique strong solution given by

$$x_t^{(u,\xi)} = x + \int_0^t b\left(s, x_s^{(u,\xi)}, u_s\right) ds + \int_0^t \sigma\left(s, x_s^{(u,\xi)}, u_s\right) dB_s + \int_0^t \lambda(t)\, d\xi_s, \tag{5.4}$$

and the reward functional J is well defined from $\mathcal{A}_{\mathcal{E}}$ into $\mathbb{R}$.

We list some notations which will be used throughout this paper.

Notations

For $\xi \in U_2^{\mathcal{E}}$, let $\nu(\xi)$ denotes the set of $\mathcal{E}_t$-adapted processes η of finite variation such that there exists $\delta > 0$ such that $\xi + \theta\eta \in U_2^{\mathcal{E}}$, for all $\theta \in [0, \delta]$. For all $u \in \mathcal{U}_1^{\mathcal{E}}$ and $0 \leq t \leq s \leq T$, we denote the following processes

$$R(t) := g'(x_T) + \int_t^T \frac{\partial f}{\partial x}(s, x_s, u_s)\, ds, \tag{5.5}$$

$$D_t(R(t)) := D_t g'(x_T) + \int_t^T D_t \frac{\partial f}{\partial x}(s, x_s, u_s)\, ds, \tag{5.6}$$

$$H_0(s, x, u) = R(s)\, b(s, x, u) + D_s R(s)\, \sigma(s, x, u), \tag{5.7}$$

$$G(t, s) := \exp\left(\int_t^s \left\{\frac{\partial b}{\partial x}(r, x_r, u_r) - \frac{1}{2}\left(\frac{\partial \sigma}{\partial x}\right)^2 (r, x_r, u_r)\right\} dr + \int_t^s \frac{\partial \sigma}{\partial x}(r, x_r, u_r)\, dB_r\right), \tag{5.8}$$

$$p(t) := R(t) + \int_t^T \frac{\partial H_0}{\partial x}(s, x_s, u_s)\, G(t, s)\, ds, \tag{5.9}$$

$$q(t) := D_t p(t). \tag{5.10}$$

We define the usual Hamiltonian of the control problem (3.1)-(3.2) by:

$$H : [0,T] \times \mathbb{R} \times U \times \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R},$$

where

$$H(t,x,u,p,q,\omega) = f(t,x,u,\omega) + p(t)\, b(t,x,u,\omega) + q(t)\, \sigma(t,x,u,\omega), \tag{5.11}$$

5.2 The stochastic maximum principle

The purpose of the stochastic maximum principle is to find necessary conditions for optimality satisfied by an optimal control. Suppose that $\left(\hat{u},\hat{\xi}\right) \in \mathcal{A}_{\mathcal{E}}$ is an optimal control and let $\hat{x}_t$ denotes the optimal trajectory, that is, the solution of (5.1) corresponding to $\left(\hat{u},\hat{\xi}\right)$. As it is well known the stochastic maximum principle is based on the computation of the derivative of the reward functional with respect to some perturbation parameter. Let us define the perturbed controls as follows.

- $u^\theta = \hat{u} + \theta v$, where v is some bounded $\mathcal{E}_t-$adapted process. We know by $(\mathbf{H_5})$ that there exists $\delta > 0$ such that $u^\theta = \hat{u} + \theta v \in \mathcal{U}_1^{\mathcal{E}}$ for all $\theta \in [-\delta, \delta]$
- $\xi^\theta = \hat{\xi} + \theta\eta$, where $\eta \in \nu(\xi)$ the set of $\mathcal{E}_t-$adapted processes of finite variation, for which there exists $\delta = \delta(\hat{\xi}) > 0$ such that $\hat{\xi} + \theta\eta \in \mathcal{U}_2^{\mathcal{E}}$.

Since $\left(\hat{u},\hat{\xi}\right)$ is an optimal control it holds that:

(1) $\lim\limits_{\theta \to 0^+} \frac{1}{\theta}\left(J\left(\hat{u},\xi^\theta\right) - J\left(\hat{u},\hat{\xi}\right)\right) \leq 0$, where $\xi^\theta = \hat{\xi} + \theta\eta$, and

(2) $\lim\limits_{\theta \to 0} \frac{1}{\theta}\left(J\left(u^\theta,\hat{\xi}\right) - J\left(\hat{u},\hat{\xi}\right)\right) \leq 0$, where $u^\theta = \hat{u} + \theta v$.

We use the two limits to obtain the variational inequalities. To achieve this goal, we need the following technical Lemmas.

We define the derivative process $\mathcal{Y}(t)$ by

$$\mathcal{Y}(t) = \lim_{\theta \to 0^+} \frac{1}{\theta}\left(x_t^{\left(\hat{u},\xi^\theta\right)} - x_t^{\left(\hat{u},\hat{\xi}\right)}\right), \tag{5.12}$$

Since that $\mathcal{Y}(0) = 0$, then

$$d\mathcal{Y}(t) = \frac{\partial b}{\partial x}(t)\,\mathcal{Y}(t)\,dt + \frac{\partial \sigma}{\partial x}(t)\,\mathcal{Y}(t)\,dB_t + \lambda(t)\,d\eta_t, \tag{5.13}$$

where we use the abbreviated notation:
$\frac{\partial b}{\partial x}(t) = \frac{\partial b}{\partial x}(t, \hat{x}_t, \hat{u}_t, \omega)$, $\frac{\partial \sigma}{\partial x}(t) = \frac{\partial \sigma}{\partial x}(t, \hat{x}_t, \hat{u}_t, \omega)$.

Lemma 5.2.1 *The solution of equation* (5.13) *is given by*

$$\mathcal{Y}(t) = Z(t)\left[\int_0^t Z^{-1}(s)\,\lambda(s)\,d\eta_s\right];\ t \in [0,T], \tag{5.14}$$

where $Z(t)$ *is the solution of the homogeneous version of* (5.13), *i.e.*

$$\begin{cases} dZ(t) = \frac{\partial b}{\partial x}(t)\,Z(t)\,dt + \frac{\partial \sigma}{\partial x}(t)\,Z(t)\,dB_t, \\ Z(0) = 1. \end{cases} \tag{5.15}$$

We set $\mathcal{Y}(t) = Z(t)\,A_t$ where

$$A_t = \int_0^t Z^{-1}(s)\,\lambda(s)\,d\eta_s.$$

By using Itô's formula for semimartingales, we get

$$\begin{aligned} d\mathcal{Y}(t) &= Z(t)\,dA_t + A_t dZ(t) + d\langle A, Z\rangle_t, \\ d\mathcal{Y}(t) &= \lambda(t)\,d\eta_t + A_t\left(\frac{\partial b}{\partial x}(t)\,Z(t)\,dt + \frac{\partial \sigma}{\partial x}(t)\,Z(t)\,dB_t\right) \\ &= \frac{\partial b}{\partial x}(t)\,\mathcal{Y}(t)\,dt + \frac{\partial \sigma}{\partial x}(t)\,\mathcal{Y}(t)\,dB_t + \lambda(t)\,d\eta_t. \end{aligned}$$

This completes the proof.

In the sequel, we use the abbreviated notation:

$$Q(t,s) = \frac{Z(s)}{Z(t)} \text{ for } t < s.$$

Lemma 5.2.2 *Let $\left(\hat{u},\hat{\xi}\right)$ be an optimal control. Then*

$$\lim_{\theta\to 0^+}\frac{1}{\theta}\left(J\left(\hat{u},\xi^{\theta}\right)-J\left(\hat{u},\hat{\xi}\right)\right)=E\left[\int_0^T\left(\lambda(t)\,\hat{P}(t)+h(t)\right)d\eta_t\right], \tag{5.16}$$

where

$$\hat{P}(t) := \hat{R}(t) + \int_t^T \frac{\partial H_0}{\partial x}(s)\,Q(t,s)\,ds, \tag{5.17}$$

$$\hat{R}(t) = R^{\left(\hat{u},\hat{\xi}\right)}(t) = g'(\hat{x}_T) + \int_t^T \frac{\partial f}{\partial x}(s)\,ds, \tag{5.18}$$

$$H_0(s,x) = R(s) + D_sR(s)\sigma(s,x). \tag{5.19}$$

We have

$$\lim_{\theta\to 0^+}\frac{1}{\theta}\left(J\left(\hat{u},\xi^{\theta}\right)-J\left(\hat{u},\hat{\xi}\right)\right)=E\left[g'(\hat{x}_T)\,\mathcal{Y}(T)+\int_0^T\frac{\partial f}{\partial x}(t)\,\mathcal{Y}(t)\,dt\right.$$
$$\left.+\int_0^T h(t)\,d\eta_t\right]. \tag{5.20}$$

We have from (5.13)

$$E\left[\int_0^T\frac{\partial f}{\partial x}(t)\,\mathcal{Y}(t)\,dt\right]=E\left[\int_0^T\frac{\partial f}{\partial x}(t)\int_0^t\left\{\mathcal{Y}(s)\frac{\partial b}{\partial x}(s)\,ds+\mathcal{Y}(s)\frac{\partial\sigma}{\partial x}(s)\,dB_s\right.\right.$$
$$\left.+\lambda(s)\,d\eta_s\right\}dt\Big].$$

Since $\mathcal{Y}(0) = 0$, we have by the duality formulae for the Malliavin derivatives,

$$E\left[\int_0^T \frac{\partial f}{\partial x}(t)\,\mathcal{Y}(t)\,dt\right] = \left[E\int_0^T\int_0^t \left\{\frac{\partial f}{\partial x}(t)\,\mathcal{Y}(s)\,\frac{\partial b}{\partial x}(s)\,ds + D_s\left(\frac{\partial f}{\partial x}(t)\right)\mathcal{Y}(s)\,\frac{\partial \sigma}{\partial x}(s)\,ds + \frac{\partial f}{\partial x}(t)\,\lambda(s)\,d\eta_s\right\} dt\right],$$

by using Fubini theorem

$$E\left[\int_0^T \frac{\partial f}{\partial x}(t)\,\mathcal{Y}(t)\,dt\right] = E\left[\int_0^T\int_s^T \left\{\frac{\partial f}{\partial x}(t)\,\mathcal{Y}(s)\,\frac{\partial b}{\partial x}(s)\,dt + D_s\left(\frac{\partial f}{\partial x}(t)\right)\mathcal{Y}(s)\,\frac{\partial \sigma}{\partial x}(s)\,dt\right\} ds + \int_0^T \left(\int_s^T \frac{\partial f}{\partial x}(t)\,\lambda(s)\,dt\right) d\eta_s\right], \tag{5.21}$$

changing the notation $s \to t$, this becomes

$$\begin{aligned} E\left[\int_0^T \frac{\partial f}{\partial x}(t)\,\mathcal{Y}(t)\,dt\right] &= E\left[\int_0^T\int_t^T \left\{\frac{\partial f}{\partial x}(s)\,\mathcal{Y}(t)\,\frac{\partial b}{\partial x}(t)\,ds + D_t\left(\frac{\partial f}{\partial x}(s)\right)\mathcal{Y}(t)\,\frac{\partial \sigma}{\partial x}(t)\,ds\right\} dt + \int_0^T \left(\int_t^T \frac{\partial f}{\partial x}(s)\,\lambda(t)\,ds\right) d\eta_t\right] \\ &= E\left[\int_0^T \left\{\left(\int_t^T \frac{\partial f}{\partial x}(s)\,ds\right)\mathcal{Y}(t)\,\frac{\partial b}{\partial x}(t) + D_t\left(\int_t^T \left(\frac{\partial f}{\partial x}(s)\right) ds\right)\mathcal{Y}(t)\,\frac{\partial \sigma}{\partial x}(t)\right\} dt + \int_0^T \left(\int_t^T \frac{\partial f}{\partial x}(s)\,ds\right)\lambda(t)\,d\eta_t\right]. \end{aligned} \tag{5.22}$$

Similary we get

$$\begin{aligned}
&E\left[g'\left(X_T\right)\mathcal{Y}\left(T\right)\right]\\
&=E\left[g'\left(X_T\right)\left\{\int_0^T \mathcal{Y}\left(t\right)\frac{\partial b}{\partial x}\left(t\right)dt+\mathcal{Y}\left(t\right)\frac{\partial \sigma}{\partial x}\left(t\right)dB_t+\lambda\left(t\right)d\eta_t\right\}\right]\\
&=E\left[\int_0^T \mathcal{Y}\left(t\right)\left\{g'\left(X_T\right)\frac{\partial b}{\partial x}\left(t\right)+D_t\left(g'\left(X_T\right)\right)\frac{\partial \sigma}{\partial x}\left(t\right)\right\}dt+g'\left(X_T\right)\lambda\left(t\right)d\eta_t\right]. \qquad (5.23)
\end{aligned}$$

Combining (5.21) and (5.22) and using the notations (5.5) and (5.7), we obtain

$$\begin{aligned}
\lim_{\theta\to 0^+}\frac{1}{\theta}\left(J\left(\hat{u},\xi^\theta\right)-J\left(\hat{u},\hat{\xi}\right)\right)&=E\left[\int_0^T \mathcal{Y}\left(t\right)\left\{R\left(t\right)\frac{\partial b}{\partial x}\left(t\right)+D_tR\left(t\right)\frac{\partial \sigma}{\partial x}\left(t\right)\right\}dt\right.\\
&\quad\left.+\left\{R\left(t\right)\lambda\left(t\right)+h\left(t\right)\right\}d\eta_t\right]\\
&=A_1\left(\eta\right)+A_2\left(\eta\right),
\end{aligned}$$

where

$$A_1\left(\eta\right)=E\left[\int_0^T \mathcal{Y}\left(t\right)\left\{R\left(t\right)\frac{\partial b}{\partial x}\left(t\right)+D_tR\left(t\right)\frac{\partial \sigma}{\partial x}\left(t\right)\right\}dt\right],$$

and

$$A_2\left(\eta\right)=\left\{R\left(t\right)\lambda\left(t\right)+h\left(t\right)\right\}d\eta_t.$$

We set

$$d\Lambda_t=\frac{\partial H_0}{\partial x}\left(t\right)dt,$$

then by using Lemma 4.1 it follows that

$$\begin{aligned} A_1(\eta) &= E\left[\int_0^T \mathcal{Y}(t)\frac{\partial H_0}{\partial x}(t)\,dt\right] \\ &= E\left[\int_0^T \mathcal{Y}(t)\,d\Lambda_t\right] \\ &= E\left[\int_0^T \left(Z(t)\int_0^t Z^{-1}(s)\lambda(s)\,d\eta_s\right)d\Lambda_t\right]. \end{aligned}$$

Hence by using Fubini's theorem we get by changing the notation $s \to t$

$$\begin{aligned} A_1(\eta) &= E\left[\int_0^T \left(\left(\int_t^T Z(s)\,d\Lambda_s\right)Z^{-1}(t)\lambda(t)\,d\eta_t\right)\right] \\ &= E\left[\int_0^T\int_t^T Q(t,s)\frac{\partial H_0}{\partial x}(s)\,ds\lambda(t)\,d\eta_t\right]. \end{aligned}$$

Finaly

$$\begin{aligned} \lim_{\theta\to 0^+}\frac{1}{\theta}\left(J\left(\hat{u},\xi^\theta\right)-J\left(\hat{u},\hat{\xi}\right)\right) &= A_1(\eta)+A_2(\eta) \\ &= E\left[\int_0^T\left(\lambda(t)\hat{P}(t)+h(t)\right)d\eta_t\right]. \end{aligned}$$

This completes the proof.

We define the derivative process $Y(t)$ by

$$Y(t) = Y^v(t) = \lim_{\theta\to 0}\frac{1}{\theta}\left(x_t^{\left(u^\theta,\hat{\xi}\right)} - x_t^{\left(\hat{u},\hat{\xi}\right)}\right), \tag{5.24}$$

then $Y(t)$ satisfies the following equation

$$\begin{aligned} dY(t) &= Y(t)\left[\frac{\partial b}{\partial x}(t)\,dt + \frac{\partial \sigma}{\partial x}(t)\,dB_t\right] \\ &\quad + v_t\left[\frac{\partial b}{\partial u}(t)\,dt + \frac{\partial \sigma}{\partial u}(t)\,dB_t\right], \\ Y(0) &= 0, \end{aligned} \tag{5.25}$$

Lemma 5.2.3 *The following identity holds*

$$\begin{aligned} &\lim_{\theta\to 0}\frac{1}{\theta}\left(J\left(u^{\theta},\hat{\xi}\right) - J\left(\hat{u},\hat{\xi}\right)\right) \\ &= E\left[\int_0^T \left\{R(t)\left\{\frac{\partial b}{\partial x}(t)\,Y(t) + \frac{\partial b}{\partial u}(t)\,v_t\right\}\right.\right. \\ &\left.\left. + D_t R(t)\left\{\frac{\partial \sigma}{\partial x}(t)\,Y(t) + \frac{\partial \sigma}{\partial u}(t)\,v_t\right\} + \frac{\partial f}{\partial u}(s)\,v_t\right\}\right] dt. \end{aligned}$$

We have

$$\begin{aligned} \frac{d}{d\theta}J\left(u^{\theta},\hat{\xi}\right)|_{\theta=0} = E\left[\int_0^T \left\{\frac{\partial f}{\partial x}(t)\,Y(t) + \frac{\partial f}{\partial u}(t)\,v_t\right\} dt\right. \\ \left. + g'\left(\hat{x}_T\right) Y(T)\right], \end{aligned} \tag{5.26}$$

where $Y(t) = Y^{v}(t)$ is the solution of the linear equation

$$\begin{cases} dY(t) = \left[\frac{\partial b}{\partial x}(t)\,Y(t) + \frac{\partial b}{\partial u}(t)\,v_t\right] dt + \left[\frac{\partial \sigma}{\partial x}(t)\,Y(t) + \frac{\partial \sigma}{\partial u}(t)\,v_t\right] dB_t \\ Y(0) = 0 \end{cases} \tag{5.27}$$

By the duality formula we get

$$
\begin{aligned}
E\left(g'\left(\hat{x}_T\right)Y(T)\right) &= E\Bigg[g'\left(\hat{x}_T\right)\int_0^T\left\{\frac{\partial b}{\partial x}(t)\,Y(t)+\frac{\partial b}{\partial u}(t)\,v_t\right\}dt \\
&\quad +g'\left(\hat{x}_T\right)\int_0^T\left\{\frac{\partial \sigma}{\partial x}(t)\,Y(t)+\frac{\partial \sigma}{\partial u}(t)\,v_t\right\}dB_t\Bigg] \\
&= E\Bigg[\int_0^T g'\left(\hat{x}_T\right)\left\{\frac{\partial b}{\partial x}(t)\,Y(t)+\frac{\partial b}{\partial u}(t)\,v_t\right\}dt \\
&\qquad +\int_0^T D_t g'\left(\hat{x}_T\right)\left\{\frac{\partial \sigma}{\partial x}(t)\,Y(t)+\frac{\partial \sigma}{\partial u}(t)\,v_t\right\}dt\Bigg].
\end{aligned}
$$

Using similar arguments and Fubini's theorem it follows that,

$$
\begin{aligned}
E\left[\int_0^T\frac{\partial f}{\partial x}(t)\,Y(t)dt\right] &= E\Bigg[\int_0^T\left(\int_0^t\frac{\partial f}{\partial x}(t)\left\{\frac{\partial b}{\partial x}(s)\,Y(s)+\frac{\partial b}{\partial u}(s)\,v_s\right\}ds\right)dt \\
&\quad +\int_0^T\left(\int_0^t D_s\frac{\partial f}{\partial x}(t)\left\{\frac{\partial \sigma}{\partial x}(t)\,Y(t)+\frac{\partial \sigma}{\partial u}(t)\,v_t\right\}ds\right)dt\Bigg] \\
&= E\Bigg[\int_0^T\left(\int_s^T\frac{\partial f}{\partial x}(t)\left\{\frac{\partial b}{\partial x}(s)\,Y(s)+\frac{\partial b}{\partial u}(s)\,v_s\right\}dt\right)ds \\
&\quad +\int_0^T\left(\int_s^T D_s\frac{\partial f}{\partial x}(t)\left\{\frac{\partial \sigma}{\partial x}(s)\,Y(s)+\frac{\partial \sigma}{\partial u}(s)\,v_s\right\}dt\right)ds\Bigg]. \qquad (5.28)
\end{aligned}
$$

Changing the notation $s\rightarrow t$, we get

$$
\begin{aligned}
&E\left[\int_0^T \frac{\partial f}{\partial x}(t)\,Y(t)dt\right] \qquad (5.29)\\
&= E\left[\int_0^T \left(\left(\int_t^T \frac{\partial f}{\partial x}(s)\,ds\right)\left\{\frac{\partial b}{\partial x}(t)\,Y(s)+\frac{\partial b}{\partial u}(t)\,v_t\right\}\right)dt\right.\\
&\left.+\int_0^T \left(\int_t^T \left(D_t\frac{\partial f}{\partial x}(s)\,ds\right)\left\{\frac{\partial \sigma}{\partial x}(t)\,Y(t)+\frac{\partial \sigma}{\partial u}(t)\,v_t\right\}\right)dt\right].
\end{aligned}
$$

Using the notation

$$R(t) := g'(X_T)+\int_t^T \frac{\partial f}{\partial x}(s)\,ds,$$

and combining (5.28) and (5.29), we get

$$
\begin{aligned}
\lim_{\theta\to 0}\frac{1}{\theta}\left(J\left(u^\theta,\hat{\xi}\right)-J\left(\hat{u},\hat{\xi}\right)\right) &= E\left[\int_0^T \left\{R(t)\left\{\frac{\partial b}{\partial x}(t)\,Y(t)+\frac{\partial b}{\partial u}(t)\,v_t\right\}\right.\right.\\
&+D_tR(t)\left\{\frac{\partial \sigma}{\partial x}(t)\,Y(t)+\frac{\partial \sigma}{\partial u}(t)\,v_t\right\}\\
&\left.\left.+\frac{\partial f}{\partial u}(t)\,v_t\right\}dt\right], \qquad (5.30)
\end{aligned}
$$

which completes the proof.

Now, we are ready to state the main result of this paper. Note that the following theorem extends in particular [47] Theorem 3.4 and [51] Theorem 2.4 to mixed regular-singular control problems.

Theorem 5.2.1 (The stochastic maximum principle) *Let $\left(\hat{u},\hat{\xi}\right)\in\mathcal{A}_{\mathcal{E}}$ be an optimal control maximizing the reward J over $\mathcal{A}_{\mathcal{E}}$ and $\hat{x}_t$ denotes the optimal trajectory, then for a.e. $t\in[0,T]$ we have:*

i) $E\left[V_{(\hat{u},\hat{\xi})}(t)/\mathcal{E}_t\right] \leq 0,$ *and* $E\left[V_{(\hat{u},\hat{\xi})}(t)/\mathcal{E}_t\right] d\hat{\xi}_t = 0$ *where*

$$V_{(\hat{u},\hat{\xi})}(t) = \lambda(t)\,\hat{p}(t) + h(t),$$

ii) $E\left[\frac{\partial H}{\partial u}(t,\hat{x}_t,\hat{u}_t)/\mathcal{E}_t\right] = 0,$ *where*

$$H(t,\hat{x}_t,\hat{u}_t,\hat{p}(t),\hat{q}(t)) = f(t,\hat{x}_t,\hat{u}_t) + \hat{p}(t)\,b(t,\hat{x}_t,\hat{u}_t) + \hat{q}(t)\,\sigma(t,\hat{x}_t,\hat{u}_t),$$

is the usual Hamiltonian.

First, we start to prove (i). By Lemma 4.2 we have

$$\lim_{\theta\to 0^+}\frac{1}{\theta}\left(J(\hat{u},\xi^\theta) - J\left(\hat{u},\hat{\xi}\right)\right) = E\left[\int_0^T V_{(\hat{u},\hat{\xi})}(t)d\eta_t\right] \leq 0,$$

for all $\eta \in \mathcal{U}_2^{\mathcal{E}}$. In particular, this holds if we choose η such that $d\eta(t) = a(t)\,dt$, where $a(t) \geq 0$ is continuous and $\mathcal{E}_t-$adapted, then

$$E\left[\int_0^T V_{(\hat{u},\hat{\xi})}(t)a(t)\,dt\right] \leq 0.$$

Since this holds for all such $\mathcal{E}_t-$adapted processes, we deduce that

$$E\left[V_{(\hat{u},\hat{\xi})}(t)/\mathcal{E}_t\right] \leq 0;\ a.e.t \in [0,T]. \tag{5.31}$$

Then, choosing $\eta_t = -\hat{\xi}_t$ we get

$$E\left[\int_0^T V_{(\hat{u},\hat{\xi})}(t)\left(-d\hat{\xi}_t\right)\right] \leq 0.$$

Next, choosing $\eta_t = \hat{\xi}_t$ we get

$$E\left[\int_0^T V_{(\hat{u},\hat{\xi})} d\hat{\xi}_t\right] \leq 0.$$

Hence

$$E\left[\int_0^T V_{(\hat{u},\hat{\xi})}(t) d\hat{\xi}_t\right] = E\left[\int_0^T E\left(V_{(\hat{u},\hat{\xi})}(t)/\mathcal{E}_t\right) d\hat{\xi}_t\right] = 0,$$

which combined with (5.31) gives

$$E\left(V_{(\hat{u},\hat{\xi})}(t)/\mathcal{E}_t\right) d\hat{\xi}_t = 0.$$

Now let us prove (ii).

We have

$$\lim_{\theta\to 0}\frac{1}{\theta}\left(J\left(u^\theta,\hat{\xi}\right) - J\left(\hat{u},\hat{\xi}\right)\right) \leq 0.$$

Then by lemma 4.3 we get

$$\begin{aligned} 0 \geq E\Bigg[\int_0^T \Big\{ R(t)\left\{\frac{\partial b}{\partial x}(t) Y(t) + \frac{\partial b}{\partial u}(t) v_t\right\} \\ + D_t R(t)\left\{\frac{\partial \sigma}{\partial x}(t) Y(t) + \frac{\partial \sigma}{\partial u}(t) v_t\right\} + \frac{\partial f}{\partial u}(s) v_t\Big\}\Bigg] dt. \end{aligned} \tag{5.32}$$

Now we apply the above to $v = v_\alpha \in \mathcal{U}_1^{\xi}$ of the form $v_\alpha(s) = \alpha 1_{[t,t+h]}(s)$, for some $t, h \in (0,T)$, $t+h \leq T$, where $\alpha = \alpha(\omega)$ is bounded and $\mathcal{E}_t$-measurable. Then $Y^{v_\alpha}(s) = 0$ for $0 \leq s \leq t$, hence (5.32) becomes

$$A_1 + A_2 \leq 0, \tag{5.33}$$

where

$$A_1 = E\left[\int_t^T \left\{R(s)\frac{\partial b}{\partial x}(s) Y(s) + D_s R(s)\frac{\partial \sigma}{\partial x}(s) Y(s)\right\} ds\right],$$

and

$$A_2 = \left[\int_t^{t+h} \left\{ R(s) \frac{\partial b}{\partial u}(t) + D_s R(s) \frac{\partial \sigma}{\partial u}(t) + \frac{\partial f}{\partial u}(s) \right\} \alpha ds \right].$$

Note that by (5.25), with $Y(s) = Y^{v_\alpha}(s)$, $s \geq t+h$ the process $Y(s)$ satisfies the following dynamics

$$dY(s) = Y(s) \left\{ \frac{\partial b}{\partial x}(s)\, ds + \frac{\partial \sigma}{\partial x}(s)\, dB_s \right\}, \tag{5.34}$$

for $s \geq t+h$ with initial condition $Y(t+h)$ at time $t+h$. An application of Itô's formula yields

$$Y(s) = Y(t+h)\, G(t+h, s)\,; \quad s \geq t+h, \tag{5.35}$$

where, for $s \geq t$,

$$G(t,s) = \exp\left(\int_t^s \left\{ \frac{\partial b}{\partial x}(r) - \frac{1}{2} \left(\frac{\partial \sigma}{\partial x} \right)^2 (r) \right\} dr + \int_t^s \frac{\partial \sigma}{\partial x}(r)\, dB_r \right).$$

Note that $G(t,s)$ does not depend on h, but $Y(s)$ does. We have by (5.7)

$$A_1 = E\left[\int_t^T \frac{\partial H_0}{\partial x}(s)\, Y(s) ds \right].$$

Differentiating with respect to h at $h = 0$ we get

$$\left.\frac{d}{dh} A_1\right|_{h=0} = \frac{d}{dh} E\left[\int_t^{t+h} \frac{\partial H_0}{\partial x}(s)\, Y(s) ds \right]\Bigg|_{h=0} + \frac{d}{dh} E\left[\int_{t+h}^T \frac{\partial H_0}{\partial x}(s)\, Y(s) ds \right]\Bigg|_{h=0}.$$

Using the fact that $Y(t) = 0$, we see that

$$\frac{d}{dh} E\left[\int_t^{t+h} \frac{\partial H_0}{\partial x}(s)\, Y(s) ds \right]_{h=0} = 0.$$

Therefore, using (5.35) and the fact that $Y(t) = 0$ it holds that,

$$\begin{aligned} &\frac{d}{dh} A_1 \Big|_{h=0} \\ &= \frac{d}{dh} E\left[\int_{t+h}^{T} \frac{\partial H_0}{\partial x}(s) Y(t+h) G(t+h,s)\, ds \right] \Bigg|_{h=0} \\ &= \int_t^T \frac{d}{dh} E\left[\frac{\partial H_0}{\partial x}(s) Y(t+h) G(t+h,s) \right] \Bigg|_{h=0} ds \\ &= \int_t^T \frac{d}{dh} E\left[\frac{\partial H_0}{\partial x}(s) G(t,s) Y(t+h) \right] \Bigg|_{h=0} ds. \end{aligned} \tag{5.36}$$

By (5.27)

$$Y(t+h) = \alpha \int_t^{t+h} \left\{ \frac{\partial b}{\partial u}(s)\, ds + \frac{\partial \sigma}{\partial u}(s)\, dB_s \right\} + \int_t^{t+h} Y_s \left\{ \frac{\partial b}{\partial x}(s)\, ds + \frac{\partial \sigma}{\partial x}(s)\, dB_s \right\}. \tag{5.37}$$

Therefore, by the duality formulae, $\frac{d}{dh} A_1 \Big|_{h=0} = \Lambda_1 + \Lambda_2$, where

$$\begin{aligned} \Lambda_1 &= \int_t^T \frac{d}{dh} E\left[\frac{\partial H_0}{\partial x}(s) G(t,s)\, \alpha \left(\int_t^{t+h} \frac{\partial b}{\partial u}(r)\, dr + \frac{\partial \sigma}{\partial u}(r)\, dB_r \right) \right] \Bigg|_{h=0} ds \\ &= \int_t^T \frac{d}{dh} E\left[F(t,s)\, \alpha \left(\int_t^{t+h} \frac{\partial b}{\partial u}(r)\, dr + \frac{\partial \sigma}{\partial u}(r)\, dB_r \right) \right] \Bigg|_{h=0} ds \\ &= \int_t^T \frac{d}{dh} E\left[\alpha \left(\int_t^{t+h} \left\{ F(t,s) \frac{\partial b}{\partial u}(r)\, dr + D_r F(t,s) \frac{\partial \sigma}{\partial u}(r) \right\} dr \right) \right] \Bigg|_{h=0} ds \\ &= \int_t^T E\left[\alpha \left\{ F(t,s) \frac{\partial b}{\partial u}(t)\, dt + D_t F(t,s) \frac{\partial \sigma}{\partial u}(t) \right\} \right] ds, \end{aligned} \tag{5.38}$$

$F(t,s) = \frac{\partial H_0}{\partial x}(s)\, G(t,s)$, and

$$\Lambda_2 = \int_t^T \frac{d}{dh} E\left[\frac{\partial H_0}{\partial x}(s)\, G(t,s) \left(\int_t^{t+h} Y_r \left\{\frac{\partial b}{\partial x}(r)\, dr + \frac{\partial b}{\partial x}(r)\, dB_r\right\}\right)\right] ds.$$

Using the fact that $Y(t) = 0$, we see that

$$\Lambda_2 = 0.$$

We conclude that

$$\frac{d}{dh} A_1|_{h=0} = \Lambda_1.$$

Moreover, we see directly that

$$\frac{d}{dh} A_2|_{h=0} = E\left[\alpha \left\{R(t) \frac{\partial b}{\partial u}(t) + D_t R(t) \frac{\partial \sigma}{\partial u}(t) + \frac{\partial f}{\partial u}(t)\right\}\right].$$

Therefore, differentiating (4.26) with respect to h at $h = 0$, gives the inequality

$$\begin{aligned} E\Bigg[\alpha \Bigg\{ &\left(R(t) + \int_t^T F(t,s)\, ds\right) \frac{\partial b}{\partial u}(t) \\ &+ D_t \left(R(t) + \int_t^T F(t,s)\, ds\right) \frac{\partial \sigma}{\partial u}(t) + \frac{\partial f}{\partial u}(t)\Bigg\}\Bigg] \leq 0. \end{aligned}$$

We can reformulate this by using the notation (5.9) and (5.10)

$$E\left[\alpha \left\{p(t) \frac{\partial b}{\partial u}(t) + q(t) \frac{\partial \sigma}{\partial u}(t) + \frac{\partial f}{\partial u}(t)\right\}\right] \leq 0.$$

Using the definition of the Hamiltonian (5.11) the last inequality can be rewritten

$$E\left[\frac{\partial H}{\partial u}(t, \hat{x}_t, \hat{u}_t)\, \alpha\right] \leq 0.$$

Since this holds for all bounded $\mathcal{E}_t$-measurable random variable α, we conclude that

$$E\left[\frac{\partial H}{\partial u}\left(t,\hat{x}_t,\hat{u}_t\right)/\mathcal{E}_t\right]=0.$$

This completes the proof.

Conclusion

The aim of our work is to establish stochastic maximum principles for partial information general regular-singular stochastic control problems by using Malliavins calculus. The control has two components, the first one is a classical regular control and the second one is a singular. We consider systems driven by random coefficients and the running and the final costs are allowed to be random. It is clear that for such systems the dynamic programming does not hold, as the state process is no longer a Markov process. We have obtained a necessary conditions for optimality satisfied by some optimal control and the adjoint process is explicitly expressed.

We point out the difference between partial information and partial observation models. Concerning the latter, the information ε_t available to the controller at time t is a noisy observation of the state. In such cases one can sometimes use filtering theory to transform the partial observation problem to a related problem with full information. The partial information problems considered in our work, however, deal with the more general cases where we simply assume that the information flow ε_t is a subfiltration of the full information $\mathcal{F}_t$. Note that the methods and results in the partial observation case do not apply to our situation. On the other hand, there are several existing works on stochastic maximum principle (either completely or partially observed) where adjoint processes are explicitly expressed . However, these works all essentially employ stochastic flows technique, over which the Malliavin calculus has an advantage in terms of numerical computations.
Following this study, several perspectives are considered. It would be interesting to use

malliavin calculus in the following problems

- Maximum principle for infinite-horizon optimal control problems.
- Maximum principle for infinite-horizon control problems with time delay.
- Infinite horizon optimal control of forward-backward stochastic differential equations.
- Infinite horizon optimal control of forward-backward stochastic differential equations with delay.

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Annexe B: Abréviations et Notations

The following notation is frequently used in this thsis

$\mathbb{R}^n$	n-dimmensional real Euclidiean space.
$\mathbb{R}^{n\times m}$	the set of all $(n\times m)$ real mtrixes.
$C([0,T];\mathbb{R}^n)$	the set of all continuous functions $\varphi:[0,T]\to\mathbb{R}^n$.
$L^p(0,T;\mathbb{R}^n)$	the of Lebegue mesurable functions $\varphi:[0,T]\to\mathbb{R}^n$ such that $\int_0^T \lvert\varphi(t)\rvert^p\,dt<\infty\ \ (p\in[1,\infty))$.
$(\Omega,\mathcal{F},P)$	probability space.
$\{\mathcal{F}_t\}_{t\geq 0}$	filtration.
$\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P\right)$	filtered probability space.
$\sigma(\mathcal{A})$	the smallest $\sigma-$ fielld containing the class $\mathcal{A}$.
$E[X]$	the expectation of the random variable X.
$L^P_{\mathcal{G}}(\Omega,\mathbb{R}^n)$	the set of $\{\mathcal{F}_t\}_{t\geq 0}-$ adapted $\mathbb{R}^n-$ valued processes $X(.)$ such that $E\int_0^T \lvert X_t\rvert^p\,dt<\infty$.
$\mathcal{U}[0,T]$	the set of all $\{\mathcal{F}_t\}_{t\geq 0}-$ adapted processes $u:[0,T]\times\Omega\to U$.
$\mathcal{U}^s_{ad}[0,T]$	the set of (stochastic) strong admissible controls.
$\mathcal{U}^w_{ad}[0,T]$	the set of (stochastic) weak admissible controls.
$D\equiv D([0,T],\mathbb{R}^n)$	the set of all functions $\zeta:[0,T]\to\mathbb{R}^n$ that are right continuous with left limits (càdlàg for short).