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Backward stochastic differential equations with two reflecting barriers and continuous with quadratic growth coefficient

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Abstract

We deal with backward stochastic differential equations with two reflecting barriers and a continuous coefficient which is, first, linear growth in (y, z) and then quadratic growth with respect to z. In both cases we show the existence of a maximal solution.

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0. Introduction

Since their introduction by Pardoux and Peng in [19], the literature on backward stochastic differential equations (BSDEs) has increased steadily. The main reason for

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that is the intervention of these equations in many fields of mathematics such as mathematical finance (see, e.g. [5,6]), stochastic control and games (see, e.g. [3,7–9,12]), partial differential equations and homogenization [18,20,21].

In [4], El-Karoui et al. have introduced the notion of one barrier reflected BSDE, which is actually a backward equation but the solution is forced to stay above a given barrier. Carrying on this work, Cvitanic and Karatzas have introduced in [1] the notion of two barrier reflected BSDE. The solution is now forced to stay between two given barriers.

Precisely a solution for that equation, associated with a coefficient f, a terminal value ξ an upper (resp. lower) barrier U (resp. L), is a quadruple of adapted processes $(Y_t, Z_t, K_t^+, K_t^-)_{t\leqslant T}$ with values in $R^{1+m+1+1}$ which mainly satisfies:

$$\begin{cases} -\mathrm{d}Y_t = f(t, Y_t, Z_t)\,\mathrm{d}t + \mathrm{d}K_t^+ - \mathrm{d}K_t^- - Z_t\,\mathrm{d}B_t, \quad t\leqslant T; \quad Y_T = \xi, \\ L_t \leqslant Y_t \leqslant U_t \quad \text{and} \quad (Y_t - L_t)\,\mathrm{d}K_t^+ = (U_t - Y_t)\,\mathrm{d}K_t^- = 0, \quad \forall t\leqslant T. \end{cases} \tag{1}$$

The process K^+ (resp. K^-) is continuous non-decreasing and its role is to keep Y above L (resp. under U). Moreover they act just when necessary. This type of equation is a powerful tool in zero-sum mixed game problems [9] and in American game options [2].

In [1], Cvitanic and Karatzas have proved the existence and uniqueness of the solution of (1) if, on the one hand, f is Lipschitz and, on the other hand, either the barriers are regular or they satisfy the so-called Mokobodski's condition which means the existence of a difference of non-negative super-martingales between L and U. However, a restrictive condition on f has been supposed when they deal with the case where the barriers are regular. In [11], Hamadène et al. consider also Eq. (1). An improvement of one of Cvitanic and Karatzas's results is obtained. They show the existence of a solution, which is not necessarily unique, when f is continuous with linear growth and when just one of the barriers is regular.

In this paper, we carry on the study of BSDEs with two reflecting barriers. First, we show the existence of a minimal and a maximal solutions for (1) when f is continuous with linear growth and under Mokobodski's condition. In a second part, we deal with the problem of existence of a solution for the same equation when f is continuous with quadratic growth with respect to z. We prove the existence of a solution in that case under either Mokobodski's condition or a regularity assumption on one of the barriers. Finally, an application related to the determination of the value function of a risk-sensitive zero-sum game on stopping times is given.

For BSDEs associated with a continuous generator satisfying a quadratic growth condition in z, but just with one reflecting barrier or without reflection, one can see, respectively, the papers by Kobylanski et al. [14], Kobylanski [13] and Lepeltier and San Martin [16].

The paper is organized as follows:

In the first section we begin to set the problem and to recall the results which provide existence/uniqueness of the solution for double barrier reflected BSDEs. A new and weak formulation of Mokobodski's condition is given.

In Section 2, we first give a comparison theorem of the solutions in the case when the coefficients are Lipschitz. We show that we can compare not only the components Y's but also the non-decreasing processes $K^{\pm}$'s of the solutions. Then using an approximation procedure we show that the two barrier reflected BSDE with a continuous and linear growth coefficient has a maximal and a minimal solutions when Mokobodski's condition is satisfied. In addition, maximal or minimal solutions can also be compared. In those proofs, the comparison of the $K^{\pm}$'s plays an important role.

Section 3 is devoted to the case when the coefficient f is continuous with quadratic growth with respect to the variable z. Using the results of Section 1, we first show the existence of a maximal solution when the coefficient satisfies a so-called *structure condition*. Then with the help of an exponential transform we turn the reflected BSDE whose coefficient is continuous with quadratic growth in z into another one whose coefficient satisfies the *structure condition*. Finally a Logarithmic transform allows us to come back to the original problem and to show the existence of a maximal solution under either Mokobodski's condition or a regularity assumption on one barrier. In the particular case of $f(t,y,z) = h(t,y) + \frac{1}{2}|z|^2$, we prove that the component Y can be identified with the value function of a risk-sensitive stopping zero-sum game. This identification could have an application in the study of American game options in a financial incomplete market with exponential utility.

1. Preliminaries and statement of the problem

Throughout this paper $(\Omega, \mathcal{F}, P)$ is a fixed probability space on which is defined a standard m-dimensional Brownian motion $B = (B_t)_{t \leqslant T}$ whose natural filtration is $(F_t^0 := \sigma\{B_s, s \leqslant t\})_{t \leqslant T}$. We denote by $(F_t)_{t \leqslant T}$ the completed filtration of $(F_t^0)_{t \leqslant T}$ with the P-null sets of $\mathcal{F}$. On the other hand, let:

- $\mathcal{P}$ be the σ-algebra on $[0,T] \times \Omega$ of F_t-progressively measurable sets,
- $\mathcal{H}^{2,k}$ be the set of $\mathcal{P}$-measurable processes $v = (v_t)_{t \leqslant T}$ with values in R^k such that $E[\int_0^T |v_s|^2 \, ds] < \infty$,
- $\mathcal{S}^2$ be the set of $\mathcal{P}$-measurable and continuous processes $Y = (Y_t)_{t \leqslant T}$ such that $E[\sup_{t \leqslant T} |Y_t|^2] < \infty$.

From now on we are given four objects:

(i) a function f from $[0,T] \times \Omega \times R^{1+m}$ into R which with (t, ω, y, z) associates $f(t, \omega, y, z)$ and such that for any $(y,z) \in R^{1+m}$, the process $(f(t,\omega,y,z))_{t \leqslant T}$ is $\mathcal{P}$-measurable and $(f(t,\omega,0,0))_{t \leqslant T}$ belongs to $\mathcal{H}^{2,1}$,

(ii) a random terminal value ξ F_T-measurable and $E[\xi^2] < \infty$,

(iii) two obstacles $U = (U_t)_{t \leqslant T}$ and $L = (L_t)_{t \leqslant T}$ which are processes of $\mathcal{S}^2$ such that P-a.s., $\forall t < T, L_t < U_t$ and $L_T \leqslant \xi \leqslant U_T$.

A solution for the reflected BSDE associated with the coefficient (or generator) f, the terminal value ξ, the upper (resp. lower) obstacle U (resp. L) is a process $(Y, Z, K^+, K^-) := (Y_t, Z_t, K_t^+, K_t^-)_{t \leqslant T}$, $\mathcal{P}$-measurable, with values in $R^{1+m+1+1}$

such that:

$$\begin{cases} Y, K^+ \text{ and } K^- \in \mathscr{S}^2, \quad Z \in \mathscr{H}^{2,m}; \quad K^+, K^- \\ \quad \text{are non-decreasing and } K_0^+ = K_0^- = 0, \\ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,\mathrm{d}s + K_T^+ - K_t^+ - K_T^- + K_t^- - \int_t^T Z_s\,\mathrm{d}B_s, \quad t \leqslant T, \\ \forall t \leqslant T, L_t \leqslant Y_t \leqslant U_t \quad \text{and} \quad \int_0^T (U_s - Y_s)\,\mathrm{d}K_s^- = \int_0^T (Y_s - L_s)\,\mathrm{d}K_s^+ = 0. \end{cases} \tag{2}$$

Let us now gather some assumptions on the data f, ξ, L and U of the problem, which we are sometimes led to assume hereafter.

(H1) There exists a constant $C \geqslant 0$ such that

$$P\text{-a.s. } |f(t,y,z) - f(t,y',z')| \leqslant C(|y-y'| + |z-z'|) \quad \text{for any } t, y, y', z, z'.$$

In this case we say that f is *uniformly Lipschitz* with respect to (y, z).

(H2) The map $(y,z) \mapsto f(t,\omega,y,z)$ is continuous. In addition there exist a constant $C \geqslant 0$ and a process $\gamma := (\gamma_t)_{t \leqslant T}$ which belongs to $\mathscr{H}^{2,1}$ such that

$$P\text{-a.s. } |f(t,y,z)| \leqslant C(\gamma_t + |y| + |z|) \quad \text{for any } t, y, z.$$

When f satisfies this assumption, it is said *continuous with linear growth* with respect to (y, z).

(H3) There exist a constant $C \geqslant 0$ and a function φ from R into R^+, which is bounded on compact subsets of R, such that

$$P\text{-a.s. } |f(t,y,z)| \leqslant C(1 + \varphi(y) + |z|^2) \quad \text{for any } t, y, z.$$

In addition the mapping $(y,z) \mapsto f(t,\omega,y,z)$ is continuous. In that case f is said *continuous with quadratic growth* with respect to z.

(H4) A process $X = (X_t)_{t \leqslant T}$ of $\mathscr{S}^2$ is called *regular* if there exists a sequence of processes $(X^n)_{n \geqslant 0}$ such that:

(i) $\forall t \leqslant T$, $X_t^n \geqslant X_t^{n+1}$ and $\lim_{n \to +\infty} X_t^n = X_t$, P-a.s.
(ii) $\forall n \geqslant 0$ and $t \leqslant T$, $X_t^n = X_0^n + \int_0^t x_n(s)\,\mathrm{d}s + \int_0^t \bar{x}_n(s)\,\mathrm{d}B_s$, where x_n, $\bar{x}_n$ are $\mathscr{F}_t$-adapted processes such that

$$\sup_n \sup_{t \leqslant T} \max\{x_n(t), 0\} \leqslant C \quad \text{and} \quad E\left[\left\{\int_0^T |\bar{x}_n(s)|^2\right\}^{1/2} \mathrm{d}s\right] < +\infty \quad \forall n \geqslant 1.$$

(H5) *Mokobodski's condition*: There exist two non-negative super-martingales $\eta := (\eta_t)_{t \leqslant T}$ and $\theta := (\theta_t)_{t \leqslant T}$ which belong to $\mathscr{S}^2$ such that $\forall t \in [0, T]$, $L_t 1_{[t<T]} + \xi 1_{[t=T]} \leqslant \eta_t - \theta_t + E[\xi | F_t] \leqslant U_t 1_{[t<T]} + \xi 1_{[t=T]}$.

(H6) The obstacles U, L and the r.v. ξ are bounded, i.e., there exists a constant $C \geqslant 0$ such that P-a.s., $\forall t \leqslant T$, $|U_t| + |L_t| + |\xi| \leqslant C$.

In this paper we have two main objectives. The first one is to show that (2) has a solution if the assumptions (H2) and (H5′) (which is a weak version of Mokobodski's condition, see Lemma 1.3 below) are fulfilled. The second is to deal with reflected BSDEs with coefficients which are continuous and with quadratic growth with respect to z. We prove that under the assumptions (H3), (H6) and some other conditions, which are linked to (H4) or (H5′), Eq. (2) has also a solution.

However, to begin with, we recall the known results which provide a solution for (2). Mainly they are of two types. Either it is assumed that Mokobodski's condition is fulfilled or that the upper barrier is *regular*. Precisely we have:

Theorem 1.1 (*Cvitanic and Karatzas [1], Hamadène and Lepeltier [9]*). *If the assumptions* (H1) *and* (H5) *hold, then the reflected BSDE* (2) *has a unique solution.*

Theorem 1.2 (*Hamadène et al. [11]*). *Under the hypothesis* (H2) *and if* U *or* $-L:=(-L_t)_{t\leqslant T}$ *satisfies* (H4), Eq. (2) *has a solution which is not necessarily unique. In addition if, instead of* (H2), f *satisfies* (H1) *then the solution is unique.*

In [11], the proof is done for the case when the upper barrier U is *regular*. However, this proof remains valid (only minor changes necessary) if the regularity assumption holds on $-L$.

Mokobodski's condition in (H5) is a bit stringent since it requires the continuity of the non-negative super-matingales η and θ which, moreover, should satisfy $\eta_T = \theta_T$. Now, when we make use of this condition in order to show the existence of a solution for Eq. (2), the continuity of η and θ is irrelevant (see e.g. [1,9]). We just need that they are right continuous with left limits (r.c.l.l. for short). Therefore, Theorem 1.1 remains valid if (H5) holds with just r.c.l.l. super-martingales. This remark allows us to weaken the hypothesis (H5) in the following way:

Lemma 1.3. *Assume that*:

(H5′) *There exist two non-negative r.c.l.l. super-martingales* $\eta=(\eta_t)_{t\leqslant T}$ *and* $\theta=(\theta_t)_{t\leqslant T}$ *such that*

$$\forall t<T,\ \ L_t\leqslant \eta_t-\theta_t\leqslant U_t \quad \text{and} \quad E\left[\sup_{t\leqslant T}\{|\eta_t|+|\theta_t|\}^2\right]<\infty.$$

Then Mokobodski's condition is satisfied.

Proof. For $t\leqslant T$, let $\tilde{\eta}_t=(\eta_t+E[\xi^-|F_t])1_{[t<T]}$ and $\tilde{\theta}_t=(\theta_t+E[\xi^+|F_t])1_{[t<T]}$. Since η and θ are non-negative super-martingales then $\tilde{\theta}$ and $\tilde{\eta}$ still non-negative super-martingales which are also r.c.l.l. . Moreover, they satisfy $L_t1_{[t<T]}+\xi 1_{[t=T]}\leqslant \tilde{\eta}_t-\tilde{\theta}_t+E[\xi|F_t]\leqslant U_t1_{[t<T]}+\xi 1_{[t=T]}$ and $E[\sup_{t\leqslant T}\{|\tilde{\eta}_t|+|\tilde{\theta}_t|\}^2]<\infty$. Thus, Mokobodski's condition is satisfied with two non-negative r.c.l.l. super-martingales. Therefore, as it is pointed out previously, the conclusion of Theorem 1.1 remains valid if (H1) and (H5′) hold. $\square$

2. Reflected BSDEs under Mokobodski's condition and linear growth

In [11], the authors show that the reflected BSDE (2) has a solution if f is continuous with linear growth and the barrier U is *regular*. In this section we are going to replace the regularity of U by Mokobodski's condition (which from now on is (H5′)) and to show, once again, that Eq. (2) has a solution. On the same subject, we are aware of a recent work of Lepeltier and San Martin [17]. They have obtained the existence of a solution for (2) when f satisfies (H3) but with a rather stronger condition, with respect to (H5′), on the barriers. In addition, the proofs are completely different.

We begin to give a comparison theorem which allows to compare the components Y's, $K^{\pm}$'s of two solutions of reflected BSDEs. This result is crucial in order to reach the linear growth case, i.e., when f satisfies (H2), from the Lipschitz case.

Let $(f'(t,\omega,y,z),\xi',L',U')$ be another quadruple such that for any $(y,z)\in R^{1+m}$, $(f'(t,y,z))_{t\leqslant T}$ is $\mathscr{P}$-measurable, $L'_t<U'_t, \forall t<T$, ξ' is F_T-measurable, square integrable and $L'_T\leqslant\xi'\leqslant U'_T$.

Theorem 2.1. *Assume that the reflected BSDE associated with* (f,ξ,L,U) (*resp.* (f',ξ',L',U')) *has a solution* $(Y_t,Z_t,K_t^+,K_t^-)_{t\leqslant T}$ (*resp.* $(Y'_t,Z'_t,K'^+_t,K'^-_t)_{t\leqslant T}$). *Then*:

(i) *if* f *satisfies* (H1), $\xi\leqslant\xi'$ *and for any* $t\leqslant T$, $L_t\leqslant L'_t$, $U_t\leqslant U'_t$, $f(t,Y'_t,Z'_t)\leqslant f'(t,Y'_t,Z'_t)$, *then we have* P-a.s. $Y\leqslant Y'$.

(ii) *if moreover*:

(a) $f(t,y,z)\leqslant f'(t,y,z)$ *for any* (t,y,z), $(f'(t,0,0))_{t\leqslant T}$ *belongs to* $\mathscr{H}^{2,1}$ *and* f' *satisfies* (H1),

(b) $L\equiv L'$, $U\equiv U'$

then we have also P-a.s., $\forall t\leqslant T$, $K_t^-\leqslant K'^-_t$ *and* $K_t^+\geqslant K'^+_t$.

Proof. First let us show that $Y\leqslant Y'$. Let us set $K_s=K_s^+-K_s^-$ and $K'_s=K'^+_s-K'^-_s$, $s\leqslant T$. Using Tanaka's formula [15,22] with $(Y-Y')^{+2}$ yields

$$\begin{aligned}
(Y_t-Y'_t)^{+2}&+\int_t^T 1_{[Y_s>Y'_s]}|Z_s-Z'_s|^2\,\mathrm{d}s\\
&=2\int_t^T (Y_s-Y'_s)^+(f(s,Y_s,Z_s)-f'(s,Y'_s,Z'_s))\,\mathrm{d}s\\
&\quad+2\int_t^T (Y_s-Y'_s)^+(\mathrm{d}K_s-\mathrm{d}K'_s)-2\int_t^T (Y_s-Y'_s)^+(Z_s-Z'_s)\,\mathrm{d}B_s\\
&\leqslant 2\int_t^T (Y_s-Y'_s)^+(f(s,Y_s,Z_s)-f(s,Y'_s,Z'_s))\,\mathrm{d}s\\
&\quad+2\int_t^T (Y_s-Y'_s)^+(\mathrm{d}K_s-\mathrm{d}K'_s)-2\int_t^T (Y_s-Y'_s)^+(Z_s-Z'_s)\,\mathrm{d}B_s
\end{aligned}$$

since $f(t,Y'_t,Z'_t)\leqslant f'(t,Y'_t,Z'_t)$. But $\int_0^t(Y_s-Y'_s)^+(\mathrm{d}K_s-\mathrm{d}K'_s)=\int_t^T(Y_s-Y'_s)^+(-\mathrm{d}K_s^--\mathrm{d}K'^+_s)\leqslant 0$ because when $Y_t>Y'_t$ we have $Y_t>L_t$ and $U_t>Y'_t$. Hence for any $t \leqslant T$,

$$
\begin{aligned}
(Y_t - Y'_t)^{+2} &+ \int_t^T 1_{[Y_s > Y'_s]} |Z_s - Z'_s|^2 \, ds \\
&\leqslant 2 \int_t^T (Y_s - Y'_s)^+ (f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)) \, ds \\
&\quad - 2 \int_t^T (Y_s - Y'_s)^+ (Z_s - Z'_s) \, dB_s.
\end{aligned}
$$

Now, since f is Lipschitz then we can write $f(t, Y_t, Z_t) - f(t, Y'_t, Z'_t) = a_t(Y_t - Y'_t) + b_t(Z_t - Z'_t)$, where $(a_t)_{t \leqslant T}$ and $(b_t)_{t \leqslant T}$ are bounded $\mathscr{P}$-measurable processes. Therefore,

$$
\begin{aligned}
(Y_t - Y'_t)^{+2} &+ \int_t^T 1_{[Y_s > Y'_s]} |Z_s - Z'_s|^2 \, ds \\
&\leqslant 2 \int_t^T (Y_s - Y'_s)^+ \{a_s(Y_s - Y'_s) + b_s(Z_s - Z'_s)\} \, ds \\
&\quad - 2 \int_t^T (Y_s - Y'_s)^+ (Z_s - Z'_s) \, dB_s.
\end{aligned}
$$

Next, using the inequality $|a.b| \leqslant \varepsilon |a|^2 + \varepsilon^{-1} |b|^2$, $\forall \varepsilon > 0$ and $a, b \in R^k$, we obtain

$$
(Y_t - Y'_t)^{+2} \leqslant C \int_t^T (Y_s - Y'_s)^{+2} \, ds - 2 \int_t^T (Y_s - Y'_s)^+ (Z_s - Z'_s) \, dB_s
$$

where C is a constant. Now since $\int_0^{\cdot} (Y_s - Y'_s)^+ (Z_s - Z'_s) \, dB_s$ is a martingale then taking expectation on both sides and using Gronwall's inequality to get $E[(Y_t - Y'_t)^{+2}] = 0, \forall t \leqslant T$ and then $Y \leqslant Y'$.

We now prove that $K'^- \geqslant K^-$. Let $\tau = \inf\{t \geqslant 0, K_t^- > K'^-_t\} \wedge T$ (hereafter we always assume that $\inf\{\emptyset\} = +\infty$). We are going to show that $P[\tau < T] = 0$ which implies that $K_t^- \leqslant K'^-_t, \forall t < T$ and then $K^- \leqslant K'^-$ by continuity.

Suppose that $P[\tau < T] > 0$. As K^- and K'^- are continuous processes then we have $K_\tau^- = K'^-_\tau$ on the set $\{\tau < T\}$.

On the other hand we also have $Y_\tau = Y'_\tau = U_\tau$ on the set $\{\tau < T\}$. Indeed, let $\omega \in \{\tau < T\}$. If $Y_{\tau(\omega)}(\omega) \neq U_{\tau(\omega)}(\omega)$, then there exists a real number $\eta(\omega) > 0$ such that $\forall t \in]\tau(\omega) - \eta(\omega), \tau(\omega) + \eta(\omega)[$ we have $Y_t(\omega) < U_t(\omega)$ which implies that $K^-_{\tau(\omega)}(\omega) = K'^-_{\tau(\omega)}(\omega) = K_t^-(\omega) \leqslant K'^-_t(\omega)$, $\forall t \in [\tau(\omega), \tau(\omega) + \eta(\omega)[$. But this contradicts the definition of $\tau(\omega)$, henceforth $Y_{\tau(\omega)}(\omega) = U_{\tau(\omega)}(\omega) = Y'_{\tau(\omega)}(\omega)$ since $Y \leqslant Y' \leqslant U$.

Now let $\delta = \inf\{t \geqslant \tau, Y_t = L_t\} \wedge T$. We have $\{\tau < T\} \subset \{\delta > \tau\}$. Indeed if ω is such that $\tau(\omega) < T$ then $Y_{\tau(\omega)}(\omega) = U_{\tau(\omega)}(\omega)$. Now if $\delta(\omega) = \tau(\omega)$ then $Y_{\delta(\omega)}(\omega) = L_{\delta(\omega)}(\omega) = U_{\tau(\omega)}(\omega) = L_{\tau(\omega)}(\omega)$ which is absurd since $U_t > L_t, \forall t < T$. Hence $\{\tau < T\} \subset \{\delta > \tau\}$ and then $P[\delta > \tau] > 0$.

Now for $t \in [\tau, \delta]$ we have $K_t^+ = K_\delta^+$ and $K'^+_t = K'^+_\delta$ since the processes K^+ (resp. K'^+) moves only when Y (resp. Y') reaches the obstacle L. It follows that, $\forall t \in [\tau, \delta]$,

$$Y_t = Y_\delta + \int_t^\delta f(s, Y_s, Z_s)\,\mathrm{d}s - (K_\delta^- - K_t^-) - \int_t^\delta Z_s\,\mathrm{d}B_s,$$

$$Y'_t = Y'_\delta + \int_t^\delta f'(s, Y'_s, Z'_s)\,\mathrm{d}s - (K'^-_\delta - K'^-_t) - \int_t^\delta Z'_s\,\mathrm{d}B_s.$$

Now let $(\bar{Y}_t, \bar{Z}_t, \bar{K}_t)_{t\leqslant\delta}$ (resp. $(\bar{Y}'_t, \bar{Z}'_t, \bar{K}'_t)_{t\leqslant\delta}$) be the unique solution on $[0, \delta]$ of the BSDE whose coefficient is f (resp. f'), the terminal value Y_δ (resp. Y'_δ) and reflected by the upper obstacle U, i.e.,

$$\bar{Y}_t = Y_\delta + \int_t^\delta f(s, \bar{Y}_s, \bar{Z}_s)\,\mathrm{d}s - (\bar{K}_\delta^- - \bar{K}_t^-) - \int_t^\delta \bar{Z}_s\,\mathrm{d}B_s$$

$$\left(\text{resp. } \bar{Y}'_t = Y'_\delta + \int_t^\delta f'(s, \bar{Y}'_s, \bar{Z}'_s)\,\mathrm{d}s - (\bar{K}'^-_\delta - \bar{K}'^-_t) - \int_t^\delta \bar{Z}'_s\,\mathrm{d}B_s, \forall t\leqslant\delta\right).$$

The comparison theorem for one upper barrier reflected BSDEs (see, e.g. [11, Proposition 2.3]) implies that $\bar{Y}\leqslant\bar{Y}'$ and $\bar{K}_t - \bar{K}_s\leqslant\bar{K}'_t - \bar{K}'_s, \forall s\leqslant t\leqslant\delta$. Now since f and f' are Lipschitz in (y,z) then $\forall t \in [\tau, \delta]$ we have $\bar{Y}_t = Y_t$, $\bar{Y}'_t = Y'_t$, $\bar{Z}_t = Z_t$ and $\bar{Z}'_t = Z'_t$. It follows that $\bar{K}_\delta^- - \bar{K}_t^- = K_\delta^- - K_t^-$ and $\bar{K}'^-_\delta - \bar{K}'^-_t = K'^-_\delta - K'^-_t, \forall t \in [\tau, \delta]$. Hence we have $K'^-_t - K'^-_s\geqslant K_t^- - K_s^-$ for any $\tau(\omega)\leqslant s\leqslant t\leqslant\delta(\omega)$. As on the set $\{\tau<T\}$, $K'^-_\tau = K_\tau^-$ then $K'^-_t(\omega)\geqslant K_t^-(\omega), \forall t \in [\tau(\omega), \delta(\omega)]$. But this contradicts the definition of τ, hence $P[\tau<T] = 0$ and then $K^-\leqslant K'^-$. In the same way we can show that P-a.s., $K^+\geqslant K'^+$, whence the desired result. $\square$

Remark 2.2. The process K^- (resp. K^+) in definition (2) stands for, in a sense, the power which is deployed in order to keep the component Y of the solution under (resp. above) the barrier U (resp. L). So since $Y\leqslant Y'$ then we can obviously guess that $K^-\leqslant K'^-$ and $K^+\geqslant K'^+$.

We now show that the reflected BSDE (2) has a solution under the assumptions (H2) and (H5′), i.e., when f is continuous with linear growth and under Mokobodski's condition.

Theorem 2.3. *Assume that* (H2) *and* (H5′) *are fulfilled. Then the reflected BSDE associated with* (f, ξ, L, U) *has a solution* $(Y_t, Z_t, K_t^+, K_t^-)_{t\leqslant T}$ *which is moreover maximal*, i.e., *if* $(Y'_t, Z'_t, K'^+_t, K'^-_t)_{t\leqslant T}$ *is another solution then* P-a.s., $Y\geqslant Y'$.

Proof. For $n\geqslant 1$ let f_n be the function defined as follows:

$$f_n(t,\omega,y,z) := \sup_{(u,v)\in R^{1+m}} \{f(t,\omega,u,v) - (n+C)(|u-y| + |v-z|)\}, \tag{3}$$

where C is the constant of linear growth of f (see (H2)). The function f_n satisfies:

$$\begin{aligned} -C(\gamma_t(\omega) + |y| + |z|) &\leqslant f_n(t,\omega,y,z) \\ &\leqslant C\gamma_t(\omega) + \sup_{(u,v)\in R^{1+m}} \{C(|u| + |v|) - (n+C)(|u-y| + |v-z|)\} \\ &\leqslant C(\gamma_t(\omega) + |y| + |z|). \end{aligned}$$

Therefore it is finite and satisfies $|f_n(t,\omega,y,z)| \leqslant C(\gamma_t(\omega)+|y|+|z|)$. On the other hand it is Lipschitz in (y,z) uniformly in (t,ω) since

$$|f_n(t,\omega,y,z)-f_n(t,\omega,y',z')| \leqslant (C+n)(|y-y'|+|z-z'|).$$

Indeed, basically this stems from the inequality $|\sup_{i\in I} a_i - \sup_{i\in I} b_i| \leqslant \sup_{i\in I} |a_i-b_i|$. Finally, $f_n \geqslant f_{n+1}$ and P-a.s. for any (t,y,z) the sequence $(f_n(t,\omega,y,z))_{n\geqslant 1}$ converges to $f(t,\omega,y,z)$. Actually, for any $n\geqslant 1$ there exits (u_n,v_n) such that $f_n(t,\omega,y,z) \leqslant f(t,\omega,u_n,v_n)-(n+C)\{|u_n-y|+|v_n-z|\}+n^{-1}$. Therefore, we have $f_n(t,\omega,y,z)+(n+C)\{|u_n-y|+|v_n-z|\} \leqslant f(t,\omega,u_n,v_n)+n^{-1}$. It implies that $\lim_{n\to\infty}(u_n,v_n)=(y,z)$ and then $\lim_{n\to\infty} f_n(t,\omega,y,z) \leqslant f(t,\omega,y,z)$. Therefore, $\lim_{n\to\infty} f_n(t,\omega,y,z) = f(t,\omega,y,z)$ since $f_n \geqslant f$.

Now according to Theorem 2.1, there exists a process $(Y^n, Z^n, K^{+,n}, K^{-,n})$ solution of the reflected BSDE associated with (f_n,ξ,L,U), i.e., which satisfies:

$$\begin{cases} Y^n, K^{+,n} \text{ and } K^{-,n} \in \mathcal{S}^2, \quad Z^n \in \mathcal{H}^{2,m}; \text{ moreover } K^{+,n}\ K^{-,n} \text{ are non-decreasing} \\ \quad (K_0^{+,n}=K_0^{-,n}=0), \\ Y_t^n = \xi + \int_t^T f_n(s,Y_s^n,Z_s^n)\,\mathrm{d}s + K_T^{+,n} - K_t^{+,n} - K_T^{-,n} + K_t^{-,n} - \int_t^T Z_s^n\,\mathrm{d}B_s, \\ \quad t \leqslant T, \\ \forall t\leqslant T,\ L_t \leqslant Y_t^n \leqslant U_t \quad \text{and} \quad \int_0^T (U_s - Y_s^n)\,\mathrm{d}K_s^{-,n} = \int_0^T (Y_s^n - L_s)\,\mathrm{d}K_s^{+,n} = 0. \end{cases}$$

As $f_n \geqslant f_{n+1}$ then according to comparison Theorem 1.1 we have $Y^n \geqslant Y^{n+1}$, $K^{+,n} \leqslant K^{+,n+1}$ and $K^{-,n} \geqslant K^{-,n+1}$. Now since for any $t\leqslant T$, $L_t \leqslant Y_t^n \leqslant U_t$ and L, U belong to $\mathcal{S}^2$ then there exists a $\mathcal{P}$-measurable process $Y := (Y_t)_{t\leqslant T}$ such that P-a.s. for any $t\leqslant T$ the sequence $(Y_t^n)_{n\geqslant 1}$ converges pointwisely to Y_t and the sequence of processes $(Y^n)_{n\geqslant 1}$ converges in $\mathcal{H}^{2,1}$ to Y.

On the other hand, let $(\underline{Y}, \underline{Z}, \underline{K}^+, \underline{K}^-)$ be the unique solution of the reflected BSDE associated with $(-C(\gamma+|y|+|z|),\xi,L,U))$. Once again, the comparison Theorem 1.1 implies that $K^{+,n} \leqslant \underline{K}^+$, $\forall n \geqslant 1$. As $E[(K_T^{-,0})^2 + (\underline{K}_T^+)^2] < \infty$, then P-a.s., for any $t\leqslant T$, the sequence $(K_t^{+,n})_{n\geqslant 1}$ (resp. $(K_t^{-,n})_{n\geqslant 1}$) converges to K_t^+ (resp. K_t^-). In addition, the process $K^+ = (K_t^+)_{t\leqslant T}$ (resp. $K^- = (K_t^-)_{t\leqslant T}$) is non-decreasing lower (resp. upper) semi-continuous and $E[(K_T^+)^2] < \infty$ (resp. $E[(K_T^-)^2] < \infty$).

Now using Itô's formula with $(Y^n)^2$ and standard calculations yield $E[\int_0^T |Z_s^n|^2\,\mathrm{d}s] \leqslant C$, where C is a constant which does not depend on n.

Let us show that Y is continuous and the sequence $(Z^n)_{n\geqslant 1}$ is convergent in $\mathcal{H}^{2,m}$. Using Itô's formula with $(Y^n - Y^m)^2$ yields, for any $t\leqslant T$,

$$\begin{aligned} &(Y_t^n - Y_t^m)^2 + E\left[\int_t^T |Z_s^n - Z_s^m|^2\,\mathrm{d}s\right] \\ &\quad = 2\int_t^T (Y_s^n - Y_s^m)(f_n(s,Y_s^n,Z_s^n) - f_m(s,Y_s^m,Z_s^m))\,\mathrm{d}s \\ &\qquad + 2\int_t^T (Y_s^n - Y_s^m)\,\mathrm{d}(K_s^{+,n} - K_s^{+,m} - K_s^{-,n} + K_s^{-,m}) \\ &\qquad - 2\int_t^T (Y_s^n - Y_s^m)(Z_s^n - Z_s^m)\,\mathrm{d}B_s. \end{aligned} \tag{4}$$

But $(\int_0^t (Y_s^n - Y_s^m)(Z_s^n - Z_s^m)\,\mathrm{d}B_s)_{t\leqslant T}$ is an (F_t, P)-martingale and

$$\begin{aligned}
&\int_t^T (Y_s^n - Y_s^m)\,\mathrm{d}(K_s^{+,n} - K_s^{+,m} - K_s^{-,n} + K_s^{-,m}) \\
&\quad = \int_t^T (Y_s^n - Y_s^m)\,\mathrm{d}(K_s^{+,n} - K_s^{+,m}) - \int_t^T (Y_s^n - Y_s^m)\,\mathrm{d}(K_s^{-,n} - K_s^{-,m}) \\
&\quad = -\int_t^T (Y_s^n - L_s)\,\mathrm{d}K_s^{+,m} + \int_t^T (L_s - Y_s^m)\,\mathrm{d}K_s^{+,n} - \int_t^T (U_s - Y_s^m)\,\mathrm{d}K_s^{+,n} \\
&\qquad + \int_t^T (Y_s^m - U_s)\,\mathrm{d}K_s^{-,n} \leqslant 0.
\end{aligned}$$

Then taking into account the linear growth of f_n, the boundedness of $(Z_n)_{n\geqslant 1}$ in $\mathscr{H}^{2,m}$ and the fact that $L \leqslant Y^n \leqslant U$ yield,

$$E\left[\int_0^T |Z_s^n - Z_s^m|^2\,\mathrm{d}s\right] \leqslant C\sqrt{E\left[\int_0^T |Y_s^n - Y_s^m|^2\,\mathrm{d}s\right]}.$$

Therefore $(Z^n)_{n\geqslant 1}$ is a Cauchy sequence in $\mathscr{H}^{2,m}$ and then converges in the same space to a process $Z = (Z_t)_{t\leqslant T}$.

Now going back to (4), taking the supremum and using the Burkholder–Davis–Gundy inequality [15,22] we obtain

$$E\left[\sup_{t\leqslant T} |Y_t^n - Y_t^m|^2\right] \leqslant C\left\{\sqrt{E\left[\int_0^T |Y_s^n - Y_s^m|^2\,\mathrm{d}s\right]} + E\left[\int_0^T |Z_s^n - Z_s^m|^2\,\mathrm{d}s\right]\right\}.$$

Henceforth the sequence $(Y^n)_{n\geqslant 1}$ converges also to Y in $\mathscr{S}^2$ and then Y is continuous.

Next we focus on the continuity of the processes $K^{\pm}$. For any $t \leqslant T$ we have,

$$K_t^{+,n} - K_t^{-,n} = Y_0^n - Y_t^n - \int_0^t f_n(s, Y_s^n, Z_s^n)\,\mathrm{d}s + \int_0^t Z_s^n\,\mathrm{d}B_s. \tag{5}$$

But there exists a subsequence of the sequence of processes $((f(t,\omega, Y_t^n, Z_t^n))_{t\leqslant T})_{n\geqslant 1}$ which converges in $L^1(\Omega \times [0,T], \mathrm{d}P \otimes \mathrm{d}t)$ to $(f(t,\omega, Y_t, Z_t))_{t\leqslant T}$. Actually for any $\delta \geqslant 1$ we have,

$$\begin{aligned}
&E\left[\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|\,\mathrm{d}s\right] \\
&\quad = E\left[\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| 1_{[|Y_s^n| + |Z_s^n| \leqslant \delta]}\,\mathrm{d}s\right] \\
&\qquad + E\left[\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| 1_{[|Y_s^n| + |Z_s^n| > \delta]}\,\mathrm{d}s\right].
\end{aligned}$$

But

$$E\left[\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| 1_{[|Y_s^n|+|Z_s^n| \leqslant \delta]} \, ds\right]$$
$$\leqslant E\left[\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n)| 1_{[|Y_s^n|+|Z_s^n| \leqslant \delta]} \, ds\right]$$
$$+ E\left[\int_0^T |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| 1_{[|Y_s^n|+|Z_s^n| \leqslant \delta]} \, ds\right].$$

The first term in the right-hand side converges to 0, as $n \to \infty$, since P-a.s, $\forall t \leqslant T$, $\sup_{|y|+|z| \leqslant \delta} |f_n(t, \omega, y, z) - f(t, \omega, y, z)| \to 0$ (thanks to Dini's theorem) and through Lebesgue dominated convergence theorem. The second term converges also to 0 at least along a subsequence. Now in order to complete the proof of the claim it is just enough to underline that we have

$$E\left[\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| 1_{[|Y_s^n|+|Z_s^n| > \delta]} \, ds\right] \leqslant \frac{C}{\sqrt{\delta}}$$

since $L \leqslant Y^n \leqslant U$, the sequence $(Z^n)_{n \geqslant 1}$ is uniformly bounded in $\mathscr{H}^{2,m}$ and finally taking into account the linear growth of f and f_n.

Therefore from (5) there exists a subsequence of $(K^{+,n} - K^{-,n})_{n \geqslant 1}$ (which we still denote by n) such that:

$$\lim_{n,m \to \infty} E\left[\sup_{t \leqslant T} |(K_t^{+,n} - K_t^{-,n}) - (K_t^{+,m} - K_t^{-,m})|\right] = 0.$$

It follows that the process $K^+ - K^-$ is continuous and once again from (5) we deduce that:

$$P\text{-a.s. } \forall t \leqslant T, \quad K_t^+ - K_t^- = Y_0 - Y_t - \int_0^t f(s, Y_s, Z_s) \, ds + \int_0^t Z_s \, dB_s \tag{6}$$

and then

$$\forall t \leqslant T, \quad K_t^+ = K_t^- + Y_0 - Y_t - \int_0^t f(s, Y_s, Z_s) \, ds + \int_0^t Z_s \, dB_s.$$

But K^+ is lower semi-continuous and K^- is upper semi-continuous. It means that K^+ and K^- are lower and upper semi-continuous in the same time therefore they are continuous and then belong to $\mathscr{S}^2$ since we know already that $E[(K_T^+)^2 + (K_T^-)^2] < \infty$.

Now from (6) we have: $\forall t \leqslant T$

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s \, dB_s.$$

In order to finish the proof it remains to show that $\int_0^T (Y_s - U_s) \, dK_s^- = \int_0^T (Y_s - L_s) \, dK_s^+ = 0$. But this is a direct consequence of the convergence of $(Y^n)_{n \geqslant 1}$, $(K^{+,n})_{n \geqslant 1}$ and $(K^{-,n})_{n \geqslant 1}$ in $\mathscr{S}^2$ respectively to Y, K^+ and K^- and since for any $n \geqslant 1$

we have $\int_0^T (Y_s^n - U_s)\,\mathrm{d}K_s^{-,n} = \int_0^T (Y_s^n - L_s)\,\mathrm{d}K_s^{+,n} = 0$. The proof of this claim can be read in [10, p. 10].

Finally, Y is the maximal solution because if (Y', Z', K'^{+}, K'^{-}) is another solution for the reflected BSDE associated with (f, ξ, L, U). Then according to Theorem 1.1 we have P-a.s., $Y^n \geqslant Y'$ and $Y \geqslant Y'$. The proof is now complete. $\square$

An example where Mokobodski's condition is satisfied is the following: assume that for $t \leqslant T$, $L_t = L_0 + \int_0^t l_s\,\mathrm{d}s + \int_0^t \tilde{l}_s\,\mathrm{d}B_s$ where the processes $(l_t)_{t\leqslant T}$ and $(\tilde{l}_t)_{t\leqslant T}$ belong to $\mathscr{H}^{2,1}$ and $\mathscr{H}^{2,m}$, respectively. Then (H5′) is satisfied with $\eta_t = E[L_T^+ + \int_t^T l_s^-\,\mathrm{d}s|F_t]$ and $\theta_t = E[L_T^- + \int_t^T l_s^+\,\mathrm{d}s|F_t]$. However it is not necessarily true that (H5) is satisfied since we do not know whether or not θ_T is equal to η_T.

Remark 2.4. In the previous theorem, the machinery works since it is possible to claim that, for every Lipschitz coefficient $\tilde{f}$, the reflected BSDE associated with $(\tilde{f}, \xi, L, U)$ has a unique solution. So if instead of (H5′) we assume that U or $-L$ satisfies (H4), in combination with (H2), then with the help of Theorem 1.2 we obtain the same result as in Theorem 2.3.

Had we approximated the function f by a non-decreasing sequence of Lipschitz functions, we would have constructed the minimal solution of the reflected BSDE. Therefore we have,

Corollary 2.5. *Assume that* (H2) *and either* (H5′) *or, U or $-L$ satisfies* (H4). *Then the reflected BSDE associated with (f, ξ, L, U) has a minimal solution $(\bar{Y}_t, \bar{Z}_t, \bar{K}_t^{+}, \bar{K}_t^{-})_{t\leqslant T}$,* i.e., *if $(\bar{Y}'_t, \bar{Z}'_t, \bar{K}_t^{'+}, \bar{K}_t^{'-})_{t\leqslant T}$ is another solution then* P-a.s., $Y \leqslant \bar{Y}'$.

We have seen in Theorem 1.1 that we can compare the solutions of reflected BSDEs in the case when, at least, one of the coefficients is uniformly Lipschitz. In the following result, which will be useful in the next section, we show that maximal solutions associated with coefficient which are of linear growth at most, can also be compared.

Proposition 2.6. Proposition *Let f, f' be two coefficients which satisfy the assumption* (H2) *and such that* P-a.s., $f(t,\omega,y,z) \leqslant f'(t,\omega,y,z)$, *for any t, y and z. Moreover assume that* (H5′) *or, U or $-L$ satisfies* (H4). *Let $(Y_t, Z_t, K_t^+, K_t^-)_{t\leqslant T}$ (resp. $(Y'_t, Z'_t, K'^{+}_t, K'^{-}_t)_{t\leqslant T}$) be the maximal solution of the reflected BSDE associated with (f, ξ, L, U) (resp. (f', ξ, L, U)), then* P-a.s., $Y \leqslant Y'$, $K^+ \geqslant K'^{+}$ *and* $K^- \leqslant K'^{-}$.

Proof. First let us point out that w.l.o.g. we can assume that the constants of linear growth of f and f' are the same. Now for $n \geqslant 1$ let f'_n be the function defined as follows:

$$f'_n(t,\omega,y,z) := \sup_{(u,v)\in R^{1+m}} \{f'(t,\omega,u,v) - (n+C)(|u-y| + |v-z|)\}.$$

So for any $n \geqslant 1$, we have $f'_n \geqslant f_n$. Now for $n \geqslant 1$ let $(Y^n, Z^n, K^{+,n}, K^{-,n})$ (resp. $(Y'^n, Z'^n, K'^{+,n}, K'^{-,n})$) be the solution of the reflected BSDE associated with

(f_n, ξ, L, U) (resp. (f'_n, ξ, L, U)). Therefore the comparison Theorem 1.1 implies that P-a.s., $Y^n \leqslant Y'^n$, $K^{+,n} \geqslant K'^{+,n}$ and $K^{-,n} \leqslant K'^{-,n}$. As $((Y^n, Z^n, K^{+,n}, K^{-,n}))_{n\geqslant 1}$ (resp. $((Y'^n, Z'^n, K'^{+,n}, K'^{-,n}))_{n\geqslant 1}$) converges to $(Y_t, Z_t, K_t^+, K_t^-)_{t\leqslant T}$ (resp. $(Y'_t, Z'_t, K'^+_t, K'^-_t)_{t\leqslant T}$) the maximal solution of the reflected BSDE associated with (f, ξ, L, U) (resp. (f', ξ, L, U)), we obtain P-a.s., $Y \leqslant Y'$, $K^+ \geqslant K'^+$ and $K^- \leqslant K'^-$. $\square$

3. Double barrier reflected BSDEs with quadratic growth with respect to z

In this section, we prove the existence of a maximal solution for a two barrier reflected BSDE with a continuous generator f which satisfies a quadratic growth condition w.r.t. z. This is done both under Mokobodski's condition as well as in the case when one of the barriers satisfies the regularity assumption (H4). However, we begin to give an intermediate result which states the existence of a maximal solution under a *structure condition* on the coefficient. Then, in the general case we use an exponential transform and we obtain a new generator which satisfies the *structure condition*. Therefore, the associated BSDE has a maximal solution. Finally, a Logarithmic transform leads to the solution of the initial problem. The change of the coefficient, in using an exponential function, is a technique which has been already used in order to study BSDEs with a generator which has the same properties as in our frame but without reflection (e.g. [13,16]) or with just one reflecting barrier (e.g. [14]).

Theorem 3.1. *Let*

(i) *η be a bounded F_T-measurable random variable with values in R,*

(ii) $\bar{L} := (\bar{L}_t)_{t\leqslant T}$ and $\bar{U} := (\bar{U}_t)_{t\leqslant T}$ be two bounded and *$\mathscr{P}$-measurable processes such that $\forall t < T, \bar{L}_t < \bar{U}_t$ and $\bar{L}_T \leqslant \eta \leqslant \bar{U}_T$. In addition there exists a constant $\alpha > 0$ such that $\forall t \leqslant T$, $\bar{L}_t \geqslant \alpha$*

(iii) *$F : [0, T] \times \Omega \times [\alpha, \infty[\times R^m \longrightarrow R$ a $\mathscr{P}$-measurable function, continuous in (y, z) and satisfying the following* structure condition:

$$\exists C > 0 \text{ such that } P\text{-a.s. } \forall t, y, z, \quad -2C^2 y - C|z|^2 \leqslant F(t, \omega, y, z) \leqslant 2C^2 y. \tag{7}$$

In addition assume that either the pair $(\bar{L}, \bar{U})$ satisfies (H5′) *or one of the processes $\bar{U}$ or $-\bar{L}$ satisfies* (H4). *Then the double barrier reflected BSDE associated with $(F, \eta, \bar{L}, \bar{U})$*

$$\begin{cases} Y_t = \eta + \int_t^T F(s, Y_s, Z_s)\,\mathrm{d}s + K_T^+ - K_t^+ - K_T^- + K_t^- - \int_t^T Z_s\,\mathrm{d}B_s, \quad \forall t \leqslant T, \\ Z \in \mathscr{H}^{2,m},\ K^{\pm} \text{ are continuous non-decreasing and } E[K_T^{\pm}] < \infty\ (K_0^{\pm} = 0), \\ \forall t \leqslant T,\ \ \bar{L}_t \leqslant Y_t \leqslant \bar{U}_t \quad \text{and} \quad \int_0^T (Y_s - \bar{L}_s)\,\mathrm{d}K_s^+ = \int_0^T (\bar{U}_s - Y_s)\,\mathrm{d}K_s^- = 0, \end{cases} \tag{8}$$

has a maximal solution $(Y_t, Z_t, K_t^+, K_t^-)_{t\leqslant T}$.

Proof. Let $M := \operatorname{ess\,sup}_{t,\omega} \bar{U}_t$ and consider the continuous and bounded function $\rho : R \to R$ such that $\rho(x) = \alpha 1_{[x<\alpha]} + x 1_{[\alpha \leqslant x \leqslant M]} + M 1_{[x \geqslant M]}$. Consider now the following reflected BSDE:

$$\begin{cases} Z \in \mathscr{H}^{2,m}; K^{\pm} \text{ are continuous non-decreasing and } E[K_T^{\pm}] < \infty \ (K_0^{\pm} = 0), \\ Y_t = \eta + \int_t^T F(s, \rho(Y_s), Z_s)\,\mathrm{d}s + K_T^+ - K_t^+ - K_T^- + K_t^- - \int_t^T Z_s\,\mathrm{d}B_s, \\ \forall t \leqslant T, \ \bar{L}_t \leqslant Y_t \leqslant \bar{U}_t \quad \text{and} \quad \int_0^T (Y_s - \bar{L}_s)\,\mathrm{d}K_s^+ = \int_0^T (\bar{U}_s - Y_s)\,\mathrm{d}K_s^- = 0. \end{cases} \tag{9}$$

We shall prove that the reflected BSDE (9) has a maximal solution $(Y_t, Z_t, K_t^+, K_t^-)_{t \leqslant T}$. Therefore it satisfies $\alpha \leqslant Y_t \leqslant M$ and then $\rho(Y) = Y$. It follows that $(Y_t, Z_t, K_t^+, K_t^-)_{t \leqslant T}$ is also a maximal solution for (8).

From now on the proof will be divided into 6 steps.

Step 1. Let us define $\tilde{F}(t, \omega, y, z) := F(t, \omega, \rho(y), z)$ and for $p \geqslant 1$, let $\kappa_p : R^m \longrightarrow R$ be a smooth function which satisfies:

$$0 \leqslant \kappa_p \leqslant 1, \ \ \kappa_p(z) = 1 \ \text{ if } |z| \leqslant p \quad \text{and} \quad \kappa_p(z) = 0 \ \text{ if } |z| \geqslant p + 1.$$

Let $\tilde{F}_p(t, \omega, y, z) := 2C^2 \rho(y)(1 - \kappa_p(z)) + \kappa_p(z) \tilde{F}(t, \omega, y, z)$. From (7) we have $\tilde{F}(t, \omega, y, z) \leqslant 2C^2 \rho(y)$ and then for any $p \geqslant 1$,

$$\tilde{F}_p(t, \omega, y, z) - \tilde{F}_{p+1}(t, \omega, y, z) = (2C^2 \rho(y) - \tilde{F}(t, \omega, y, z))(\kappa_{p+1}(z) - \kappa_p(z)) \geqslant 0$$

since the sequence $(\kappa_p)_{p \geqslant 1}$ is increasing. It means that the sequence of functions $(\tilde{F}_p)_{p \geqslant 1}$ is decreasing and $\lim_{p \to \infty} \downarrow \tilde{F}_p = \tilde{F}$. In addition, $\tilde{F}_p$ is bounded since for any (t, y, z) we have $|\tilde{F}_p(t, \omega, y, z)| \leqslant C_1(1 + |p|^2)$. Therefore, Theorem 2.3 (resp. Remark 2.4) implies that the reflected BSDE associated with $(\tilde{F}_p, \eta, \bar{L}, \bar{U})$ has a maximal solution $(Y^p, Z^p, K^{p+}, K^{p-})$ since the pair $(\bar{L}, \bar{U})$ (resp. one of the processes $\bar{U}$ or $-\bar{L}$) satisfies (H5′) (resp. (H4)). So we have

$$\begin{cases} Z^p \in \mathscr{H}^{2,m}, K^{p\pm} \text{ belong to } \mathscr{S}^2 \text{ and non-decreasing } (K_0^{p\pm} = 0), \\ Y_t^p = \eta + \int_t^T \tilde{F}_p(s, Y_s^p, Z_s^p)\,\mathrm{d}s + K_T^{p+} - K_t^{p+} - K_T^{p-} + K_t^{p-} - \int_t^T Z_s^p\,\mathrm{d}B_s, \\ \forall t \leqslant T, \ \bar{L}_t \leqslant Y_t^p \leqslant \bar{U}_t \quad \text{and} \quad \int_0^T (Y_s^p - \bar{L}_s)\,\mathrm{d}K_s^{p+} = \int_0^T (\bar{U}_s - Y_s^p)\,\mathrm{d}K_s^{p-} = 0. \end{cases} \tag{10}$$

Now the comparison theorem of maximal solutions (Proposition 2.6) implies that $M \geqslant Y^p \geqslant Y^{p+1} \geqslant \alpha$, $K^{p+} \leqslant K^{(p+1)+}$ and $K^{p-} \geqslant K^{(p+1)-}$, since $\tilde{F}_{p+1} \leqslant \tilde{F}_p$. Therefore there exists a process $Y := (Y_t)_{t \leqslant T}$ such that P-a.s. $\forall t \leqslant T$, $Y_t = \lim_{p \to \infty} Y_t^p$ and $Y = \mathscr{H}^{2,1} - \lim_{p \to \infty} Y^p$. In addition P-a.s. $\forall t \leqslant T, \alpha \leqslant Y_t \leqslant M$.

Step 2. There exists a positive constant $\tilde{c}$ such that $E[\int_0^T |Z_s^p|^2\,\mathrm{d}s] \leqslant \tilde{c}$, $\forall p \geqslant 1$.

Let $\Psi(x) = \mathrm{e}^{-3Cx}$. By Itô's formula we have

$$\begin{aligned} &\Psi(Y_t^p) + \frac{1}{2} \int_t^T \Psi''(Y_s^p) |Z_s^p|^2\,\mathrm{d}s \\ &\quad = \Psi(Y_T^p) + \int_t^T \Psi'(Y_s^p) \tilde{F}_p(s, Y_s^p, Z_s^p)\,\mathrm{d}s + \int_t^T \Psi'(Y_s^p)\,\mathrm{d}K_s^{p+} \\ &\qquad - \int_t^T \Psi'(Y_s^p)\,\mathrm{d}K_s^{p-} - \int_t^T \Psi'(Y_s^p) Z_s^p\,\mathrm{d}B_s. \end{aligned}$$

It is easily seen that $E[|\int_t^T \Psi'(Y_s^p)\,\mathrm{d}K_s^{p-}|] \leqslant M_1 E[K_T^{p-}]$ where $M_1 := \sup_{\alpha \leqslant x \leqslant M} |\Psi'(x)|$. Now since $\Psi' < 0$ and K_t^{p+} is an increasing process, $\int_t^T \Psi'(\ Y_s^p)\,\mathrm{d}K_s^{p+} \leqslant 0$. Finally, taking expectation on both sides of the last inequality yields

$$
\begin{aligned}
&E[\Psi(Y_0^p)] + \frac{1}{2}E\left[\int_0^T \Psi''(Y_s^p)|Z_s^p|^2\,\mathrm{d}s\right] \\
&\quad \leqslant E[\Psi(Y_T^p)] + E\left[\int_0^T \Psi'(Y_s^p)\tilde{F}_p(s, Y_s^p, Z_s^p)\,\mathrm{d}s\right] + M_1 E[K_T^{p-}].
\end{aligned}
$$

Now let $A := 2C^2 \max\{\alpha, M\}$. Using the inequality (7) and the fact that $\Psi' < 0$ we obtain

$$
\begin{aligned}
&E[\Psi(Y_0^p)] + \frac{1}{2}E\left[\int_0^T \Psi''(Y_s^p)|Z_s^p|^2\,\mathrm{d}s\right] \\
&\quad \leqslant E[\Psi(Y_T^p)] - E\left[\int_0^T \Psi'(Y_s^p)(A + C|Z_s^p|^2)\,\mathrm{d}s\right] + M_1 E[K_T^{p-}].
\end{aligned}
$$

Henceforth,

$$
\begin{aligned}
&E\big[\Psi(Y_0^p)\big] + E\left[\int_0^T \left(\frac{1}{2}\Psi''(Y_s^p) + C\Psi'(Y_s^p)\right)|Z_s^p|^2\,\mathrm{d}s\right] \\
&\quad \leqslant E[\Psi(Y_T^p)] - AE\left[\int_0^T \Psi'(Y_s^p)\,\mathrm{d}s\right] + M_1 E[K_T^{p-}].
\end{aligned}
$$

As the function Ψ satisfies $\frac{1}{2}\Psi'' + C\Psi' = \frac{3}{2}C^2\Psi$ we get

$$
\begin{aligned}
&E\big[\Psi(Y_0^p)\big] + E\left[\int_0^T \frac{3}{2}C^2\Psi(Y_s^p)|Z_s^p|^2\,\mathrm{d}s\right] \\
&\quad \leqslant E[\Psi(Y_T^p)] - AE\left[\int_0^T \Psi'(Y_s^p)\,\mathrm{d}s\right] + M_1 E[K_T^{p-}]
\end{aligned}
$$

and then

$$
\begin{aligned}
&E[\Psi(Y_0^p)] + E\left[\int_0^T \frac{3}{2}C^2 \mathrm{e}^{-3CY_s^p}|Z_s^p|^2\,\mathrm{d}s\right] \\
&\quad \leqslant E[\Psi(Y_T^p)] - AE\left[\int_0^T \Psi'(Y_s^p)\,\mathrm{d}s\right] + M_1 E[K_T^{p-}].
\end{aligned}
$$

Finally, taking into account the fact that $K^{p-} \geqslant K^{(p+1)-}$, it holds that

$$
\frac{3}{2}C^2\mathrm{e}^{-3CM}E\left[\int_0^T |Z_s^p|^2\,\mathrm{d}s\right] \leqslant E[\Psi(\xi)] + 3CAT\mathrm{e}^{-3C\alpha} + M_1 E[K_T^{1-}],
$$

which implies the existence of a constant $\tilde{c}$ such that $E[\int_0^T |Z_s^p|^2\,\mathrm{d}s] \leqslant \tilde{c},\ \forall p \geqslant 1$.

Step 3. There exists a subsequence of $(Z^p)_{p\geqslant 1}$ which converges strongly in $\mathscr{H}^{2,m}$.

The sequence $(Z^p)_{p\geqslant 1}$ is bounded in $\mathscr{H}^{2,m}$ then there exists a subsequence of $(Z^p)_{p\geqslant 1}$ which we still denote $(Z^p)_{p\geqslant 1}$ which converges weakly in $\mathscr{H}^{2,m}$ to a process

$Z:=(Z_t)_{t\leqslant T}$. Now let $\theta=\max\{C,4C^2M\}$, $\Psi(x)=(e^{12\theta x}-1/12\theta)-x$ and p,q be two positive integers such that $p\leqslant q$. Then we have $Y^p\geqslant Y^q$. On the other hand using Itô's formula with $\psi(Y^p-Y^q)$ and taking expectation yields

$$
\begin{aligned}
&E[\Psi(Y_0^p-Y_0^q)]+\frac{1}{2}E\left[\int_0^T \Psi''(Y_s^p-Y_s^q)|Z_s^p-Z_s^q|^2\,\mathrm{d}s\right]\\
&\quad =I_1(p,q)+I_2(p,q)+I_3(p,q),
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(p,q)&=E\left[\int_0^T \Psi'(Y_s^p-Y_s^q)\{\tilde F_p(s,Y_s^p,Z_s^p)-\tilde F_q(s,Y_s^q,Z_s^q)\}\,\mathrm{d}s\right],\\
I_2(p,q)&=E\left[\int_0^T \Psi'(Y_s^p-Y_s^q)\,\mathrm{d}(K_s^{p+}-K_s^{q+})\right]\quad\text{and}\\
I_3(p,q)&=-E\left[\int_0^T \Psi'(Y_s^p-Y_s^q)\,\mathrm{d}(K_s^{p-}-K_s^{q-})\right].
\end{aligned}
$$

First note that

$$
\begin{aligned}
I_2(p,q)=&E\left[\int_0^T \Psi'(Y_s^p-Y_s^q)\mathbf{1}_{\{Y_s^p=L_s\}}\,\mathrm{d}K_s^{p+}\right]\\
&-E\left[\int_0^T \Psi'(Y_s^p-Y_s^q)\mathbf{1}_{\{Y_s^q=L_s\}}\,\mathrm{d}K_s^{q+}\right]
\end{aligned}
$$

since K^{p+} (resp. K^{q+}) moves only when Y^p (resp. Y^q) reaches the lower obstacle L. But $Y^q\leqslant Y^p$ then $\{Y^p=L\}\subset\{Y^q=L\}$. As $\Psi'(0)=0$ then the first term in the right-hand side is null. On the other hand we have $E[\int_0^T \Psi'(Y_s^p-Y_s^q)\mathbf{1}_{\{Y_s^q=L_s\}}\,\mathrm{d}K_s^{q+}]\geqslant 0$ since $\Psi'(x)\geqslant 0$ when $x\geqslant 0$. It follows that $I_2(p,q)\leqslant 0$. In the same way we can show that $I_3(p,q)\leqslant 0$. Thus

$$
\begin{aligned}
&\frac{1}{2}E\left[\int_0^T \Psi''(Y_s^p-Y_s^q)|Z_s^p-Z_s^q|^2\,\mathrm{d}s\right]\\
&\quad\leqslant E\left[\int_0^T \Psi'(Y_s^p-Y_s^q)\{\tilde F_p(s,Y_s^p,Z_s^p)-\tilde F_q(s,Y_s^q,Z_s^q)\}\,\mathrm{d}s\right]
\end{aligned}
\tag{11}
$$

since $\Psi(Y_0^p-Y_0^q)\geqslant 0$. But we have

$$
\begin{aligned}
\tilde F_p(s,Y_s^p,Z_s^p)-\tilde F_q(s,Y_s^q,Z_s^q)&=2C^2\rho(Y_s^p)(1-\kappa_p(Z_s^p))+\kappa_p(Z_s^p)\tilde F(s,Y_s^p,Z_s^p)\\
&\quad-\tilde F_q(s,Y_s^q,Z_s^q)\\
&\leqslant 2C^2\rho(Y_s^p)(1-\kappa_p(Z_s^p))+\kappa_p(Z_s^p)\times 2C^2\rho(Y_s^p)\\
&\quad-\tilde F(s,Y_s^q,Z_s^q)\\
&\leqslant 2C^2\rho(Y_s^p)+2C^2\rho(Y_s^q)+C|Z_s^q|^2\\
&\leqslant 4C^2M+C|Z_s^q|^2\leqslant\theta(1+|Z_s^q|^2)\\
&\leqslant 3\theta(1+|Z_s^q-Z_s^p|^2+|Z_s^p-Z_s|^2+|Z_s|^2).
\end{aligned}
$$

As $\Psi'(Y_s^p - Y_s^q) \geqslant 0$, then plugging this latter inequality in (11) yields

$$E\left[\int_0^T \left(\frac{\Psi''}{2} - 3\theta\Psi'\right)(Y_s^p - Y_s^q)|Z_s^p - Z_s^q|^2\,\mathrm{d}s\right]$$
$$\leqslant 3\theta E\left[\int_0^T \Psi'(Y_s^p - Y_s^q)\{1 + |Z_s^p - Z_s|^2 + |Z_s|^2\}\,\mathrm{d}s\right]. \tag{12}$$

However $\Psi''(x)/2 - 3\theta\Psi'(x) = 3\theta e^{12\theta x} + 3\theta$, then the process $(\Psi''/2 - 3\theta\Psi')^{1/2}(Y^p - Y^q)$ converges, as q tends to ∞, strongly in $\mathscr{H}^{2,1}$ to the uniformly bounded process $(\Psi''/2 - 3\theta\Psi)^{1/2}(Y^p - Y)$. Hence $(\Psi''/2 - 3\theta\Psi')^{1/2}(Y^p - Y^q) \times (Z^p - Z^q)$ converges weakly to $(\Psi''/2 - 3\theta\Psi')^{1/2}(Y^p - Y) \times (Z^p - Z)$. Therefore by (12) we obtain

$$E\left[\int_0^T \left(\frac{\Psi''}{2} - 3\theta\Psi'\right)(Y_s^p - Y_s)|Z_s^p - Z_s|^2\,\mathrm{d}s\right]$$
$$\leqslant \liminf_{q\to\infty} E\left[\int_0^T \left(\frac{\Psi''}{2} - 3\theta\Psi'\right)(Y_s^p - Y_s^q)|Z_s^p - Z_s^q|^2\,\mathrm{d}s\right]$$
$$\leqslant 3\theta E\left[\int_0^T \Psi'(Y_s^p - Y_s)(1 + |Z_s^p - Z_s|^2 + |Z_s|^2)\,\mathrm{d}s\right]$$

since for any sequence $(x_n)_{n\geqslant 1}$ of $\mathscr{H}^{2,1}$ which converges weakly to x we have $\|x\|^2 \leqslant \liminf_{n\to\infty} \|x_n\|^2$. It implies that

$$E\left[\int_0^T \left(\frac{\Psi''}{2} - 6\theta\Psi'\right)(Y_s^p - Y_s)|Z_s^p - Z_s|^2\,\mathrm{d}s\right]$$
$$\leqslant 3\theta E\left[\int_0^T \Psi'(Y_s^p - Y_s)(1 + |Z_s|^2)\,\mathrm{d}s\right].$$

Now since $(\Psi''/2 - 6\theta\Psi') = 6\theta$, thanks to Lebesgue dominated convergence theorem, we deduce that $\lim_{p\to\infty} E\int_0^T |Z_s^p - Z_s|^2\,\mathrm{d}s = 0$ which is the desired result.

Step 4. The process $Y := (Y_t)_{t\leqslant T}$ is continuous.

We shall prove that Y^p converges uniformly to Y in $L^2(\Omega, \mathrm{d}P)$. Let p, q be two positive integers such that $p \leqslant q$. Applying Itô's formula with $(Y^p - Y^q)^2$ yields:

$$|Y_t^p - Y_t^q|^2 + \int_0^T |Z_s^p - Z_s^q|^2\,\mathrm{d}s = I_1(p,q) + I_2(p,q) + I_3(p,q) + I_4(p,q),$$

where

$$I_1(p,q) = 2\int_0^T (Y_s^p - Y_s^q)(\tilde{F}_p(s, Y_s^p, Z_s^p) - \tilde{F}_q(s, Y_s^q, Z_s^q))\,\mathrm{d}s,$$

$$I_2(p,q) = 2\int_0^T (Y_s^p - Y_s^q)\,\mathrm{d}(K_s^{p+} - K_s^{q+}),$$

$$I_3(p,q) = -2\int_0^T (Y_s^p - Y_s^q)\,\mathrm{d}(K_s^{p-} - K_s^{q-}) \quad \text{and}$$

$$I_4(p,q) = -2\int_0^T (Y_s^p - Y_s^q)(Z_s^p - Z_s^q)\,\mathrm{d}B_s.$$

But as in Step 3 we have $I_2(p,q)\leqslant 0$ and $I_3(p,q)\leqslant 0$. Now applying the Burkholder–Davis–Gundy inequality [15,22] we obtain for some constant $\bar{C}$,

$$E\left[\sup_{0\leqslant t\leqslant T}|Y_t^p-Y_t^q|^2\right]+E\left[\int_0^T|Z_t^p-Z_t^q|^2\,\mathrm{d}t\right]$$
$$\leqslant \bar{C}E\left[\int_0^T|Y_s^p-Y_s^q||\tilde{F}_p(s,Y_s^p,Z_s^p)-\tilde{F}_q(s,Y_s^q,Z_s^q)|\,\mathrm{d}s\right].$$

As both $\tilde{F}$ and $\tilde{F}_p$ are continuous and $\tilde{F}_p$ converges decreasingly to $\tilde{F}$, then by Dini's theorem $\tilde{F}_p(t,\omega,.,.)$ converges to $F(t,\omega,.,.)$ uniformly on compact subsets of R^{1+m} for each fixed (t,ω). On the other hand, since $(Z^p)_{p\geqslant 1}$ converges strongly in $\mathscr{H}^{2,m}$ to Z then there exists $\tilde{Z}\in\mathscr{H}^{2,m}$ and a subsequence, which we still denote $(Z^p)_{p\geqslant 1}$, such that $(Z^p)_{p\geqslant 1}$ converges to Z, $\mathrm{d}t\otimes\mathrm{d}P$-a.e and $\sup_{p\geqslant 1}|Z_t^p|\leqslant\tilde{Z}_t$. Then $\tilde{F}_p(s,Y_s^p,Z_s^p)$ converges to $\tilde{F}(s,Y_s,Z_s)$, $\mathrm{d}t\otimes P$-a.e and moreover $|\tilde{F}_p(s,Y_s^p,Z_s^p)|\leqslant C_1(1+|Z_s^p|^2)\leqslant C_1(1+|\tilde{Z}_s|^2)$ for some constant C_1. Finally, since the sequence $(Y^p)_{p\geqslant 1}$ is uniformly bounded, the Lebesgue dominated convergence theorem implies that $E[\int_0^T|Y_s^p-Y_s^q||\tilde{F}_p(s,Y_s^p,Z_s^p)-\tilde{F}_q(s,Y_s^q,Z_s^q)|\,\mathrm{d}s]$ tends to 0 as p,q tend to infinity. Henceforth the sequence $(Y^p)_{p\geqslant 1}$ converges uniformly to Y in $L^2(\Omega,\mathrm{d}P)$ and then Y is continuous.

Step 5. Construction of the continuous processes K^+ and K^-.

For any $p\geqslant 1$ and $t\leqslant T$ we have,

$$Y_t^p=Y_0^p-\int_0^t\tilde{F}_p(s,Y_s^p,Z_s^p)\,\mathrm{d}s-K_t^{p+}+K_t^{p-}-\int_0^t Z_s^p\,\mathrm{d}B_s \tag{13}$$

and then

$$K_T^{p+}\leqslant K_T^{p-}+|Y_T^p|+|Y_0^p|+\int_0^T C_1(1+|\tilde{Z}_s|^2)\,\mathrm{d}s+\left|\int_0^T Z_s^p\,\mathrm{d}B_s\right|. \tag{14}$$

The sequence of increasing processes $(K^{p-})_{p\geqslant 1}$ is non-increasing then it is convergent to a process $(K_t^-)_{t\leqslant T}$ which moreover is increasing, upper semi-continuous and integrable since $E[K_T^-]\leqslant E[K_T^{0-}]<\infty$.

Next, inequality (14) implies that for any $p\geqslant 1$, $E[K_T^{p+}]\leqslant C$ for some constant C since the sequences $(Y^p)_{p\geqslant 1}$ and $(Z^p)_{p\geqslant 1}$ are so in their respective spaces. On the other hand the sequence of increasing processes $(K^{p+})_{p\geqslant 1}$ is increasing then, in combination with Fatou's Lemma, it converges also to a process $(K_t^+)_{t\leqslant T}$ which moreover is lower, semi-continuous and satisfies $E[K_T^+]<\infty$.

Now as there exists a subsequence of $((\tilde{F}(t,Y_t^p,Z_t^p))_{t\leqslant T})_{p\geqslant 1}$ which converges in $L^1(\Omega\times[0,T],\mathrm{d}P\otimes\mathrm{d}t)$ to $(\tilde{F}(t,Y_t,Z_t))_{t\leqslant T}$ then working with Eq. (13) and with the same subsequence we deduce that $(K^{p+}-K^{p-})_{p\geqslant 1}$ converges uniformly in $L^1(\Omega,\mathrm{d}P)$ to K^+-K^-. Therefore the process K^+-K^- is continuous and once again from (13) we deduce that P-a.s. for any $t\leqslant T$ we have

$$K_t^-=K_t^++Y_t-Y_0+\int_0^t\tilde{F}(s,Y_s,Z_s)\,\mathrm{d}s+\int_0^t Z_s\,\mathrm{d}B_s. \tag{15}$$

It follows that the processes K^- and K^+ are upper and lower semi-continuous in the same time; therefore they are continuous and once again through Dini's theorem we have

$$P-\text{a.s.} \lim_{p\to\infty} \sup_{t\leqslant T}\{|K_t^{p+}-K_t^+|+|K_t^{p-}-K_t^-|\}=0. \tag{16}$$

Step 6. The process $(Y_t, Z_t, K_t^+, K_t^-)_{t\leqslant T}$ satisfies (8) and (9) and is a maximal solution.

Regarding (15), in order to show that the quadruple $(Y, Z, K^{\pm})$ satisfies (8)–(9), it remains to show that

$$\int_0^T (Y_s-\bar{U}_s)\,\mathrm{d}K_s^- = \int_0^T (\bar{L}_s - Y_s)\,\mathrm{d}K_s^+ = 0. \tag{17}$$

But this is true as a direct consequence of the uniform convergence of $(Y^p(\omega))_{p\geqslant 1}$ and $(K^{\pm}(\omega))_{p\geqslant 1}$ to $Y(\omega)$ and $K^{\pm}(\omega)$, respectively, and the following properties:

$$\forall p\geqslant 1, \quad \int_0^T (Y_s^p-\bar{U}_s)\,\mathrm{d}K_s^{p-} = \int_0^T (\bar{L}_s - Y_s^p)\,\mathrm{d}K_s^{p+} = 0.$$

One can see the proof of this claim in [10, p. 10].

Let us now show that this solution is maximal. Let $(\bar{Y}, \bar{Z}, \bar{K}^+, \bar{K}^-)$ be another solution for (8), which of course is also a solution for (9). Now for any $p\geqslant 1$ and $l\geqslant 1$, let $\tilde{F}_p^l$ be the function defined as follows:

$$\tilde{F}_p^l(t,\omega,y,z) := \sup_{(u,v)\in R^{1+m}} \{\tilde{F}_p(t,\omega,u,v) - l(|u-y|+|v-z|)\}.$$

Like for the definition of f_n in (3), since we have $|\tilde{F}_p(t,\omega,y,z)|\leqslant C_1(1+|p|^2)$ for any $(y,z)\in R^{1+m}$ the function $\tilde{F}_p^l$ is defined, Lipschitz with respect to (y,z) and converge decreasingly and pointwisely to $\tilde{F}_p$ as $l\to\infty$. Now let $(Y_p^l, Z_p^l, K_p^{l,+}, K_p^{l,-})$ be the solution of the reflected BSDE associated with $(\tilde{F}_p^l, \eta, \bar{L}, \bar{U})$. Since $\tilde{F}_p^l\geqslant\tilde{F}_p\geqslant\tilde{F}$, $Y_p^l\geqslant\bar{Y}$ for any $p,l\geqslant 1$. But for any $p\geqslant 1$ we have $\lim_{l\to\infty} Y_p^l = Y_p$ (see theconstruction of the maximal solution in Theorem 2.3). Therefore $Y_p\geqslant\bar{Y}$ and finally $Y\geqslant\bar{Y}$. It implies that the solution we have constructed is maximal. $\square$

We are now going to give the main result of this part. Basically, it is based on the use of an exponential transform which turns the reflected BSDE with a quadratic coefficient into another one whose coefficient satisfies the *structure condition*.

So from now on we assume that f satisfies (H3) and ξ, L and U satisfy (H6). Let $m=\inf_{t,\omega} L_t(\omega)$, $M=\sup_{t,\omega} U_t(\omega)$ and φ the function from R into R such that $\varphi(x)=m1_{[x<m]}+x1_{[m\leqslant x\leqslant M]}+M1_{[x>M]}$. Henceforth, there exists a constant $\tilde{C}\geqslant 0$ such that $|f(t,\omega,\varphi(y),z)|\leqslant\tilde{C}(1+|z|^2)$ for any (t,y,z).

Now let $\alpha=e^{2\tilde{C}m}$ and let us set

$$\forall (t,y,z)\in[0,T]\times[\alpha,\infty[\times R^m,$$

$$F(t,\omega,y,z)=2\tilde{C}y\left[f\left(t,\omega,\frac{Lny}{2\tilde{C}},\frac{z}{2\tilde{C}y}\right)-\frac{|z|^2}{4\tilde{C}y^2}\right].$$

Then the function F satisfies the *structure condition*. Indeed, first, it is easily seen that

$$2\tilde{C}y\left\{f\left(t,\omega,\frac{Lny}{\tilde{C}},\frac{z}{2\tilde{C}y}\right)-\frac{|z|^2}{4\tilde{C}y^2}\right\}\leqslant 2\tilde{C}^2y.$$

In addition,

$$2\tilde{C}y\left\{f\left(t,\omega,\frac{Lny}{\tilde{C}},\frac{z}{2\tilde{C}y}\right)-\frac{|z|^2}{4\tilde{C}y^2}\right\}\geqslant -2\tilde{C}^2y-\frac{|z|^2}{y}\geqslant -2\tilde{C}^2y-\frac{|z|^2}{\alpha}.$$

Then there exists a constant C such that

$$P\text{-a.s. } \forall(t,y,z)\in[0,T]\times[\alpha,\infty[\times R^m,\quad -2Cy^2-C|z|^2\leqslant F(t,\omega,y,z)\leqslant 2Cy^2,$$

i.e., F satisfies the *structure condition*. Note that the coefficient F is the one we obtain when we apply the exponential transform ($x\longmapsto e^{2\tilde{C}x}$) to the BSDE associated with (f,ξ,L,U).

We are now ready to give the main theorem of this section.

Theorem 3.2. *Assume that*:

(i) f *satisfies* (H3) *and* L, U *and* ξ *satisfy* (H6), i.e., *they are bounded*,
(ii) *either the pair* $(e^{2\tilde{C}L},e^{2\tilde{C}U})$ *satisfies* (H5′) *or one of the processes* $e^{2\tilde{C}U}$, $-e^{2\tilde{C}L}$ *satisfies* (H4).

Then there exists a quadruple of $\mathscr{P}$*-measurable processes* $(Y,Z,K^+,K^-):=(Y_t,Z_t,K_t^+,K_t^-)_{t\leqslant T}$ *solution of the reflected BSDE associated with* (f,ξ,L,U), i.e., *which satisfies*

$$\begin{cases} Z\in\mathscr{H}^{2,m}, K^{\pm}\text{ continuous non-decreasing and } E[K_T^{\pm}]<\infty (K_0^{\pm}=0),\\ Y_t=\xi+\int_t^T f(s,Y_s,Z_s)\,ds+K_T^+-K_t^+-K_T^-+K_t^- -\int_t^T Z_s\,dB_s,\quad t\leqslant T,\\ \forall t\leqslant T,\ L_t\leqslant Y_t\leqslant U_t\quad\text{and}\quad \int_0^T(U_s-Y_s)\,dK_s^-=\int_0^T(Y_s-L_s)\,dK_s^+=0.\end{cases}\tag{18}$$

Moreover it is maximal.

Proof. First let us notice that in (18), unlike to (2), we just require $E[K_T^{\pm}]<\infty$ and not $E[(K_T^{\pm})^2]<\infty$. Now as it is said previously the function F satisfies the *structure condition*. Therefore, according to Theorem 3.1, the double obstacle reflected BSDE associated with $(F,e^{2\tilde{C}\xi},e^{2\tilde{C}L},e^{2\tilde{C}U})$ has a maximal solution $(\tilde{Y}_t,\tilde{Z}_t,\tilde{K}_t^+,\tilde{K}_t^-)_{t\leqslant T}$.

Now for $t\leqslant T$, let us set

$$Y_t=\frac{Ln\tilde{Y}_t}{2\tilde{C}},\quad Z_t=\frac{\tilde{Z}_t}{2\tilde{C}\tilde{Y}_t}\quad\text{and}\quad dK_t^{\pm}=\frac{d\tilde{K}_t^{\pm}}{2\tilde{C}\tilde{Y}_t}.$$

Since $\tilde{Y} \geqslant e^{2\tilde{C}m}$ then these processes are well defined. Henceforth using Itô's formula we obtain,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,\mathrm{d}s + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s\,\mathrm{d}B_s, \quad t \leqslant T.$$

On the other hand, Z belongs to $\mathscr{H}^{2,m}$ since $\tilde{Z}$ is so and $\tilde{Y} \geqslant e^{2\tilde{C}m}$. In addition, for the same reason, we have $E[K_T^{\pm}] < \infty$. Now K^+ (resp. K^-) is a continuous process which satisfies $\int_0^T (Y_s - L_s)\,\mathrm{d}K_s^+ = 0$ (resp. $\int_0^T (U_s - Y_s)\,\mathrm{d}K_s^- = 0$) since $\int_0^T (\tilde{Y}_s - e^{2\tilde{C}L_s})\,\mathrm{d}\tilde{K}_s^+ = 0$ (resp. $\int_0^T (e^{2\tilde{C}U_s} - \tilde{Y}_s)\,\mathrm{d}\tilde{K}_s^- = 0$). It follows that $(Y_t, Z_t, K_t^+, K_t^-)_{t \leqslant T}$ satisfies (18). Finally let us show that this solution is also maximal. Let $(\bar{Y}_t, \bar{Z}_t, \bar{K}_t^+, \bar{K}_t^-)_{t \leqslant T}$ be another solution. Then $(e^{2\tilde{C}\bar{Y}_t}, 2\tilde{C}e^{2\tilde{C}\bar{Y}_t}Z_t, \int_0^t 2\tilde{C}e^{2\tilde{C}\bar{Y}_s}\,\mathrm{d}K_s^+, \int_0^t 2\tilde{C}e^{2\tilde{C}\bar{Y}_s}\,\mathrm{d}K_s^-)_{t \leqslant T}$ is a solution for the reflected BSDE associated with $(F, e^{2\tilde{C}\xi}, e^{2\tilde{C}L}, e^{2\tilde{C}U})$. Henceforth we have $e^{2\tilde{C}\bar{Y}} \leqslant \tilde{Y}$ and then $\bar{Y} \leqslant Y$. The proof is now complete. $\square$

Remark 3.3. We give below two examples where the assumptions of Theorem 2.3 are verified:

(i) let g be a bounded C^2-function from R^m into R such that $D_x g$ is of polynomial growth and there exists a constant c such that $\sum_{i=1,m} |D_i g(x)|^2 + \sum_{i,j=1,m} D_{ij} g(x) \leqslant c$. If $U_t = g(B_t)$, $t \leqslant T$, then $(e^{2\tilde{C}g(B_t)})_{t \leqslant T}$ satisfies the assumption (H4).

(ii) Assume that there exists a constant a such that $\forall t \leqslant T$, $L_t \leqslant a \leqslant U_t$ then the pair $(e^{2\tilde{C}L_t}, e^{2\tilde{C}U_t})$ satisfies (H5$'$) with e.g. $\eta = e^{2\tilde{C}a}$ and $\theta = 0$. $\square$

Finally, let us deal with a particular case of the coefficient f. Actually assume that $f(t, y, z) = h(t, y) + \frac{1}{2}|z|^2$. Then there exists a link between the component Y of the solution $(Y_t, Z_t, K_t^+, K_t^-)_{t \leqslant T}$ of the reflected BSDE associated with (f, ξ, L, U) and the value function of a risk-sensitive zero-sum game on stopping times. Indeed we have

$$\begin{aligned} e^{Y_t} = &\operatorname*{essinf}_{\tau \geqslant t} \operatorname*{esssup}_{v \geqslant t} E\left[\exp\left\{\int_t^{\tau\wedge v} h(s, Y_s)\,\mathrm{d}s + U_\tau 1_{[\tau < v]}\right.\right. \\ &\left.\left. + L_v 1_{[v \leqslant \tau < T]} + \xi 1_{[v = \tau = T]}\right\} | F_t\right] \\ = &\operatorname*{esssup}_{v \geqslant t} \operatorname*{essinf}_{\tau \geqslant t} E\left[\exp\left\{\int_t^{\tau\wedge v} h(s, Y_s)\,\mathrm{d}s + U_\tau 1_{[\tau < v]}\right.\right. \\ &\left.\left. + L_v 1_{[v \leqslant \tau < T]} + \xi 1_{[v = \tau = T]} \int_t^{\tau\wedge v} \right\} | F_t\right], \quad \forall t \leqslant T, \end{aligned}$$

where v and τ are F_t-stopping times whose values are in $[t, T]$. Actually for $t \leqslant T$, let $\tilde{Y}_t = \exp\{Y_t + \int_0^t h(s, Y_s)\,\mathrm{d}s\}$. Then there exist processes $\tilde{K}^+$, $\tilde{K}^-$ and $\tilde{Z}$ such that the quadruple $(\tilde{Y}, \tilde{Z}, \tilde{K}^+, \tilde{K}^-)$ is solution of an appropriate reflected BSDE. Namely it satisfies:

$$-\mathrm{d}\tilde{Y}_t = \mathrm{d}\tilde{K}_t^+ - \mathrm{d}\tilde{K}_t^- - \mathrm{d}\tilde{Z}_t\,\mathrm{d}B_t,\ \ t\leqslant T;\quad \tilde{Y}_T = \exp\left\{\int_0^T h(s,Y_s)\,\mathrm{d}s+\xi\right\},$$

$$\tilde{L}_t\leqslant \tilde{Y}_t\leqslant \tilde{U}_t \quad\text{and}\quad \int_0^T (\tilde{Y}_t-\tilde{L}_t)\,\mathrm{d}\tilde{K}_t^+ = \int_0^T (\tilde{U}_t-\tilde{Y}_t)\,\mathrm{d}\tilde{K}_t^- = 0,$$

where $\tilde{L}_t=\exp\{L_t+\int_0^t h(s,Y_s)\,\mathrm{d}s\}$ and $\tilde{U}_t=\exp\{U_t+\int_0^t h(s,Y_s)\,\mathrm{d}s\}$, $t\leqslant T$. Now according to [1], Theorem 3.1 or [9], Theorem 4, the process $\tilde{Y}$ is the value function of a zero-sum game on stopping times, i.e.,

$$\begin{aligned}\tilde{Y}_t =& \operatorname*{essinf}_{\tau\geqslant t}\ \operatorname*{esssup}_{\nu\geqslant t} E\Bigg[\exp\left\{\int_0^T h(s,Y_s)\,\mathrm{d}s+\xi\right\}1_{[\nu=\tau=T]}\\ &+\tilde{U}_\tau 1_{[\tau<\nu]}+\tilde{L}_\nu 1_{[\nu\leqslant\tau<T]}|F_t\Bigg]\\ =& \operatorname*{esssup}_{\nu\geqslant t}\ \operatorname*{essinf}_{\tau\geqslant t} E\Bigg[\exp\left\{\int_0^T h(s,Y_s)\,\mathrm{d}s+\xi\right\}1_{[\nu=\tau=T]}\}\\ &+\tilde{U}_\tau 1_{[\tau<\nu]}+\tilde{L}_\nu 1_{[\nu\leqslant\tau<T]}|F_t\Bigg].\end{aligned}$$

The result now follows obviously.

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References

[1] J. Cvitanic, I. Karatzas, Backward SDEs with reflection and Dynkin games, Ann. Probab. 24 (4) (1996) 2024–2056.
[2] J. Cvitanic, J. Ma, Reflected backward–forward SDEs and obstacle problems with boundary conditions, J. Appl. Math. Stochastic Anal. 14 (2) (2001) 113–138.
[3] N. El-Karoui, S. Hamadène, BSDEs and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations, Stochastic Process. Appl. 107 (2003) 145–169.
[4] N. El-Karoui, C. Kapoudjian, E. Pardoux, S. Peng, M.C. Quenez, Reflected solutions of backward SDEs and related obstacle problems for PDEs, Ann. Probab. 25 (2) (1997) 702–737.
[5] N. El-Karoui, E. Pardoux, M.C. Quenez, Reflected BSDEs and American options, in: L. Roberts, D. Talay (Eds.), Numerical Methods in Finance, Cambridge University Press, Cambridge, 1997, pp. 215–231.
[6] N. El-Karoui, S. Peng, M.C. Quenez, Backward stochastic differential equations in finance, Math. Finance 7 (1997) 1–71.
[7] S. Hamadène, J.P. Lepeltier, Zero-sum stochastic differential games and backward equations, Systems Control Lett. 24 (1995) 259–263.

[8] S. Hamadène, J.P. Lepeltier, Backward equations, stochastic control and zero-sum stochastic differential games, Stochastics Stochastic Rep. 54 (1995) 221–231.
[9] S. Hamadène, J.-P. Lepeltier, Reflected BSDEs and mixed game problem, Stochastic Process. Appl. 85 (2000) 177–188.
[10] S. Hamadène, Y. Ouknine, Backward stochastic differential equations with jumps and random obstacle, Electron. J. Probab. 8 (2) (2003) 1–20 http://www.math.washington.edu/.
[11] S. Hamadène, J.P. Lepeltier, A. Matoussi, Double barrier reflected backward stochastic differential equations with continuous coefficient, in: N. El-Karoui, L. Mazliak (Eds.), Pitman Research Notes in Mathematics Series, vol. 364, 1997, pp. 115–128.
[12] S. Hamadène, J.P. Lepeltier, S. Peng, BSDEs with continuous coefficients and application to Markovian nonzero sum stochastic differential games, in: N. El-Karoui, L. Mazliak (Eds.), Pitman Research Notes in Mathematics Series, vol. 364, 1997, pp. 161–175.
[13] M. Kobylanski, BSDEs and PDE's with quadratic growth, Ann. Probab. 28 (2000) 558–602.
[14] M. Kobylanski, J.P. Lepeltier, M.C. Quenez, S. Torres, Reflected BSDE with super-linear quadratic coefficient, Probab. Math. Statist. 1 (22) (2002) 51–83.
[15] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, Springer, New York, 1991.
[16] J.P. Lepeltier, J. San-Martin, Existence for BSDE with superlinear-quadratic coefficient, Stochastic Stochastic Rep. 63 (1998) 227–240.
[17] J.P. Lepeltier, J. San-Martin, Backward SDEs with two reflecting barriers and continuous coefficient: an existence result, J. Appl. Probab. 41 (1) (2004) 162–175.
[18] E. Pardoux, BSDEs weak convergence and homogenization of semilinear PDEs, in: F. Clarke, R. Stern (Eds.), Nonlinear Analysis, Differential Equations and Control, Kluwer Academic Publishers, Netherlands, 1999, pp. 503–549.
[19] E. Pardoux, S. Peng, Adapted solutions of backward stochastic differential equations, Systems Control Lett. 14 (1990) 51–61.
[20] E. Pardoux, S. Peng, Backward stochastic differential equations and quasilinear parabolic partial differential equations, in: B. Rozovskii, R. Sowers (Eds.), Stochastic Differential Equations and their Applications, Lecture Notes in Control and Information Sciences, vol. 176, Springer, Berlin, 1992, pp. 200–217.
[21] S. Peng, Probabilistic interpretation for systems of quasilinear parabolic partial differential equations, Stochastics 14 (1991) 61–74.
[22] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, Springer, Berlin, 1991.