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BELOUERGHI Malika

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Chaotic Dynamical Systems: Control and Synchronization

Publicly defended, In 07/03/2024, front of the jury members ;

Mr. KHALFALLAH Nabil	Professor	University of Biskra	President
Mr. MENACER Tidjani	Professor	University of Biskra	Supervisor
Mr. LOZI Rene	Professor	University of Nice	Co-Supervisor
Mr. LAADJAL Baya	MCA	University of Biskra	Examiner
Mr. ABDELOUAHAB M.Salah	Professor	University Center of Mila	Examiner

Chaotic Dynamical Systems: Control and Synchronization

Presented by:
BELOUERGHI Malika

MOHAMED KHIDER UNIVERSITY BISKRA
FACULTY OF EXACT SCIENCES OF NATURE and LIFE
DEPARTEMENT OF MATHEAMTICS

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Abstract

Chaos is a typical phenomenon of nonlinear systems and is currently widely studied, because of its features and many potential applications.

The main aim of this thesis concerns principally two major subjects. the first is to deploy a method for uncovering a hidden bifurcation in a multispiral Chua system with a sine function that is based on the famous paper by Menacer et al. This method is based on the core idea of Leonov and Kuznetsov for searching hidden attractors while keeping ε *constant, a new bifurcation parameter is introduced. We end this part with the introduction of a novel method based on the duration of integration for unveiling hidden patterns of an even number of spirals.*

In the second part, We used the Routh-Hurwitz Criteria for studying the stability of the Chua system at equilibrium point E_0 *concerning* ε*. Furthermore, we made a theoretical and numerical study of bifurcation and chaos control on Multispiral Chua's system.*

Keywords: *Chaos, Hidden Attractor, Bifurcation, Equilibrium point, Stability, Chua Chaotic system, Integration duration, Control, Routh-Hurwitz criterion, Synchronization.*

Dedication

"*To my loving parents and my husband whose courage me and gave me a chance for a better life, and to our angel* ***Rinad*** *...*"

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Introduction

"So far as the laws of mathematics refer to reality, they are not certain, and so far as they are certain, they do not refer to reality"

A.EINSTEIN

Chaotic dynamical systems have been shown to emerge from natural phenomena such as the weather or from designed engineering problems such as the movement of a rigid body in three-dimensional (3D) space. These systems have several beneficial characteristics, including random-like trajectories, great sensitivity to initial conditions, and ergodicity, according to chaos theory [KO21].

In 1963, **Edward Lorenz** described a simple mathematical model consisting of three non-linear differential equations giving rates of change in temperature and wind speed. Some surprising results were shown, such as the complex behavior of supposedly simple equations; also, this behavior is very sensitive to changes in the initial conditions: if there are errors in the measurement or observation of the initial state of the system (which is inevitable in real systems), then the prediction of the future of the system, in this case, is impossible. Lorenz referred to these systems with a noticeable dependence on the initial conditions as having the 'butterfly effect': this unique name came from the proposition that a butterfly flapping its wings in Hong Kong can affect the course of a tornado in Texas [Elh06]

The fundamental work of Lorenz in 1963 gave a scientific overview of the concept of a new type of behavior called Chaos. **Chaos** is a phenomenon that appears to be which is deterministic and very sensitive to initial conditions. It defines a special state of a system whose unpredictable behavior is never repeated. These systems may be divided into two types based on the type of calculus used: continuous-time and discrete-time. We are interested in continuous-time chaotic systems in this thesis. Chaos theory is useful in many fields such as data encryption [Zho+11], financial systems [Las00], biology [Vai15], and biomedical engineering [Boz97].

The primary aim of this thesis is to extend; in the framework of the qualitative theory of dynamic systems, the name "bifurcation" was coined by Henri Poincaré in 1885 [Poi85]. For a given value of a parameter, a bifurcation is seen when the number of solutions to a family of differential equations increases from one to two, like the pitchfork of a branch of a tree [Sat73], or when the topological structure of the solution is changed from steady state to periodic function (Hopf bifurcation [MM76], or from periodic to quasi-periodic function (secondary Hopf bifurcation).

Lorenz [Lor63], extended the scope of the bifurcation theory by identifying the first chaotic strange attractor in 1963. Chua invented the first differential equation modeling a real electronic device system with a chaotic asymptotic attractor (hence the strange attractor) while he was invited Professor at Matsumoto's laboratory, Waseda University, Tokyo [CHU92].

Nowadays, Chua's attractor is widely used, due to both its realizations: electronic circuit and its mathematical model. The electronic circuit and the system of differential equations may be combined to reach multiple objectives by Duan et al. [DWH04]. One can consider Chua's circuit as the simplest electronic circuit presenting chaos and possessing a highly interesting dynamical behavior. This was checked in many laboratory experiments for Zhong and Ayrom [ZA85], computer simulations for Matsumoto [Mat84a] and rigorously done mathematics [CKM86], [LU93], [BL00].

Chua's circuit is, surely the most extensively studied chaotic electrical system. It has the extraordinary feature of being able to generate a large variety of dynamic behaviors with just a few self-electronic components. In particular, it consists of two linear capacitors (a resistor and an inductor) and one non-linear resistor known as Chua's diode. By appropriately choosing these components, the circuit becomes chaotic, and the trajectories tend to a limit set called a strange attractor. The best-known attractor generated by Chua's circuit is called double scroll [SV91], [Are+96]. It has been generalized in several ways, replacing continuous piecewise-linear functions with some smooth functions, such as polynomials [KRC93], [SHI94], [Hua+96], and [HSD03], etc. In the work of Tang et al. [Tan+01], it is demonstrated that the n-scroll attractors can be generated using a simple sine or a cosine function.

In 2011, Leonov et al. [LKV11a], [LK11b] introduced a new classification of attractors of non-linear dynamical systems; in which attractors are dispatched in self-excited or hidden attractors. The first ones can be localized numerically via a standard computational procedure. After a transient process, a trajectory starts from a point of the unstable manifold in a neighborhood of equilibrium to reach a state of oscillation. Hence, one can easily identify it. In contrast, for hidden attractors, a basin attraction does not intersect with small neighborhoods of equilibria. To localize them, it is necessary to develop special procedures. Therefore, there are no transient processes leading to such attractors. The name hidden comes from this property.

An effective method for the numerical localization of hidden attractors in multidimensional dynamical systems has been proposed by the previously cited authors [Leo09], [LVK10]. Their method is based on numerical continuation: a sequence of similar systems is constructed such that for the first (starting) system the initial data for numerical computation of possible oscillating solution (starting oscillation) can be obtained analytically and its transformation from one system to another is followed numerically. One can find the first example of a hidden chaotic strange attractor in Chua's attractor [LKV12a].

The modified multispiral Chua system with a sine function is presented in [Tan+01]. In [MLC16], Menacer et al. using the Kuznetsov and Leonov method in another way, found a sequence of hidden bifurcations.

In another part of the thesis, we are interested in another branch developed in the field of dynamic systems that is attracting the interest of scientific researchers: Control and synchronization. The Control refers to the adaptive control of a given chaotic system with the aim of forcing its states to be asymptotically stable, usually converging towards zero [Khe+19], [FEA06], [Boc+00]. Many chaotic control methods have been presented. The principal methods to the control of a chaotic system are : OGY method [Ban+15], adaptive feedback method [Mat11] [SC08], backstepping design method [HS13], etc. In paper of hwag et al [YZ15] proposed a new feedback control on a modified chau system in ordre to have a better performance thans previous controllers. [SC08] considered the feedback control of the modified chua's circuit. Yassen [Yas03] presented the adaptive control and synchronization of Chua's circuit with unknown system parameters by used the Routh Hurwitz Criteria and Lyapunov direct method for study the asymptotic stability states.

The synchronization phenomenon has become an active subject of research linked to the development of telecommunications, It underwent remarkable improvements at the beginning of the 20th century. In 1990, Carroll and Pecora, pioneers of synchronization, came up with the idea of using a chaotic signal between two dynamic systems. chaotic signal between two identical dynamic systems. The first system that produces the chaotic signal is called the transmitter system (master), and the second is the receiver system (slave), this is identical synchronization. Chaos synchronization, an important topic in applications of nonlinear sciences, has been developed and widely studied in recent years, as they can be applied to wide areas of engineering and information sciences, notably in secure communication and cryptology [KA].

Our thesis is divided into two parts. We start with preliminary chapters: 1

and 2, and the second part presents our main results. The content of each chapter is outlined as follows:

Chapter 1

We present some mathematical preliminaries, We start with foundation definitions like an orbit, phase portrait, and notion of stability ...., which also contain definitions of attractors and bifurcation. So, this chapter gives you an introduction to the Chaotic dynamical system.

Chapter 2

This chapter is mainly devoted to the basics of self-excited, hidden attractors and presents effective analytical-numerical algorithms for hidden oscillation localization in the example of a circuit for the Chua attractor.

Chapter 3

The chapter deals with hidden bifurcations of the multi-spiral Chua Chaotic attractor generated by the sine function by applying an effective method proposed by Menacer et. al for uncovering hidden bifurcation; based on a core idea of the authors Leonov and Kuznetsov, where presented in the second chapter. Furthermore, We demonstrate the impact of the integration duration approach on the search for hidden patterns with an odd number of spirals, with present numerical results. The content of this chapter has been the subject of an international publication **"Integration Duration-Based Method for Unveiling Hidden Patterns of Even Number of Spirals of Chua Chaotic Attractor. Indian Journal of Industrial and Applied Mathematics, 10(1), 13-33"**.

Chapter 4

The main objective of this chapter is to study the stability of the original equilibrium point and adaptive control of the multi-spiral Chua attractor by using the Routh-Hurwitz criteria. The content of this chapter has been the subject of an international publication **"The Routh Hurwitz Criteria for Studying The Stability and Bifurcation in Multispiral Chua Chaotic Attractor. In MENDEL (Vol. 29, No. 1, pp. 71-83"**.

Part I

Preliminary Theory

1 Introduction to Chaotic Dynamical Systems

"***The world is chaos, and its disorder exceeds anything we would like to remedy***"

P.CORNEILLE

1.1 Introduction

Nonlinear systems are very interesting to engineers, physicists, and mathematicians because most real physical systems are inherently nonlinear in nature. In mathematics, a nonlinear system is problem, where the variables to be solved cannot be written as a linear combination of independent components. If the equation contains a nonlinear function (power or cross-product), the system is nonlinear as well. The system is nonlinear if there is some typical nonlinearity , for instance, as saturation, hysteresis, etc. These characteristics are basic properties of the nonlinear systems. Nonlinear equations are difficult to be solved by analytical methods and give rise to interesting phenomena such as bifurcation and chaos. Even simple nonlinear (or piecewise linear) dynamical systems can exhibit completely unpredictable behavior, the so-called deterministic chaos.
In this chapter, we define some of the basic conceptions, in order that we can leverage them in the next.

1.2 Dynamical System

There are many different types of mathematical dynamical systems, including ordinary differential equations, partial differential equations, ergodic systems, and discrete dynamical systems.
In the literature, a dynamic system is a structure that evolves over time in both :

1. Causal, where its future depends solely on past or present phenomena.
2. Deterministic, i.e. exact knowledge of the system's state at t_0 enables us to calculate its evolution at any other moment.

From a mathematical approach, dynamic systems can be divided into two categories:

- Continuous dynamic systems.
- Discrete dynamic systems.

1.2.1 Continuous and Discrete Dynamic Systems

- the continuous-time evolution of a dynamic system in the integer case is represented by a system of differential equations of the form:

$$\dot{x} = f(x, t, \gamma), x \in \mathbb{R}^m, \gamma \in \mathbb{R}^r, \tag{1.1}$$

  where $x \in E($ E a non-empty set of $\mathbb{R}^m$ called a phace space) is the vector of state, $\gamma \in \mathbb{R}^r$ is a vector of the parameters and $f : \mathbb{R}^m \times \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}^m$ is the vector field, which represents the dynamics of the system (1.1)

- The general form of a discrete-time dynamical system is described by an application (iterative function):

$$x_{n+1} = f(x_n, \gamma) \text{ where } x_n \in \mathbb{R}^m \text{ and } \gamma \in \mathbb{R}^r,\ n = 1, 2, ... \tag{1.2}$$

  where $f : \mathbb{R}^m \times \mathbb{Z}^+ \longrightarrow \mathbb{R}^m$ indicates system dynamics in discrete time.

► **Remark 1.1.** In this thesis, we will concentrate only on continues dynamical systems. ◄

1.2.2 Autonomous and Non-Autonomous Systems

When the vector field f does not explicitly depend on time, the system(1.1) is said to be autonomous. Otherwise, it is said to be non-autonomous. Using an appropriate change of variable, we can easily transform a non-autonomous system of dimension m into an equivalent autonomous system of dimension $m + 1$ by posing:

$$\begin{cases} X_{m+1} &= & t, \\ \dot{X}_{m+1} &= & f_{m+1}(X, \gamma) = 1. \end{cases}$$

- A non-autonomous system can always be transformed into an autonomous system (where t does not appear explicitly)

Phase Space

▶ **Definition 1.2.** Phase space is an often multi-dimensional space that can be used to interpret geometrically the motion of a dynamic system described by differential equations with respect to time. ◀

State Space

▶ **Definition 1.3.** The state space is the set of coordinates necessary for a complete description of a system. ◀

Flow

▶ **Definition 1.4.** Let M be any set and G an additive group ($\mathbb{R}$ or $\mathbb{Z}$). Consider $\{\varphi_t\}_{t\in G}$ a group of one-parameter applications M in M indexed by the group G.

- The term **flow** refers to the $(M, \{\varphi_t\}_{t\in G})$. The preceding set M constitutes the space of the phases of the flow. Any point x in this space represents a state of the dynamical system.

◀

Orbits and Trajectory

▶ **Definition 1.5.** We call the **trajectory** of a point x of M the application defined on G and with values in M by :

$$\begin{array}{rcl} \varphi : G & \to & M, \\ t & \to & \varphi^t(M). \end{array} \tag{1.3}$$

The image of the **trajectory** [Mar03]originating from x is called the **orbit** of a point x, i.e. the subset $\gamma(x)$ of the phase space defined by :

$$\gamma(x) = \varphi^t(x),\ t \in G.$$

◀

▶ **Definition 1.6.** A **trajectory** is a solution of the differential system. We consider the system dynamic system (1.1) the orbit is defined by:

$$O(x_0) = \{x(t); -\infty < t < +\infty\}.$$

◀

Phase Portrait

▶ **Definition 1.7.** Each initial condition corresponds to a different trajectory. The set of trajectories constitutes the **phase portrait** of the system. ◀

Periodic Points and Limit Cycles

▶ **Definition 1.8.** a point x is said to be T-periodic: $\forall t \in \mathbb{R} : x(t+T) = x(t)$ and $x(t+\bar{T}) = x(t)$; for $0 < \bar{T} < T$; is then called the period of the solution ◀

▶ **Remark 1.9.** A periodic orbit $O(x)$ is always a sequence of periodic points. All these points are called **periodic points** of period T. ◀

▶ **Definition 1.10.** **Limit Cycles** Consider the system:

$$\begin{cases} \dot{x} & = & P(x,y), \\ \dot{y} & = & Q(x,y), \end{cases}$$

such that P and Q are polynomials in x and y with real coefficients of degree d.

A **limit cycle** C is an isolated closed trajectory in space, i.e. no other closed orbit can be found in its neighbourhood. ◀

▶ **Remark 1.11.** The width of a limit cycle is the maximum value of the variable x over the limit cycle. ◀

Invariant Set

▶ **Definition 1.12.** Let A be a subset of the phase space; A is said to be **invariant** (resp. positively invariant) by a flow φ_t, if for any t in $\mathbb{R}$ (resp. in $[0; +\infty[$), $\varphi_t(A)$ is included in A.

- The trajectory of an autonomous system in state space is a set of invariants.

◀

1.2.3 Conservative Systems and Dissipative Systems

Physicists define a conservative system as one that conserves total energy. On the other hand, a dissipative system is one that dissipates energy.
So the conservative system has a first integral (or constant) of motion, and the second has at least one rate-dependent term. But let us not forget that the systems under consideration are deterministic systems, so to specify this definition more

precise, we can say that a deterministic system is conservative if and only if the dynamics of the system associated with each initial condition x_0 has one and only one final state $x(t)$, for this there must exist a bijective application of the phase space

$$\begin{array}{rcl} \varphi : X \times \mathbb{R} & \longrightarrow & X, \\ (x,t) & \longrightarrow & \varphi_t(x) = \varphi(x,t), \end{array}$$

which is called a **flow** and has the following properties:

$$\begin{array}{rcl} \varphi_t(x_0) & = & x_0, \\ \varphi_{t+s}(x_0) & = & \varphi_t(\varphi_s(x_0)),\ t,s \in \mathbb{R}^+. \end{array}$$

► **Remark 1.13.** If the system is dissipative, the **flow** is not bijective and there are generally one (or more) attractors in the phase space of the system. ◄

1.3 Chaotic Dynamical System

Nonlinear, or simply piecewise linear, dynamic systems can exhibit completely unpredictable behaviors, which may even seem random (although these are perfectly deterministic systems). This unpredictability is called **chaos**. In general, there is no formal definition of **chaos**. However, there are several possible approaches to defining chaos. These approaches are not all equivalent, but they converge toward certain common properties characterizing chaos [KA]. For example, the definition of chaos according to **Li-Yorke** [Via09] is as follows:

A continuous application $f : [0,1] \longrightarrow [0,1]$ is chaotic, if there is an enumerable set $S \subset [0,1]$, such that the trajectories of two different points $x_1, x_2 \in S$ are proximal, and not asymptotic; i.e :

$$\lim_{n\to\infty} \inf(|f^{(n)}(x_1) - f^{(n)}(x_2) = 0|) \text{ and } \lim_{n\to\infty} \sup(|f^{(n)}(x_1) - f^{(n)}(x_2) > 0|).$$

► **Remark 1.14.** Li and Yorke introduced the first mathematical definition of chaos. They established a very simple criterion: "The presence of three periods implies chaos". This criterion plays a very important role in the analysis of chaotic dynamical systems. ◄

One of the most popular definitions of chaos is the one given by **Devaney** [Dev86]: a dynamical system is chaotic if and only if :

a) It is topologically transitive, i.e. if we consider two arbitrary neighborhoods of two distinct states of a dynamical system, then there exists a trajectory from one to the other.

b) It has a dense set of periodic orbits.

c) It has the property of sensitivity to initial conditions.

1.3.1 Characterisation of Chaotic Dynamical System

There are a number of properties that sum up the characteristics observed in chaotic systems. They are regarded as mathematical criteria that define chaos. The most well-known are :

Sensitivity to Initial Conditions

Chaotic systems are extremely sensitive to initial conditions. Very small perturbations to the initial state of a system can ultimately lead to strictly different behaviour in its final state. Figure. 1.1 illustrates the temporal evolution of a trajectory of the system of Lü [KA] with three very close different initial conditions.

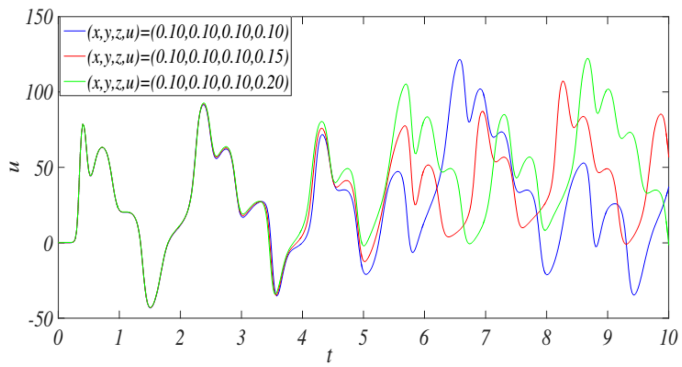


Figure 1.1: The temporal evolution of a trajectory of the system of Lü with different initial conditions

Non-linearity

A chaotic system is a non-linear dynamic system. A linear system cannot be chaotic. Non-linearity is one of the fundamental characteristics of chaotic systems.

Determinism

A chaotic system is deterministic (rather than probabilistic), i.e. it is subject to laws that completely describe its motion. subject to laws that completely describe its movement. The notion of determinism therefore means the ability to predict the future state of a phenomenon from a past event. However, in the case of random phenomena, it is impossible to predict the trajectories of any given particle.

Fractal Dimension

Berge et. al [Min+90] impose an additional condition of a dimensional type [Ham].

▶ **Definition 1.15.** A strange attractor is characterized by sensitivity to initial conditions and having a fractal dimension. ◀

Strange Attractors

▶ **Definition 1.16.** A **strange attractor** has an attractor which displays sensitive dependence on initial conditions. We note that an attractor is a bounded region of phase space to which all sufficiently close trajectories from the so-called basin of attraction are attracted asymptotically for long enough times ◀

▶ **Definition 1.17.** The particular geometric figure that represents the attractor of a chaotic system in phase space as a function of time, is called the **chaotic attractor**. Thus, this attractor is produced by two simultaneous operations, namely stretching, which is responsible for the sensitivity to initial conditions and instability, and folding, which is responsible for the strangeness. ◀

Example of Continuous chaotic Dynamical Systems

To demonstrate the characteristics of a chaotic system, let's take the Rössler model as an example [Dim12]:

$$\begin{cases} \dot{x} &= -x - z, \\ \dot{y} &= x + \alpha y, \\ \dot{z} &= b + z(x - \gamma). \end{cases}$$

For parameter values ($\alpha = 0.398$, $\beta = 2$ and $\gamma = 4$), the Rössler system operates in a chaotic regime. Figure. 1.2 represents the evolution as a function of time of the chaotic trajectories of the Rössler system and illustrates a complex, non-periodic. This is the random aspect of chaotic systems. The evolution of a chaotic trajectory over time appears to be random, but observation of the trajectory in phase space,

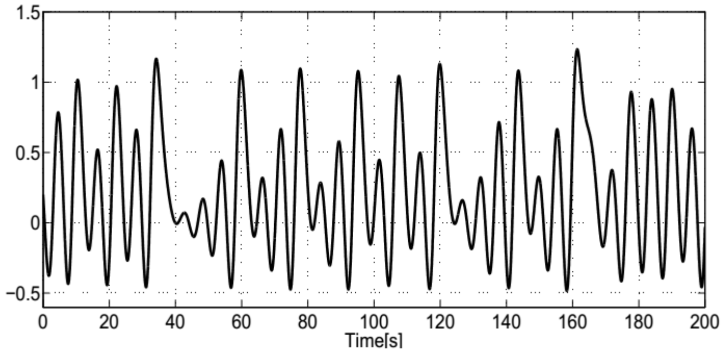


Figure 1.2: Time evolution of the solution y.

as t tends towards infinity, reveals a particular form that has a fractal structure: this is the strange attractor, see figure. 1.3 There are several other models of chaotic

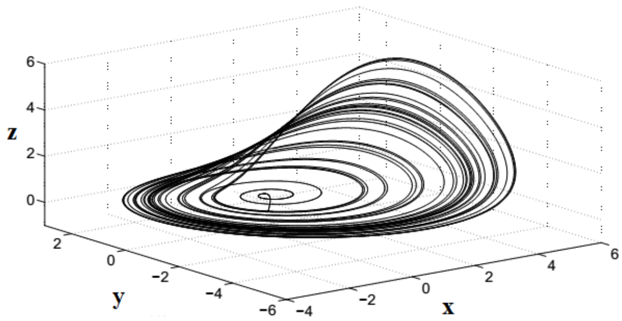


Figure 1.3: Rössler chaotic system attractor.

systems that have similar characteristics to the Rössler system. As examples, we mention the Lorenz model, the Chua circuit, Lü's system, etc.

1.4 Qualitative Study of Dynamic Systems

The qualitative study allows us to see how the solutions behave without having to solve the equation. In particular, it allows the local study of solutions around points of equilibrium. This study will begin with a search for the fixed points of the system (1.1).

1.4.1 Equilibrium Points

▶ **Definition 1.18.** An equilibrium point (or critical point, singular point, or stationary point) in the system (1.1 is a point x_{eq} in phase space where $f(x_{eq}) = 0$. ◀

▶ **Remark 1.19.** The geometric equilibrium point is an intersection between the curve of our function $f(x)$ with the x-axis. ◀

In phase space, an equilibrium point is represented by a point. Its value is determined by the initial condition chosen. Furthermore, for different initial conditions, we can find several equilibrium points. Furthermore, these stable or unstable, i.e. convergence or divergence between neighboring trajectories [Cha+07].
For a complete study of a dynamic system, we generally expect the environment to behave in a stationary manner. This is presented by the disappearance of transient phenomena by canceling out the transition function or the vector field. In this case, the system will have one of two states:

- Equilibrium state (fixed points, periodic points).
- Chaotic state.

To facilitate this study, we use the properties of linear algebra on the equations that describe our dynamic systems. describe our dynamic systems, where as the majority of dynamic systems associated with natural phenomena are not linear, so we have to linearise them.

1.4.2 Linearization of Dynamic Systems

Consider the nonlinear dynamic system defines by:

$$\dot{x} = f(x(t)), \tag{1.4}$$

where $x = (x_1, x_2, \ldots, x_n)$, $f = (f_1, f_2, \ldots, f_n)$, and x_{eq} a fixed point of this system.

Suppose a small upset $\varepsilon(t)$ is applied in the neighborhood of the fixed point. The function f can be developed in a series of Taylor in the neighborhood of point x_{eq} as follows:

$$\dot{\varepsilon}(t) + x_{eq} = f(\varepsilon(t) + x_{eq}) \simeq f(x_{eq}) + J_f(x_{eq}).\varepsilon(t), \tag{1.5}$$

with $J_f(x_{eq})$ is the Jacobian matrix of the function f defined by:

$$J_f(x_{eq}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}_{x=x_{eq}} . \tag{1.6}$$

As $f(x_{eq}) = 0$, then equation (1.5) becomes again:

$$\dot{\varepsilon}(t) = J_f(x_{eq}).\varepsilon(t). \tag{1.7}$$

The writing (1.7) means that the system (1.4) is linearized.

1.4.3 Hartmann-Groβman Theorem

▶ **Definition 1.20.** Two flows φ_t and ψ_t are said to be topologically equivalent in a neighborhood of the equilibrium point if there exists a homeomorphism h which sends the equilibrium point of the first flow to the point of equilibrium of the second flow and conjugates the flows, i.e. $h \circ \varphi_t = \psi_t \circ h$. ◀

Let x_{eq} be an equilibrium point of the system (1.4) and let $J_f(x)$ be the Jacobian matrix at point x_{eq}. The following theorem follows:

▶ **Theorem 1.21.** [Lu91] If $J_f(x_{eq})$ admits pure non-zero or imaginary eigenvalues, then there exists a homeomorphism that transforms the orbits of the nonlinear flow into those of the flow linear in some neighborhood U of x_{eq}. This theorem will allow us to link the dynamics of the nonlinear system (1.4) to the dynamics of the linearity system (1.7). ◀

1.4.4 Concept of Stability

The notion of stability of a dynamic system characterizes the behavior of its trajectories in the vicinity of its equilibrium points. Analyzing the stability of a dynamic system allows us to study the evolution of its state trajectory when the initial state is very close to an equilibrium point. Stability theory in the Lyapunov sense is valid for any differential equation. This notion means that the solution of an equation initialized in the vicinity of an equilibrium point always remains sufficiently close to it.

Stability in the Lyapunov Sense

▶ **Definition 1.22.** The equilibrium point x_0 of the system (1.1) is:

1. **Stable** if [SL+91]:

$$\forall \varepsilon > 0, \exists\, \delta > 0 : \|x(t_0) - x_0\| < \delta \Rightarrow \|x(t, x(t_0)) - x_0\| < \varepsilon, \forall\, t \geq t_0. \quad (1.8)$$

2. **Asymptotically stable** if [SL+91]:

$$\exists\, \delta > 0 : \|x(t_0) - x_0\| < \delta \Rightarrow \lim_{t \to \infty} \|x(t, x(t_0)) - x_0\| = 0. \quad (1.9)$$

3. **Exponentially stable**: if there exist two strictly positive constants a, b as: [SL+91]:

$$\forall \varepsilon > 0, \exists\, \delta > 0 : \|x(t_0) - x_0\| < \delta \Rightarrow \|x(t, x(t_0)) - x_0\| < a \exp(-bt), \ \forall\, t > t_0. \quad (1.10)$$

4. **Unstable** if:

$$\exists\, \varepsilon > 0, \ \forall\, \delta > 0 : \|x(t_0) - x\| < \delta \text{ and } \|x(t, x(t_0)) - x_0\| > \varepsilon, \forall\, t \geq t_0, \quad (1.11)$$

wich mean that equation (1.8) is not satisfied.

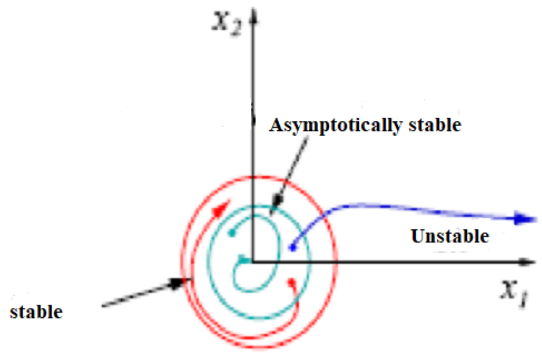


Figure 1.4: Different types of Lyapunov stability.

▶ **Remark 1.23.** Noticeable that using the previous definitions to achieve stability of the system (1.1), in the neighborhood of its equilibrium point, requires the explicit resolution of the system (1.1), which is often very difficult in most cases. For this reason, the following two methods of Lyapunov allow us to get around this obstacle. ◀

1. Lyapunov's First Method (Direct Method)

As we have seen, Lyapunov's first method is simple to apply, but it allows us to analyze the stability of equilibria only very partially. Besides, it gives no indication of the size of the basins of attraction. The second method is more difficult to implement, but, on the other hand, it is far-reaching and more general. It is founded on the definition of a specific function, denoted $V(x)$ and known as the Lyapunov function, which decreases along the trajectories of the system within the attraction basin. This theorem will summarize this method [SL+91].

▶ **Theorem 1.24.** The system's equilibrium point x_{eq} (1.1) is stable if a function $V(x) : D \longrightarrow \mathbb{R}$ continuously differentiable having the following properties :

a) D is an open of $\mathbb{R}^n$ and $x_{eq} \in D$,

b) $V(x) > V(x_{eq}) \; \forall \; x \neq x_{eq}$ in D,

c) $\dot{V}(x) \leq 0 \; \forall \; x \neq x_{eq}$ in D.

◀

There is no method to find a Lyapunov function. But in mechanics and for electrical systems, one can often use the total energy as a Lyapunov function.

2. Lyapunov's Second Method (Indirect Method)

Lyapunov's first method is based on examining the linearization around the equilibrium point x_{eq} of the system (1.1). More precisely, we examine the eigenvalues λ_i of the Jacobian matrix evaluated at the equilibrium point. According to this method, the properties of stability of x_{eq} are expressed as follows [SL+91]:

1- If all the eigenvalues of the Jacobian matrix have a **strictly negative real** part, then x_{eq} is **exponentially stable**.

2- If the Jacobian matrix has at least one eigenvalue with a **strictly positive real** part, x_{eq} is **unstable**.

▶ **Remark 1.25.** This method does not allow us to say if the equilibrium is stable or unstable when the matrix Jacobian has at least one zero eigenvalue and no eigenvalue with an exactly positive real part. In this case, the trajectories of the system converge to a subspace (a manifold) whose dimension is the number of zero eigenvalues of the Jacobian matrix, and the stability of the equilibrium can be studied in this subspace by the first method. ◀

Routh-Hurwitz Criterion

To demonstrate that a point of equilibrium is asymptotically stable, it is therefore necessary a priory to calculate the n eigenvalues λ_i of A and check that $\forall i$ $\mathrm{Re}(i) < 0$. An algebraic method has been developed by **Routh-Hurwitz**, based on the calculation of particular determinants known as **Routh-Hurwitz** determinants [Mer96]-[AEE06].

Let's assume the system (1.1) , its linearisation is written as : $\dot{x} = Ax$. The eigenvalues of A are solutions of the characteristic equation :

$$p(\lambda) = det(A - \lambda I) = 0 \Leftrightarrow \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_{n-1}\lambda + a_n = 0.$$

The Routh-Hurwitz determinants are defined as follows:

$$H_1 = |a_1|$$

$$H_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix}$$

$$H_3 = \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}$$

$$H_j = \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ a_5 & a_4 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2j-1} & a_{2j-2} & a_{2j-3} & \cdots & a_j \end{vmatrix}$$

▶ **Proposition 1.26.** The equilibrium point x_{eq} is asymptotically stable $\Leftrightarrow \forall\lambda_i, Re(\lambda_i) < 0 \Leftrightarrow \forall(H_i) > 0$. ◀

▶ **Theorem 1.27. (Routh-Hurwitz Criteria)** Let $p(\lambda)$ be a polynomial : For P to be asymptotically stable, the main determinants of the Hurwitz matrix must be strictly stable. ◀

Let's consider the system for $n = 3 : \dot{x} = Ax$. The characteristic equation is:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3.$$

The Routh-Hurwitz determinants are:

$$H_1 = |a_1| = a_1,$$
$$H_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} = a_1a_2 - a_3,$$
$$H_3 = \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} = a_3,$$

Thus the conditions of stability of the equilibrium point are: $H_1 > 0$, $H_2 > 0$, $H_3 > 0$. So the point is asymptotically stable.

1.5 Attractors and Attraction Basin

1.5.1 Attractors

▶ **Definition 1.28.** The region of phase space to which all trajectories of a dissipative dynamical system converge is called an "attractor". Attractors are therefore geometric shapes that characterize the long-term evolution of dynamical systems. ◀

▶ **Definition 1.29.** set A is an attractor if :

1. A is a compact, flow-invariant set φ_t (i.e. $\varphi_t(A) = A$ for all t)
2. For any neighborhood U of A, there exists a neighborhood V of A such that any solution $X(t, X_0) = \varphi_t(X_0)$ will remain in U if $X_0 \in V$
3. $\cap\varphi_t(V) = A$, $t \geq 0$
4. There exists a dense orbit in A.

◀

1.5.2 Attraction Basin

▶ **Definition 1.30.** When A is an attractor, the set :

$$B(A) = \left\{ x \in \mathbb{R}^n / \lim_{t \to \infty} \varphi(x) = A \right\}$$

is called the **Basin of Attraction** of A. It is, therefore, the set of points for which the trajectories converge asymptotically to A. ◀

1.5.3 Attractor Properties

1. An attractor is indecomposable, i.e. the union of two attractors is not an attractor.

2. There exists a set $B \supset A$, such that for any neighborhood of A, the trajectory that originates in B origin in B is found after a finite time in this neighborhood of A. In other words, any trajectory that originates in B tends towards the attractor; this "zone of influence" is the **Basin of Attraction**.

3. A bounded subset A of the space is of zero volume invariant by flow φ_t. In other words, any point in the state space that belongs to an attractor remains inside that attractor for all t.

1.5.4 Types of Attractors

There are two types of Attractors: Regular Attractors and Strange or Chaotic Attractors :

1. **Regular Attractors**

   Regular attractors characterize the evolution of non-chaotic systems and can take three forms, as shown in **Figure(1.5)**:

   - **The Fixed Point:** is the simplest attractor. It is represented by a point in phase space. It is therefore a constant, stationary solution.
   - **The Periodic Limit Cycle:** it's a closed trajectory that attracts all neighboring orbits. It is therefore a periodic solution of the system.

     The limit cycle of a dynamic system is any isolated periodic solution in the set of all periodic solutions of this system.

- **The pseudo-Periodic Limit Cycle(Invariant Tori)**: it corresponds to a sum of periodic solutions, whose period ratio is an irrational number. A quasi-periodic regime can be represented in state space by a torus.

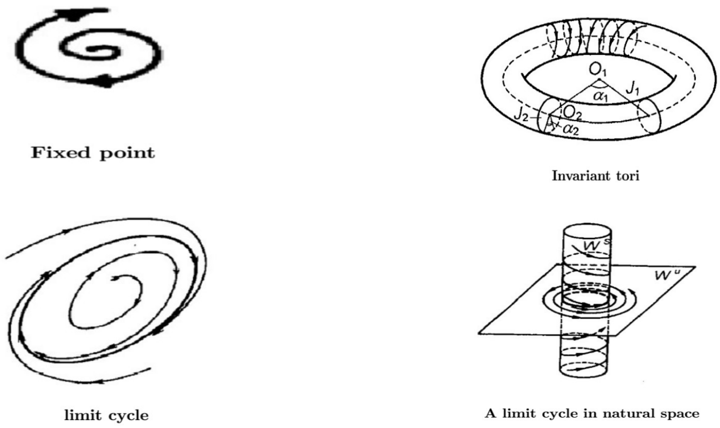


Figure 1.5: The Fixed point, Invariant Tori and Limit cycle attractor.

2. **Strange Attractors**

   A chaotic attractor has far more complex geometric shapes that characterize the evolution of chaotic systems: after a certain period of time, all points in phase space (and belonging to the attractor's basin of attraction) gives rise to trajectories that tend to form the strange attractor, and is characterized by:

   - Sensitivity to initial conditions (two initially neighboring attractor trajectories always end up moving away from each other, reflecting chaotic behavior).
   - A zero volume in phase space.
   - A fractal (non-integer) dimension d, $2 < d < n$, where n is the dimension of the phase space.
   - An exponentially fast separation of two initially close trajectories.

1.6 Bifurcation

The fundamental aspect of the study of dynamic systems is the notion of bifurcation. It means a division in two, a splitting a part, a change. This term was introduced

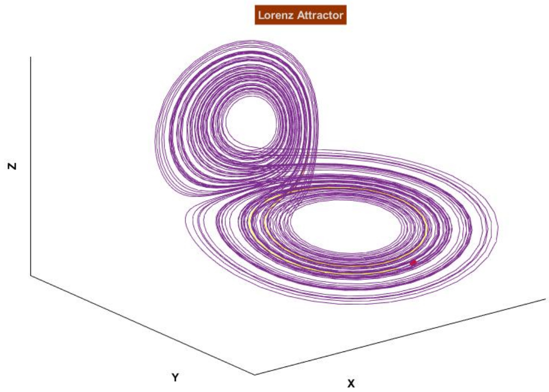


Figure 1.6: Chaotic Lorenz system attractor.

by Henri Poincaré at the beginning of the 20th century in his work on differential systems. For certain critical values of the system's control parameters, the solution of the differential equation changes qualitatively: this is known as a bifurcation. A first approach to studying dynamic systems is to look for equilibrium points, i.e. stationary solutions that do not exhibit time evolution. The next step is to vary the system's control parameters. We then look at what happens to the equilibrium points, in particular those that were stable before changing the system parameters, and the bifurcations that appear. For the parameter values at which such qualitative changes occur, known as bifurcation values, the construction of the phase portrait requires adapted tools.

▶ **Definition 1.31.** The non-linear dynamic system defined as follows:

$$\frac{dx}{dt} = g(x, t, \delta), \tag{1.12}$$

with the control parameter δ, and let x_0 be the solution.

A bifurcation is a qualitative change in the solution x_0 of the system (1.12) when δ is modified, and more exactly the absence or change in stability and the appearance of new solutions. ◀

▶ **Definition 1.32.** If the portrait of a system's phase portrait does not change when its parameters are changed, the system is said to be structurally stable. As a result, a bifurcation is associated with a loss of structural stability (the value of the parameter for which the system (1.12) is not structurally stable). ◀

1.6.1 Bifurcation Diagrams

▶ **Definition 1.33.** A bifurcation occurs when such a qualitative change in the structure of a system occurs at the same time as a quantitative change in one of its parameters (called a bifurcation value). The graphs that represent these bifurcations are called bifurcation diagrams. The bifurcation diagram is therefore a very important tool for evaluating the possible behaviours of a system as a function of bifurcation values. ◀

▶ **Example 1.34.** Bifurcation diagram of the Rössler system described by (see the Figure.1.7) [Ame07] :

$$\begin{cases} \dot{x} &= -(y+z), \\ \dot{y} &= x+ay, \\ \dot{z} &= b+z(x-c). \end{cases}$$

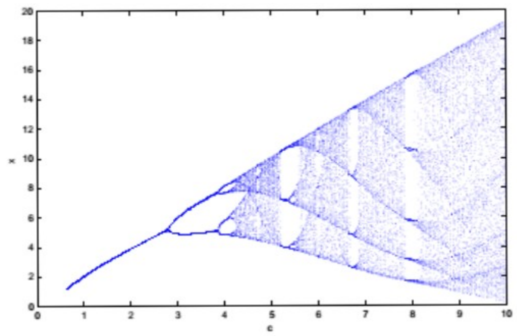


Figure 1.7: Bifurcation diagram of the Rössler system for a=0.2 and b=0.2.

◀

1.6.2 Bifurcations in One-Dimensional Systems

Bifurcations of a one-dimensional system are associated with the stabilities of its equilibrium points. Such bifurcations are known as local bifurcations as they occur in the neighborhood of the equilibrium points. Such types of bifurcations are occurred in the population growth model, outbreak insect population model, chemical kinetics model, bulking of a beam, etc. There are several types of local bifurcation, including four important bifurcations,which depend on a single parameter δ, namely the saddle-node, pitchfork, transcritical, and hopf bifurcations are discussed for one-dimensional systems [Lay+15],[GH13].

Saddle-Node Bifurcation

A linear function does not change the number of roots. The simplest polynomial which changes the number of roots depending on the parameter δ is the quadratic polynomial [HK12]. In the one-dimensional dynamical system (1.12) depending on one parameter δ, and the function g can be rewritten as :

$$g(x,\delta) = \delta - x^2; \qquad \delta, x \in \mathbb{R} \tag{1.13}$$

We call the function (1.13) the normal form of the saddle-node bifurcation. The equilibrium points of (1.13) are the solutions of the equation

$$g(x,\delta) = 0 \Rightarrow \delta - x^2 = 0 \Rightarrow x^2 = \delta. \tag{1.14}$$

We have three possibilities depending on the sign of the parameter δ :
If $\delta < 0$, the equation $g(x,\delta) = 0$, as no point of equilibrium. And if $\delta > 0$, there are two solutions, so two equilibrium points : $x_{1,2} = \pm\sqrt{\delta}$.

Their stability is determined by :$\frac{dg(x,\delta)}{dx} = -2x$ so $\frac{dg(x,\delta)}{dx}\ |_{x_1} = 2\sqrt{\delta} > 0$ and $\frac{dg(x,\delta)}{dx}\ |_{x_2} = -2\sqrt{\delta} < 0$.

Depending on the signs of $g(x)$, we see that $x_1 = 2\sqrt{\delta}$ is stable, while $x_2 = -\sqrt{\delta}$ is unstable. The bifurcation diagram is shown in Fig. 1.8

▶ **Remark 1.35.** Same study done when $g(x,\delta) = \delta + x^2$, $g(x,\delta) = -\delta - x^2$, $g(x,\delta) = -\delta + x^2$. But in all cases, there is a transition at $\delta = 0$ between the existence of no fixed point and the existence of two fixed points, one of which is stable and the other unstable. ◀

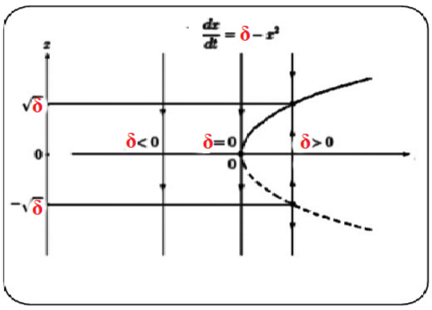


Figure 1.8: Saddle-node bifurcation diagram.

Transcritical Bifurcation

Many parameter-dependent physical systems have an equilibrium point that must exist for all values of a system parameter and can never disappear. However, if the parameter changes, the stability character may change. The transcritical bifurcation is one type of bifurcation in which the stability characteristics of the fixed points change as the parameter values change.
Consider the following one-dimensional dynamical system:

$$\dot{x}(t) = g(x, \delta) = \delta x - x^2; \quad x, \delta \in \mathbb{R} \tag{1.15}$$

The equilibrium points of this system are obtained as:

$$g(x, \delta) = 0 \Rightarrow x(\delta - x) = 0 \Rightarrow \begin{cases} x_1 = 0, \\ x_2 = \delta. \end{cases}$$

The equation $g(x, \delta) = 0$ admits two equilibrium points x_1, x_2. We calculate:

$$\frac{dg(x,\delta)}{dx} = \delta - 2x \text{ so } \frac{dg(x,\delta)}{dx}\,|_{x_1} = \delta\ ,\ \frac{dg(x,\delta)}{dx}\,|_{x_2} = -\delta.$$

- $\delta < 0$: two fixe points for $x_1 = 0$ stable and $x_2 = \delta$ unstable.
- $\delta = 0$: one semi stable fixed point for $x = 0 = \delta$.
- $\delta > 0$: two fixe points for $x_1 = 0$ unstable and $x_2 = \delta$ .stable

This is referred to as a transcritical bifurcation. An exchange of stabilities has occurred between the system's two fixed points in this bifurcation.The bifurcation diagram is presented in Fig. 1.9

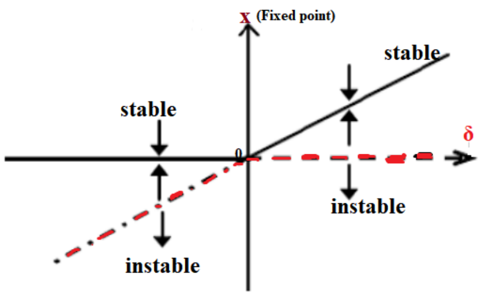


Figure 1.9: Bifurcation diagram transcritical.

Pitchfork Bifurcation

We now discuss pitchfork bifurcation in a one-dimensional system which appears when the system is symmetric in the left and right directions. The stability of an equilibrium point changes in favor of the birth of a pair of equilibrium points. There are two kinds of this bifurcation :

1. **Supercritical**: consider a normal form:

$$g(x,\delta) = \delta x - x^3. \tag{1.16}$$

The equilibrium points of the system are obtained as:

$$g(x,\tau) = 0 \Rightarrow x(\delta - x^2) = 0,$$

$$\Rightarrow \begin{cases} x = 0, \\ \text{or} \\ \delta - x^2 = 0, \end{cases} \Rightarrow \begin{cases} x = 0, \\ \text{or} \\ x^2 = \delta. \end{cases}$$

*If $\delta < 0$, we have a single point of equilibrium at $x = 0$.

*If $\delta > 0$, we have three equilibrium points: $x_1 = 0$, $x_{2,3} = \pm\sqrt{\delta}$.

We study the stability of these equilibrium points:

$$\frac{dg(x,\delta)}{dx} = \delta - 3x^2 \text{ so } \begin{cases} \frac{dg(x,\delta)}{dx}|_{x_1} = \delta, \\ \frac{dg(x,\delta)}{dx}|_{x_{2,3}} = -2\delta. \end{cases}$$

As a result :

*if $\delta = 0$: the system has only one equilibrium point in nature, where $x = 0$ is semi-stable.

*If $\delta < 0$: we have the only equilibrium point at the origin, where $x = 0$ is stable.

*If $\delta > 0$ we have three equilibrium points:

$$\begin{cases} x = 0 \text{ is unstable,} \\ x = \pm\sqrt{\delta} \text{ is stable.} \end{cases}$$

2. **Subcritical**: having a normal form :

$$g(x,\delta) = \delta x + x^3.$$

The same calculation yields, the equilibrium points of the system are obtained as :

$$g(x,\delta) = 0 \Rightarrow \delta x + x^3 = 0 \Rightarrow x(\delta + x^2) = 0.$$

$$\Rightarrow \begin{cases} x = 0, \\ \text{ou} \\ \delta + x^2 = 0, \end{cases} \Rightarrow \begin{cases} x = 0, \\ \text{ou} \\ x^2 = -\delta. \end{cases}$$

This system has three equilibrium pointsif $x_1 = 0$ and $x_{2,3} = \pm\sqrt{-\delta}$.

We study the stability of these equilibrium points :$\frac{dg(x,\delta)}{dx} = \delta + 3x^2$ so $\begin{cases} \frac{dg(x,\delta)}{dx}|_{x_1} = \delta, \\ \frac{dg(x,\delta)}{dx}|_{x_{2,3}} = -2\delta. \end{cases}$ .As a result :

*If $\delta > 0$ we have the only equilibrium point where $x_1 = 0$ which is unstable.

*If $\delta < 0$ we have the equilibrium point: $\begin{cases} x_1 = 0 \text{ is stable}, \\ x_{2,3} = \pm\sqrt{\delta} \text{ is unstable}. \end{cases}$

The bifurcation diagram is presented in Fig. 1.10

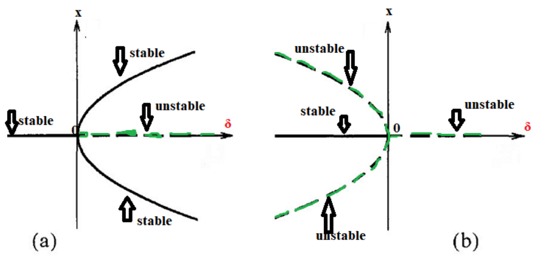


Figure 1.10: Pitchfork bifurcations diagram : (a) supercritical, (b) subcritical.

1.6.3 Hopf Bifurcation

While all the bifurcations we have described are stationary, the Hopf bifurcation gives rise to oscillatory solutions, and the phase space now has two components, and its form is expressed in the complex plane. Normal form:

Set the standard representation of a Hopf bifurcation :

$$\frac{dZ}{dt} = \delta Z - |Z|^2 Z. \tag{1.17}$$

Posing $\delta = \delta_R + i\delta_I$ and $Z = x\exp(i\theta)$, we get:

$$\begin{cases} \frac{dx}{dt} = \delta_R x - x, \\ \frac{dx}{dt} = \delta_I. \end{cases}$$

We therefore obtain a pitchfork bifurcation for the amplitude while the phase rotates at speed δ_I. The solution is therefore periodic, and the trajectories describe a spiral drawn towards an asymptotic curve called the limit cycle. Naturally, the bifurcation of Hopf can also be subcritical if the coefficient of the term $|Z|^2 Z$ has a positive sign, then a negative term is needed in $|Z|^4 Z$ to obtain a non-linear saturation. We will now focus on the step that follows the temporal regularity. According to Landau the bifurcation of a point from a stationary behavior (equilibrium point) towards a periodic behavior (limit cycle) and then biperiodic (a torus) constitutes the first stages of the green transition turbulence. The latter presents a very interesting phenomenon that we call chaos, which has long been synonymous with disorder and confusion and is opposed to order and method. Many researchers in science have been interested in so-called chaotic movements. They confirmed that contrary to what deterministic thought has hammered home for ages, there could be an equilibrium in the disequilibrium, organization in the disorganization.

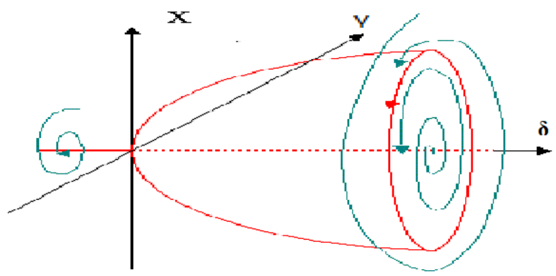


Figure 1.11: Hopf bifurcation diagram .

2 Hidden and Self-excited Attractors

"We must see the present state of the universe as the effect of its previous state and as the cause of the state that will follow."

P – S.Laplace

2.1 Introduction

Recently, modeling numerous chaotic phenomena in the form of nonlinear dynamical systems, with some special features, examples being particular dynamic behaviors and specific properties related to the system equilibria has drawn the attention of many researchers. Generally speaking, regardless of the type of system (continuous time or discrete-time dynamical systems) chaos can appear in the form of "self-excited attractors" or "hidden attractors", which is a new classification that is defined by Leonov and Kuznetsov in [LKV12b]. The self-excited attractors are attractors for which the initial conditions are located close to the saddle points of the chaotic flow [Leo+14]. On the other hand, hidden chaotic attractors are attractors in unusual systems, for example, systems with no equilibria or with only one stable equilibrium [Wei11]; for which the initial conditions can only be found via extensive numerical search [LK13a] [Sha+15]. Consequently, these types of chaotic attractors are difficult to discover [PSJ18].
These studies have proven the significant role of hidden attractors in theoretical problems and engineering applications. For example, hidden attractors can generate unexpected and potentially disastrous responses to perturbations in a structure like a bridge or an airplane wing.

2.2 Self-Excited Attractors

In a dynamical system, an oscillation can be easily numerically localized if beginning conditions from its open neighborhood lead to long-time behavior that approaches the oscillation. This type of oscillation (or group of oscillations) is known as an attractor, and its attracting set is known as the basin of attraction. As a result, self-excited attractors may be numerically localized using the traditional computational approach, in which a trajectory, starting from an unstable manifold in a region of unstable equilibrium, is attracted to the state of oscillation and follows there after a brief process. As a result, self-excited attractors are easily seen.

▶ **Definition 2.1.** An attractor is said to be self-excited if its basin of attraction exceeds an arbitrarily small open neighborhood of unstable equilibrium. ◀

During the first half of the twentieth century, during the early stages of the development of nonlinear oscillation theory (Krylov, 1936; Andronov et al., 1966; Stoker, 1950), much attention was focused on the analysis and synthesis of oscillating systems, for which the problem of the existence of oscillations can be studied relatively easily. These researches were motivated by applying periodic oscillation research in electronics, chemistry, biology, and other fields. The structure of many applied systems (see, for example, van der Pol (van der Pol, 1926), Tricomi (Tricomi, 1933) and BelousovZhabotinsky (Belousov, 1959) systems) was such that the presence of oscillations was almost clear since the oscillations were produced from unstable equilibria (self-stimulated oscillations).
Consider some famous visualizations of self-excited attractors are shown in Figure. 2.1

Example for Van der Pol Oscillator

Consider the oscillations that occur within an electrical circuit, specifically those generated by the van der Pol(1926) oscillator [Pol26].

$$\ddot{x} + \theta(x^2 - 1) + x = 0,$$

The result was determined when $\theta = 2$.

Example for Belousov–Zhabotinsky Reaction

Belousov observed the first oscillations in chemical processes in a liquid phase in 1951 [Bel58]. Consider one of the dynamic models developed by BelousovZhabotinsky.

$$\eta\dot{x} = x(1-x) + \frac{\lambda(\theta - x)}{\theta + x}\dot{z},$$
$$\dot{z} = x - z,$$

Then execute a computer simulation using typical parameters: $\lambda = \frac{2}{3}$, $\eta = 4 \times 10^{-2}$, $\theta = 8 \times 10^{-4}$. The existence of chaotic oscillations was discovered numerically in the

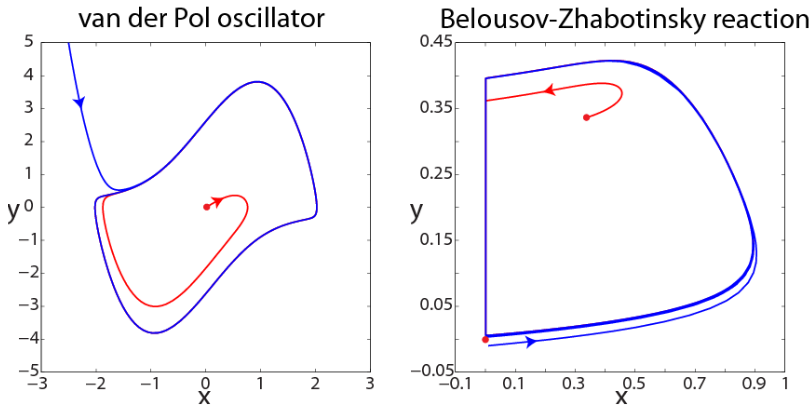


Figure 2.1: Standard computation of classical self-excited chaotic attractors.

middle of the twentieth century (Ueda et al., 1973; Lorenz, 1963), which were also excited from unstable equilibria and could be calculated using ordinary computing procedures. Many additional famous self-excited attractors were discovered later (Rössler 1976; Chua et al. 1986; Sprott 1994; Chen and Ueta 1999; Lu and Chen 2002). There are already an enormous number of articles devoted to the computation and study of various self-excited chaotic oscillations (see, for example, recent publications Zelinka et al. [Zel+13], Zhang et al. [Zha+14] and many others). the computation of classical self-excited chaotic attractors: Lorenz system, system (Rossler, 1976), and “double-scroll” attractor in Chua’s circuit is shown in Figure.2.2.

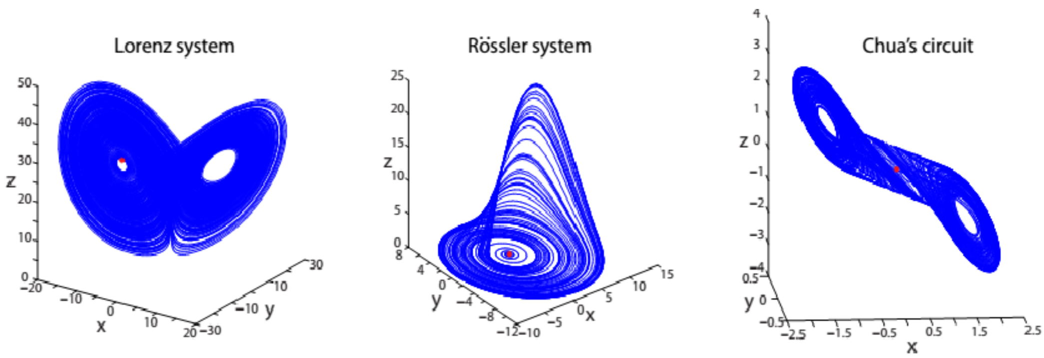


Figure 2.2: Numerical localization of the chaotic attractor in the Lorenz, Rössler and Chua system

2.3 Hidden Attractors

Further research revealed that the self-excited periodic and chaotic oscillations did not provide comprehensive information on the different kinds of oscillations. Thus, the hidden attractors cannot be computed by using this standard procedure. Hidden attractors are attractors in systems with no equilibria or only stable equilibrium (a specific case of multistable systems and the coexistence of attractors) in the middle of the twentieth century. Examples of periodic and chaotic oscillations of a different sort were discovered, then called [Leonov et al. [LK11a]] Hidden oscillations and hidden attractors, in which the basin of attraction does not connect with small areas of equilibria.
The topic of analyzing hidden oscillations initially appeared in the second half of Hilbert's 16th problem (1900) on the number and possible placements of limit cycles in two-dimensional polynomial systems, as well as in the authors' publications [LKV11b] [LK13b]). Even for a basic class of quadratic systems, the problem is far from solved.
Later, in engineering difficulties, the difficulty of analyzing concealed oscillations developed. The examination of Aizerman's and Kalman's conjectures on absolute stability in the previous century resulted in the discovery of hidden oscillations in automated control systems with a unique stable stationary point and a nonlinearity that corresponds to the sector of linear stability (see, for example, the works [KLV10] [LK13b]) .
Further investigations on hidden oscillations were greatly encouraged by the present authors' discovery ([KLV10][LK13b][Leo+14] and [LK11c])

▶ **Definition 2.2.** hidden attractors have a basin of attraction that does not contain neighborhoods of equilibria [LKV11b]. ◀

Hidden attractors are important in engineering applications because it can explain perturbations in a a structure like a bridge or an aircraft wing, convective fluid motion in a rotating cavity [LKM15] and a model of a drilling system actuated by an induction motor [Leo+14].

In 2009-2010 (for the first time), a chaotic hidden attractor was discovered in Chua's circuits [Mat84b] and[Chu93], which is the most basic electrical circuit that exhibits chaos. Consider one of the basic Chua circuits shown in Figure.2.3 [Kis+18] [Kuz+23].

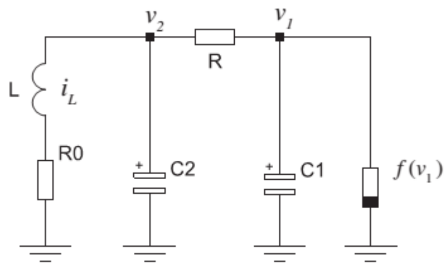


Figure 2.3: Chua Circuit

The circuit consists of passive resistors (R and R_0), capacitors (C_1 and C_2), conductor L, and one nonlinear element with characteristics $f(.)$, called Chua diode. Figure. 2.3 is transformed and represented by the following form below:

$$\begin{cases} \dot{x} &= \alpha(y - x - f(x)), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y - \gamma z, \end{cases} \tag{2.1}$$

where the function

$$f(x) = \eta_1 x + \frac{1}{2}(\eta_0 - \eta_1)(|x+1| - |x-1|). \tag{2.2}$$

For the simulation of this system, we use the following parameters: $\alpha = 9.35$, $\beta = 14.79$, $\gamma = 0.016$, $\eta_0 = -1.1384$, $\eta_1 = 0.7225$.(see figure (2.4))

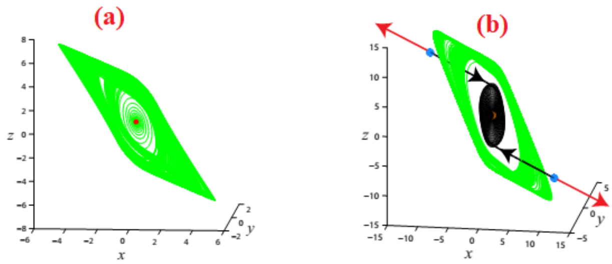


Figure 2.4: (a) Self-excited, (b) Hidden Chua attractor with similar shapes

Nowadays, Chua's attractor is widely used, due to both its realizations: electronic circuit or its mathematical model

2.4 An Analytical-Numerical Method for Searching Hidden Attractor Localization

Leonov [Leo09] and Leonov et al. [LVK10], [LK13a] suggested the method for searching attractors of multidimensional nonlinear dynamical systems with scalar nonlinearity.

Their method is based on numerical continuation: a sequence of linked systems is constructed such that for the first (starting) system the initial data for numerical computation of possible oscillating solution (starting oscillation) can be obtained analytically and the transformation of this starting solution oscillation when is passing from one system to another is followed numerically. This suggested approach is generalized in [LKV11a], [LK11b], [LKV12a] to the system of the form

$$\dot{U} = MU + qH(r^T U), \quad U \in \mathbb{R}^n \tag{2.3}$$

where M is a constant $n \times n$-matrix, $H(\sigma)$ is a continuous piecewise differentiable function satisfying the condition $H(0) = 0$, q, r are constant $n-$dimensional vectors and ^{T} denote transpose operation.

We present here their method in the simplified case $n = 3$. Therefore we consider the equation

$$\dot{U} = MU + qH(r^T U), \quad U \in \mathbb{R}^3 \tag{2.4}$$

where $H(\sigma)$ is a continuous nonlinear function.

They then define a coefficient of harmonic linearization χ (suppose that such χ exists) in such a way that the matrix

$$M_0 = M + \chi q r^T \tag{2.5}$$

of the linear system

$$\dot{U} = MU \tag{2.6}$$

has a pair of purely imaginary eigenvalues $\pm i\omega_0$, ($\omega_0 > 0$) and the other eigenvalue has negative real part. In practice, to determine χ and ω_0 they use the transfer function $W(m)$ of system(2.3)

$$W(m) = r(M - mI)^{-1} q \tag{2.7}$$

where m is a complex variable and I is a unit matrix. The number $\omega_0 > 0$ is determined from the equation $\operatorname{Im} W(i\omega_0) = 0$ and χ is calculated by the formula

$\chi = -\operatorname{Re} W(i\omega_0)^{-1}$.

Therefore, system (2.3) can rewriten as

$$\dot{U} = M_0 U + qg(r^T U), \quad U \in \mathbb{R}^3 \tag{2.8}$$

where $g(\sigma) = H(\sigma) - \chi\sigma$.

Following that, they introduce a finite sequence of continous functions $g^0(\tau)$, $g^1(\tau), ..., g^m(\tau)$ in such a way that the graphs of neighboring functions $g^j(\tau)$ and $g^{j+1}(\tau)$, $(j = 0, ..., m-1)$ in a sense, slightly differ from each other, the function $g^0(\tau)$ is small and $g^m(\tau) = g(\tau)$. The smallness of function $g^0(\tau)$, allows to apply the method of harmonic linearization (describing function method) to the system

$$\dot{U} = M_0 U + qg^0(r^T U), \quad U \in \mathbb{R}^3 \tag{2.9}$$

if the stable periodic solution $U^0(t)$ close to harmonic one is determined. Then for the localization of an attractor of the original system (2.8), one can follow numerically the transformation of this periodic solution (a starting oscillating is an attractor, not including equilibria, denoted further by A_0) simply increasing j.

By nonsingular linear transformation S $(U = SZ)$ the system (2.9) can be reduced to the form

$$\begin{cases} \dot{z}_1(t) &= -\omega_0 z_2(t) + b_1 g^0(z_1(t) + c_3^T z_3(t)) \\ \dot{z}_2(t) &= \omega_0 z_1(t) + b_2 g^0(z_1(t) + c_3^T z_3(t)) \\ \dot{z}_3(t) &= a_3 z_3(t) + b_3 g^0(z_1(t) + c_3^T z_3(t)) \end{cases} \tag{2.10}$$

where $z_1(t)$, $z_2(t)$, $z_3(t)$ are scalar values, a_3, b_1, b_2, b_3, c_3 are real numbers and $a_3 < 0$.

The describing function G is defined as

$$G(\tau) = \int_0^{\frac{2\pi}{\omega_0}} g(\cos(\omega_0 t)\tau)\cos(\omega_0 \tau)dt. \tag{2.11}$$

▶ **Theorem 2.3.** [LVK10] If it can be found a positive τ_0 such that

$$G(\tau_0) = 0, \quad b_1 \frac{dG(\tau)}{d\tau} \Big|_{\tau=\tau_0} < 0.$$

has a stable periodic solution with initial data $U^0(0) = S(z_1(0), z_2(0), z_3(0))^T$ at the initial step of algorithm one has $z_1(0) = \tau_0 + O(\varepsilon)$, $z_2(0) = 0$, $z_3(0) = O_{n-2}(\varepsilon)$, where $O_{n-2}(\varepsilon)$ is an $(n-2)$-dimensional vector such that all its components are $O(\varepsilon)$. ◀

2.5 Example for Hidden Chaotic Attractors in Chua's System Circuit

Attractors in classical Chua's theory are those that are excited by unstable equilibria [Chu93], [Chu95] [BP08]. But recently, A hidden chaotic attractor was discovered for the first time in Chua's circuit in 2010 by Kuznetsov et al [KLV10] [LKV11b] characterized by a three-dimensional dynamical system. The authors utilized the above approach to find hidden oscillations. In the following section, we will show how to use the above-mentioned approach to locate hidden chaotic attractors in Chua's system. For this purpose, write Chua's system (2.1-2.2) in the form (2.4)

$$\frac{dU}{dt} = MU + qH(r^T U), \qquad U \in \mathbb{R}^3. \tag{2.12}$$

Here,

$$M = \begin{pmatrix} -\alpha(\eta_1 + 1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix}, \quad q = \begin{pmatrix} -\alpha \\ 0 \\ 0 \end{pmatrix}, \quad r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and $H(\sigma) = f(\sigma)$.

Introduce the coeffïcent χ and small parameter ε, and represent system (2.12) as

$$\frac{dU}{dt} = M_0 U + q\varepsilon\varphi(r^T U), \tag{2.13}$$

where

$$M_0 = M + \chi q r^T = \begin{pmatrix} -\alpha(\eta_1 + 1 + \chi) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix}, \quad \lambda_{1,2}^{M_0} = \pm i\omega_0, \quad \lambda_3^{M_0} = -d,$$

$g(\sigma) = H(\sigma) - \chi\sigma$.

By nonsingular linear transformation $U = SZ$ system (2.13) is compressed into the form

$$\frac{dZ}{dt} = AZ + B\varepsilon g(C^T Z), \tag{2.14}$$

where

$$A = \begin{pmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & -d \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ 1 \end{pmatrix}, \quad Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \text{and} \quad C = \begin{pmatrix} 1 \\ 0 \\ -h \end{pmatrix}.$$

The transfer function$W_A(m)$ of system (2.14) can be represented as

$$W_A(m) = \frac{-b_1 m + b_2\omega_0}{m^2 + \omega_0^2} + \frac{h}{m+d}.$$

Further, using the equality of transfer functions of systems (2.13) and (2.14), we obtain

$$W_A(m) = q^T(M_0 - mI)^{-1} r.$$

This implies the relations indicated below:

$$\begin{aligned} \chi &= \tfrac{-\alpha(\eta_1+\eta_1\gamma+\gamma)+\omega_0^2-\gamma-\beta}{\alpha(1+\gamma)}, \\ d &= \tfrac{\alpha+\omega_0^2-\beta+1+\gamma+\gamma^2}{1+\gamma}, \\ h &= \tfrac{\alpha(\gamma+\beta-(1+\gamma)d+d^2)}{\omega_0^2+d^2}, \\ b_1 &= \tfrac{\alpha(\gamma+\beta-\omega^2-(1+\gamma)d)}{\omega_0^2+d^2}, \\ b_2 &= \tfrac{\alpha((\gamma+\beta)d+(1+\gamma-d)\omega_0^2)}{\omega_0\left(\omega_0^2+d^2\right)}. \end{aligned} \tag{2.15}$$

Since system (2.13) can be reduced to the form (2.14) by the nonsingular linear transformation $U = SZ$, for the matrix S the following relations

$$A = S^{-1}MS, B = S^{-1}q, \ C^T = r^T S, \tag{2.16}$$

are valid. The entries of this matrix are obtained by solving these matrix equations:

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & ZS_{33} \end{pmatrix}.$$

where

$$S_{11} = 1, \quad S_{12} = 0, \quad S_{13} = -h,$$

$$S_{21} = \eta_1 + 1 + \chi, \quad S_{22} = \frac{-\omega_0}{\alpha}, \quad S_{23} = -\frac{h(\alpha(\eta_1 + 1 + \chi) - d}{\alpha},$$
$$S_{31} = \frac{\alpha(\eta_1 + \chi) - \omega_0^2}{\alpha}, \quad S_{32} = \frac{\alpha(\beta + \gamma)(\eta_1 + \chi) + \alpha\beta + -\omega_0^2}{\alpha\omega_0},$$
$$S_{33} = h\frac{\alpha(\eta_1 + \chi)(d - 1) + d(1 + \alpha - d)}{\alpha}.$$

We determine initial data for the first step of a multistage localization procedure for small enough ε, as

$$U(0) = SZ(0) = \begin{pmatrix} \tau_0 S_{11} \\ \tau_0 S_{21} \\ \tau_0 S_{31} \end{pmatrix}. \tag{2.17}$$

The starting condition for the system (2.1-2.2) is provided by this:

$$U(0) = (x(0) = \tau_0,\ y(0) = \tau_0(1 + \eta_1 + \chi),\ \ z(0) = \tau_0\frac{\alpha(1 + \eta_1) - \omega_0^2}{\alpha}). \tag{2.18}$$

Consider system (2.13) with the parameters

$$\alpha = 8.4562,\ \beta = 12.0732,\ \gamma = 0.0052,\ \eta_0 = -0.1768, \eta_1 = -1.1468. \tag{2.19}$$

There are three equilibria in the system for the parameter values under consideration: a locally stable zero equilibrium and two saddle equilibria. Let's now employ the hidden attractor localization process described above to Chua's system (2.12) with parameters (2.19). Calculate a beginning frequency and a harmonic linearization coefficient for this.

$$\omega_0 = 2.0392, \quad \chi = 0.2098. \tag{2.20}$$

Then, we compute solutions of system (2.13) with the nonlinearity $\varepsilon(H(x) - \chi x)$sequentially increasing ε from the value $\varepsilon_1 = 0.1$ to $\varepsilon_{10} = 1$ with step 0.1. By (2.15) and (2.18), the initial data can be obtained

$$x(0) = 9.4287, \quad y(0) = 0.5945, \ z(0) = -13.4705,$$

for the initial phase of a multi-stage process. For $\varepsilon = 0.1$, the computation approaches the beginning oscillation $U^1(t)$ following a transitory process. Additionally, the set of hidden is calculated for the original Chua's system (2.12) using numerical methods and the sequential transformation $U^j(t)$ with increasing pa-

rameter ε_j, the set A_{hidden} is computed for the original Chua's system.
This set is seen in Figure (2.5): two symmetric hidden chaotic attractors (blue), trajectories (red), $M^{ns}_{\pm}$ of two saddle points $S_{\pm}$ are either attracted to locally stable zero equilibrium F_0 or tend to infinity; trajectories (black) from stable manifolds $M^{st}_{\pm}$ end to F_0 or $S_{\pm}$. As a result, stable separation manifolds of saddles exist in the system's phase space.

The previous results, combined with the observation that a locally stable zero equilibrium F_0 exists in the system, attracting the stable manifolds $M^{st}_{\pm}$ of two symmetric saddles $S_{\pm}$, lead to the conclusion that in A_{hidden} a hidden strange attractor is computed.

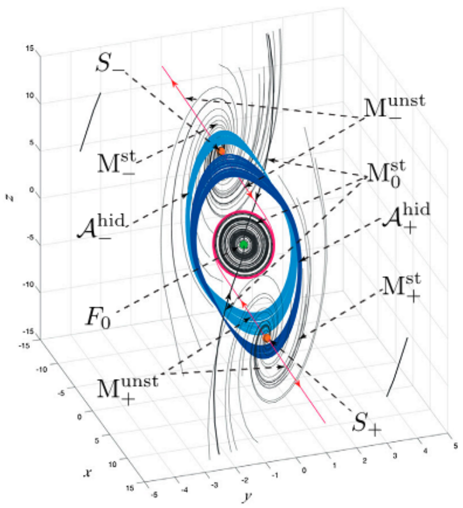


Figure 2.5: Equilibrium, saddles, and the localization of hidden attractors.

► **Remark 2.4.** In the papers [Nat+18][NG20][DWY20][WWH21], we can discover hidden attractors for different chaotic systems. ◄

Part II

Main Contribution

3 A Technique Based on Integration Duration to Uncover Hidden Spiral Patterns in the Chua Chaotic Attractor

“***The essence of mathematics is not to make simple things complicated but to make complicated things simple.***”

S.Gudder

3.1 Introduction

This part is divided into two chapters, the first presents a novel method for unveiling hidden patterns of an even number of spirals in the multispiral Chua Chaotic attractor. This method based on the duration of integration of this system, allows displaying every pattern containing from 1 *to* $c + 1$ *spirals of this attractor. After having given the equation of the multispiral Chua Chaotic attractor, the Menacer-Lozi-Chua (MLC) method for uncovering hidden bifurcations is explained again and a numerical example of the route of bifurcation is given. In the chosen example, it is shown that the attractors found along this hidden bifurcation route display an odd number of spirals. With the novel method, during the integration process, before reaching the asymptotical attractor which possesses and odd the number of spirals, the number of spirals increases one by one until it reaches the maximum number corresponding to the value fixed by* ε, *unveiling both patterns of even and odd number of spirals. Joint work with T. Menacer and R. Lozi [BML19]*

3.2 Multiple-spiral Attractors in Chua's Sine Function System

Complex chaotic attractors with n-double scrolls using cellular neural networks with a piecewise-linear output function were highlighted in [HSD03]. Another simpler mechanism for generating n-scroll attractors was introduced by Tang in [Tan+01], using sine or cosine functions. Since then, in the last decades, multi-scroll chaotic attractors generation has been extensively studied due to their promising applications in various real-world chaos-based technologies including secure and digital communications, random bit generation, etc.

The system of differential equations, describing the behavior of Chua's circuits, that we consider in this article is three-dimensional with a combination of piecewise-linear and sinusoidal nonlinearity [CHU92], [Tan+01], [Mad93](see **Fig. 3.1**)

$$\begin{cases} \dot{x}(t) &=& \alpha(y(t) - f(x(t))), \\ \dot{y}(t) &=& x(t) - y(t) + z(t), \\ \dot{z}(t) &=& -\beta y(t), \end{cases} \tag{3.1}$$

where $\dot{x}(t) = \frac{dx(t)}{dt}$, $\dot{y}(t) = \frac{dy(t)}{dt}$, $\dot{z}(t) = \frac{dz(t)}{dt}$.

$$f(x(t)) = \begin{cases} \frac{b\pi}{2a}(x(t) - 2ac) & \text{if} \quad x(t) \geq 2ac, \\ -b\sin(\frac{\pi x(t)}{2a} + d) & \text{if} \quad -2ac < x(t) < 2ac, \\ \frac{b\pi}{2a}(x(t) + 2ac) & \text{if} \quad x(t) \leq -2ac, \end{cases} \tag{3.2}$$

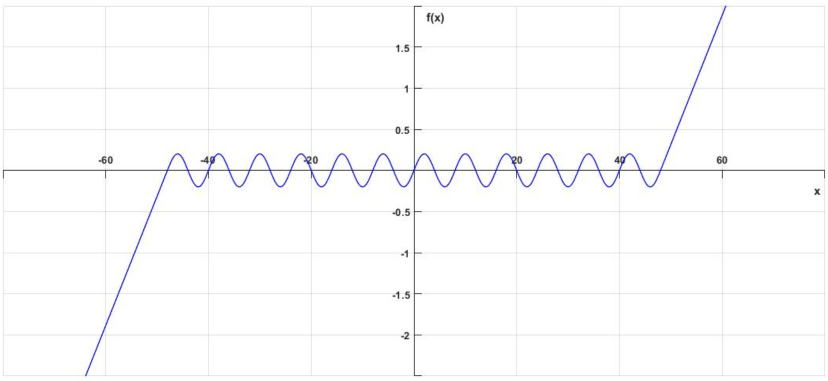


Figure 3.1: Proposed sine function $f(x)$ with parameters values $a = 2$, $b = 0.2$, $c = 12$, $d = \pi$.

The goal of this part is to analyze the general shape of the attractors and their global geometric features, which can be described in terms of the number of spirals. In this chapter, the real parameter values have been fixed to $\alpha = 11$, $\beta = 15$, $a = 2$, $b = 0.2$ for which topologically chaotic attractors have been found. They are equivalent to the attractors found in electronic devices in [Tan+01].

The parameter c is an integer which governs the number n of spirals according to the following formula

$$n = c + 1 \tag{3.3}$$

and d is chosen such that

$$d = \begin{cases} \pi \text{ if } c \text{ is even.} \\ 0 \text{ if } c \text{ is odd.} \end{cases} \quad (3.4)$$

A straightforward computation gives the equilibrium points of (3.1) which are $(-x_{eq}, 0, x_{eq})$ with $x_{eq} = 2ak$, $k = 0, \pm1, \pm2, ..., \pm c$ [Tan+01]. In the case $c = 12$, one obtains 13 spirals attractor (see **Fig. 3.2**). For a complete study of this model with other values of c see [MLC16].

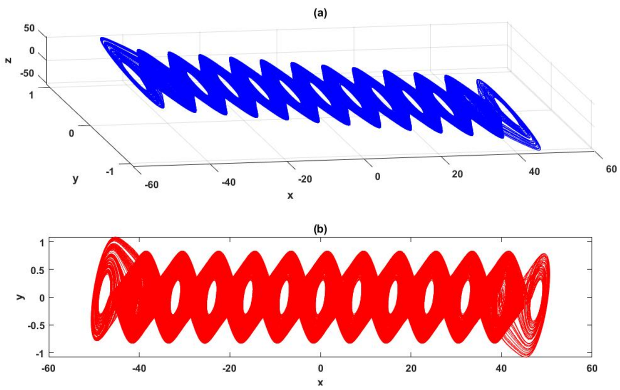


Figure 3.2: The 13-spiral attractor generated by Equation (3.1) and (3.2) for $c = 12$. (a) 3−dimensional figure, (b) Projection into the plane $(x - y)$.

In order to compare the new integration duration-based method versus the method previously introduced by Menacer et al. [MLC16] we need a benchmark sequence of hidden bifurcations. In this goal, we limit our study to the case where parameter c is set to the value 12 which gives a 13 spirals attractor (**Fig. 3.2**). This number of scrolls is enough to obtain a significant comparison.

3.3 The MLC Method for Uncovering Hidden Bifurcations

The method introduced by Menacer et al. [MLC16] to find hidden bifurcations is based on the core idea of Leonov and Kuznetsov for searching hidden attractors (i.e. homotopy and numerical continuation (see the **section (2.4)**). While keeping c constant, a new bifurcation parameter ε is introduced. This method is briefly recalled and applied to (3.1)-(3.2) in this section. The value of parameters are fixed to $\alpha = 11$, $\beta = 15$, $a = 2$, $b = 0.2$, $c = 12$, $d = \pi$.

3.3.1 The Method

The system(3.1)-(3.2) is rewritten in the Lure's form [Lur57]

$$\dot{U} = MU + qH(r^T U), \quad U = (x, y, z) \in \mathbb{R}^3. \tag{3.5}$$

where

$$M = \begin{pmatrix} 0 & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}, \quad q = \begin{pmatrix} -\alpha \\ 0 \\ 0 \end{pmatrix}, \quad r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \text{ and } H(\sigma) = f(\sigma).$$

In order to transform system (3.5) into the form similar to the system (2.8, see in the **section (2.4)**), a new coefficient χ, and a small parameter ε, are introduced as follows

$$\dot{U} = M_0 U + q\varepsilon g^0(r^T U), \quad U = (x, y, z) \in \mathbb{R}^3. \tag{3.6}$$

where

$$M_0 = M + \chi q r^T = \begin{pmatrix} -\alpha\chi & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix},$$

and

$$g(\sigma) = H(\sigma) - \chi\sigma = -\alpha(f(\sigma) - \chi\sigma).$$

Therefore, (3.6) is written as

$$\begin{cases} \dot{x}(t) &= -\alpha(\chi x(t) - y(t)) + \varepsilon g(x(t)), \\ \dot{y}(t) &= x(t) - y(t) + z(t), \\ \dot{z}(t) &= -\beta y(t). \end{cases} \tag{3.7}$$

Then considering the transfer function $W(m)$ (2.7, see in the **section (2.4)**) and solving the equations $\operatorname{Im} W(i\omega_0) = 0$ and $\chi = -\operatorname{Re} W(i\omega_0)^{-1}$, one obtains $\omega_0 = 2.1018$ and $\chi = 0.03796$.

Using the nonsingular linear transformation $U = SZ$ defined in the **Section (2.4)**, the system (3.7) is reduced to the form similar to equation (2.9)

$$\dot{Z} = AZ + B\varepsilon g^0(C^T Z), \quad Z = (z_1, z_2, z_3) \in \mathbb{R}^3 \tag{3.8}$$

where

$$A = \begin{pmatrix} 0 & -\omega_0 & 0 \\ -\omega_0 & 0 & 0 \\ 0 & 0 & -d_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 0 \\ -h \end{pmatrix}$$

The transfer function $W_A(m)$ of system (3.8) reads

$$W_A(m) = \frac{-b_1 m + b_2 \omega_0}{m^2 + \omega_0^2} + \frac{h}{m + d_1}.$$

Then, using the equality of transfer of both functions $W_A(m)$ and $W_{M_0}(m) = r^T(M_0 - mI)^{-1}q$ to systems (??) and (3.8) one obtains the following relations

$$\begin{aligned} \chi &= \frac{a + \omega_0^2 - \beta}{\alpha}? \\ d_1 &= \alpha + \omega_0^2 - \beta + 1? \\ h &= \frac{\alpha(\beta - d_1 + d_1^2)}{\omega_0^2 + d_1^2}? \\ b_1 &= \frac{\alpha(\beta - \omega_0^2 - d_1)}{\omega_0^2 + d_1^2}? \\ b_2 &= \frac{\alpha(1 - d_1)(\omega_0 - \beta d_1)}{\omega_0^2(\omega_0^2 + d_1^2)}. \end{aligned}$$

▶ **Remark 3.1.** The matrix S is defined by the following relations easily computed

$$\begin{aligned} A = S^{-1}M_0 S, \quad B = S^{-1}q, \quad C^T = r^T S, \\ S = \begin{pmatrix} 1 & 0 & -h \\ \chi & \frac{-\omega_0}{-\alpha} & \frac{hd - h\alpha\chi}{\alpha} \\ \frac{-\beta}{\alpha} & \frac{\chi\beta}{\omega_0} & \frac{\beta h(d - \alpha\chi)}{d\alpha} \end{pmatrix}. \end{aligned} \tag{3.9}$$

◀

3.3.2 Numerical Example of Hidden Bifurcation Route

In this section we present a numerical example of hidden bifurcations route. For the values of parameters fixed in this section, we obtain

$$\chi = 0.03796, \quad d_1 = 1.4176, \quad h = 26.686, \quad b_1 = 15.686, \quad b_2 = 3.1003,$$

therefore, the matrix (3.9), is $S = \begin{pmatrix} 1 & 0 & -26.686 \\ 0.03795 & -0.19107 & -2.4264 \\ -1.3636 & -0.27084 & -25.674 \end{pmatrix}$.

Using *theorem* 1 of the **Section (2.4)**, for ε small enough, one obtains the initial condition

$$U^0(0) = SZ(0) = S\begin{pmatrix} \tau_0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \tau_0 s_{11} \\ \tau_0 s_{21} \\ \tau_0 s_{31} \end{pmatrix}, U = (x, y, z) \in \mathbb{R}^3 \tag{3.10}$$

Using the notation of Sec. 3.2, one obtains for the determination of the initial condition of starting solution for the multistage localization procedure, Chua system

$$x(0) = \tau_0, \quad y(0) = \tau_0 \chi, \quad z(0) = -\tau_0 \frac{\beta}{\alpha}, \tag{3.11}$$

Then, the localization procedure described in the **Sec.2.4** is applied to the system (3.5)-(3.7). The starting frequency ω_0 and the coefficient of harmonic linearization χ have been already computed in Sec. 3.3.1. Equation(3.11), allows to obtain the initial conditions for the first step.

Then, the solutions of system (3.7) with the nonlinearity $\varepsilon g(x) = \varepsilon(H(x) - \chi x)$ are computed, by increasing sequentially ε from the value $\varepsilon = 0.1$ to $\varepsilon = 1$, with step size 0.1, except between $\varepsilon = 0.9$ and $\varepsilon = 1$, where one uses 0.001 as increasing step.

All the points of the stable periodic solution $U^1(t)$ corresponding-to $\varepsilon = 0.1$ belong to the domain of attraction of the stable periodic solution $U^2(t)$ corresponding to $\varepsilon = 0.2$. This stable periodic solution $U^2(t)$ is then simply obtained numerically by solving system (3.5) with $\varepsilon = 0.2$ and $U^1(t_{\max})$ as initial point, where $t_{\max}$ represents the last value of time of integration after discarding the transitory regime. The same numerical procedure is reitered by increasing the ε-value, to obtain the next periodic solutions $U^3(t), U^4(t), ..., U^j(t), ...$ corresponding to the next values of ε. When $\varepsilon = 0.8$, the first chaotic solution one scroll is found.

The initial conditions for recovering the solutions for increasing values of ε as shown in the **Table 3.1.**

Table 3.1: Initials conditions according to the values of ε

ε	$X^j(0)$	x_0	y_0	z_0
0.1	$U^1(0) = U^0(t_{\max})$	38.3269	-0.0257	-37.9339
0.2	$U^2(0) = U^1(t_{\max})$	-3.7487	0.1619	-5.0818
0.3	$U^3(0) = U^2(t_{\max})$	3.3406	-0.2062	-4.9772
0.4	$U^4(0) = U^3(t_{\max})$	-3.6009	-0.2981	4.7285
0.5	$U^5(0) = U^4(t_{\max})$	3.4797	-0.1484	-5.0577
0.6	$U^6(0) = U^5(t_{\max})$	3.6919	0.0202	-5.1617
0.7	$U^7(0) = U^6(t_{\max})$	4.1512	-0.0232	-5.5316
0.8	$U^8(0) = U^7(t_{\max})$	1.3915	-0.5758	-2.6930
0.86	$U^9(0) = U^8(t_{\max})$	-0.4213	-0.3265	0.2085
0.9785	$U^{10}(0) = U^9(t_{\max})$	1.8233	0.2554	-2.6074
0.989	$U^{11}(0) = U^{10}(t_{\max})$	1.4667	0.7801	-0.6740
0.993	$U^{12}(0) = U^{11}(t_{\max})$	3.2919	0.5697	-3.9526
0.9953	$U^{13}(0) = U^{12}(t_{\max})$	15.5612	-0.3315	-16.0715
0.9994	$U^{14}(0) = U^{13}(t_{\max})$	15.5612	-0.3315	-16.0715
1	$U^{15}(0) = U^{14}(t_{\max})$	-33.7244	-0.0092	35.1636

Using these initial conditions, we get the solutions $U^8(t)$ (**Fig. 3.3(a)**) with one spiral to $U^{13}(t)$ (**Fig. 3.4(g)**) with 13 spirals. In each figure, there are a different odd number of spirals in the attractor. The number of spirals increases by 2 at each step as displayed on **Table 3.2** from 1 to 13 spirals. The values of ε in this table are exactly the values of bifurcation points with respect to ε. Therefore **Fig. 3.3** and **Fig. 3.4** display the hidden bifurcations of the multi-spiral Chua's attractor.

Table 3.2: Values of the parameter ε at the bifurcation points for $c = 12$ (13 spirals).

ε						
0.80	0.86	0.9785	0.989	0.993	0.9953	0.9994
$U^8(0)$	$U^9(0)$	$U^{10}(0)$	$U^{11}(0)$	$U^{12}(0)$	$U^{13}(0)$	$U^{14}(0)$
1 spiral	3 spirals	5 spirals	7 spirals	9 spirals	11 spirals	13 spirals

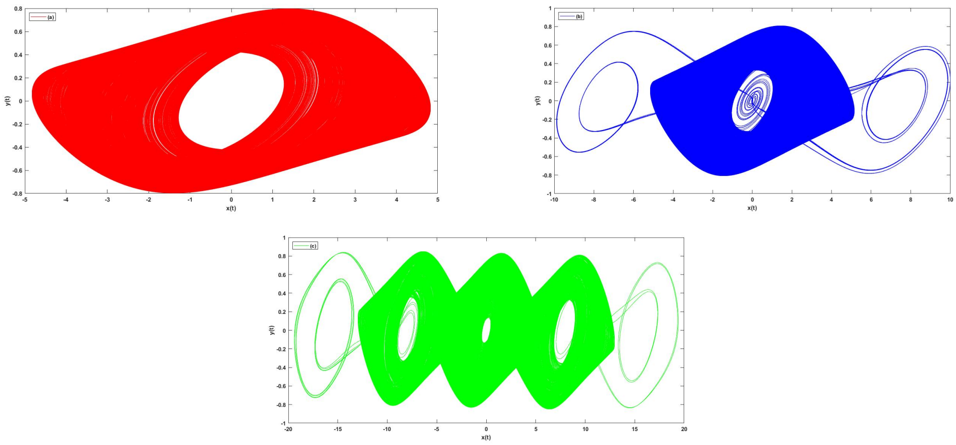


Figure 3.3: The increasing number of spirals of system (3.7) with respect to increasing ε values. (a) 1-spiral for $\varepsilon = 0.80$, (b) 3-spirals for $\varepsilon = 0.86$, (c) 5-spirals for $\varepsilon = 0.9785$.

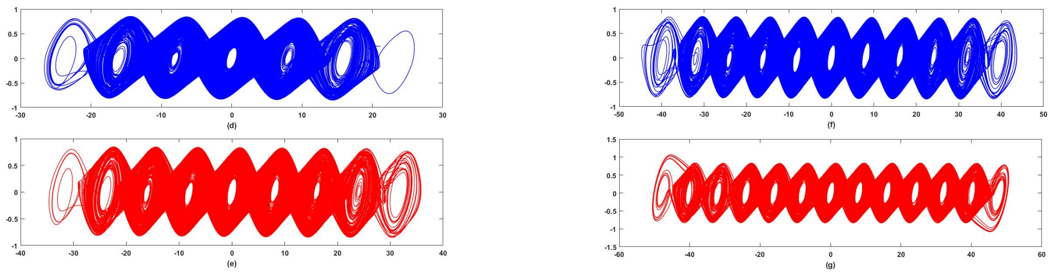


Figure 3.4: (continued) The increasing number of spirals of system (3.7) with respect to increasing ε values. (d) 7-spirals for $\varepsilon = 0.989$, (e) 9-spirals for $\varepsilon = 0.993$, (f) 11-spirals for $\varepsilon = 0.9953$, (g) 13-spirals for $\varepsilon = 0.9994$.

3.4 Integration Duration Method for Unveiling Hidden Patterns of Even Number of Spirals

In the previous section, it is shown that the attractors along the hidden bifurcation route display an odd number of spirals. In this section, we introduce a novel method for unveiling hidden patterns of an even number of spirals.

This method is based on the duration of integration of System (3.7). For this novel method, we fix ε and we increase the duration of integration $t_{\max}$. During the integration process, before reaching the asymptotical attractor which possesses and odd the number of spirals, the number of spirals increases until it reaches the maximum number corresponding to the value fixed by ε.

As a numerical example, we set the value of ε to 0.9953. For this value, 13 spirals belong to the asymptotic attractor (**Fig. 3.4g**).
All the patterns are displayed in **Fig. 3.5 and Fig. 3.6**, and the value of $t_{step\,\max}$ is given in **Table.3.3**. In each figure, the number of spirals increases by 1 (instead of 2 in the MLC method). MATLAB's standard solver for ordinary differential equations ode45 is used in order to integrate Chua's system. This function implements a Runge–Kutta method with a variable time step. Therefore, it is difficult to know $t_{\max}$. The value $t_{step\,\max}$ (shown in **Table.3.3**) which is the maximum number of steps used (i.e. corresponding to the duration of integration).

Table 3.3: Values of $t_{step\,\max}$ for $\varepsilon = 0.9994$ and $c = 12$.

$t_{step\,\max}$	90	900	1450	2220	3480	4000	4480
Number of spirals	1	2	3	4	5	6	7
Figure	(a)	(b)	(c)	(d)	(e)	(f)	(g)
$t_{step\,\max}$	8020	8780	10125	15000	20000	900000	
Number of spirals	8	9	10	11	12	13	
Figure	(h)	(i)	(j)	(k)	(l)	(m)	

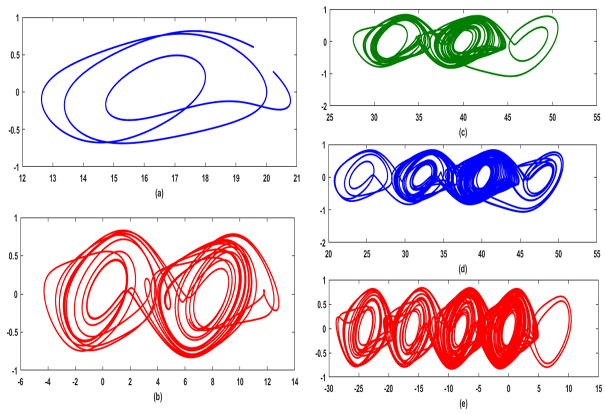


Figure 3.5: The increasing number of spirals for the same values of $\varepsilon = 0.9994$ and various values of $t_{step\,\max}$. (a) 1-spiral for $t_{step\,\max} = 90$,(b) 2-spirals for $t_{step\,\max} = 900$, (c) 3-spirals for $t_{step\,\max} = 1450$, (d) 4-spirals for $t_{step\,\max} = 2220$, (d) 5-spirasl for $t_{step\,\max} = 3480$.

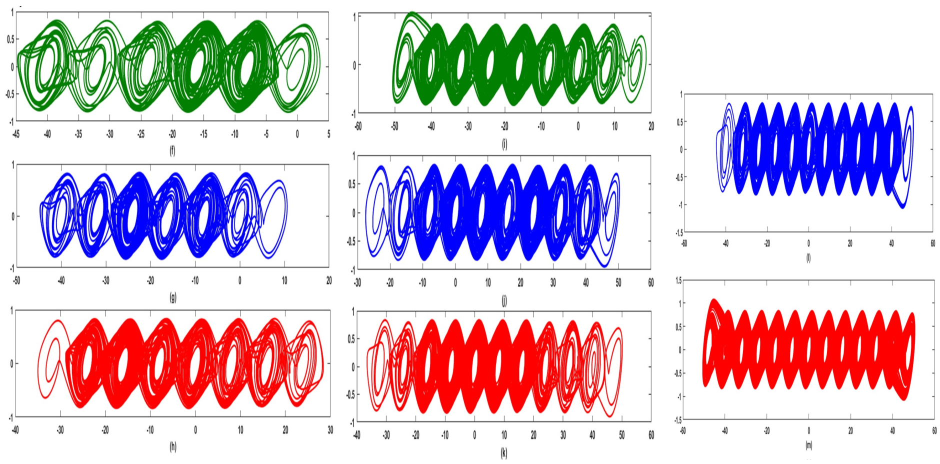


Figure 3.6: (continued) The increasing number of spirals for the same values of $\varepsilon = 0.9994$ and various values of $t_{step\,\max}$. (f) 6-spirals for $t_{step\,\max} = 4000$,(g) 7-spirals for $t_{step\,\max} = 4480$, (h) 8-spirals for $t_{step\,\max} = 8020$, (i) 9-spirals for $t_{step\,\max} = 8780$, (j) 10-spirals for $t_{step\,\max} = 10125$, (k) 11-spirals for $t_{step\,\max} = 15000$, (l) 12-spirals for $t_{step\,\max} = 20000$, (m) 13-spirals for $t_{step\,\max} = 900000$.

In order to study the effect of this integration duration on the basis of this method, we repeat the same procedure for all values of ε of **Tab.** 3.2, we are going to observe the change of scroll number and we resume our results in **Table.**3.7.

$\varepsilon = 0.80$	$t_{stepmax}$	50000				
	Nb of spirals	1				
$\varepsilon = 0.86$	$t_{stepmax}$	7200	19000	41430		
	Nb of spirals	1	2	3		
$\varepsilon = 0.97885$	$t_{stepmax}$	195	680	20000	90000	100000
	Nb of spirals	1	2	3	4	5
$\varepsilon = 0.989$	$t_{stepmax}$	95	350	1000	1380	6000
	Nb of spirals	1	2	3	4	5
	$t_{stepmax}$	80000	90110			
	Nb of spirals	6	7			
$\varepsilon = 0.993$	$t_{stepmax}$	85	2260	2300	2340	9310
	Nb of spirals	1	2	3	4	5
	$t_{stepmax}$	9410	36090	36322	90000	
	Nb of spirals	6	7	8	9	
$\varepsilon = 0.9953$	$t_{stepmax}$	144	2050	2070	2700	6090
	Nb of spirals	1	2	3	4	5
	$t_{stepmax}$	6700	9590	14000	60000	88210
	Nb of spirals	6	7	8	9	10
	$t_{stepmax}$	200000				
	Nb of spirals	11				
$\varepsilon = 0.9994$	$t_{stepmax}$	91	910	1435	2250	3590
	Nb of spirals	1	2	3	4	5
	$t_{stepmax}$	4450	5610	8070	8773	10125
	Nb of spirals	6	7	8	9	10
	$t_{stepmax}$	15000	20000	70000		
	Nb of spirals	11	12	13		

Figure 3.7: Values of $t_{step\,\max}$ for $\varepsilon = 0.80$ to $\varepsilon = 0.9994$ and $c = 12$.

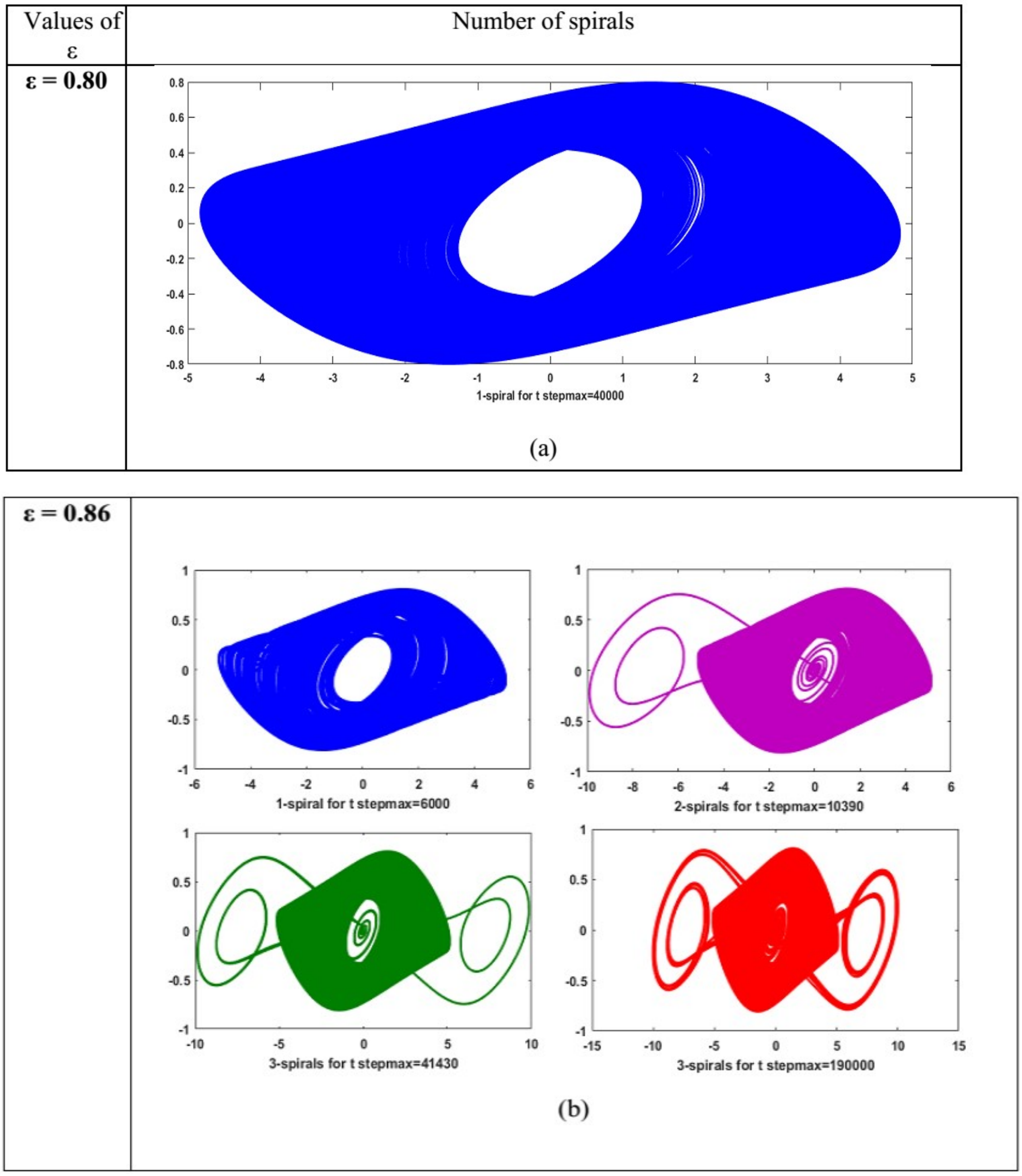


Figure 3.8: The increasing number of spirals for the same values of $\varepsilon = 0.80$ and $\varepsilon = 0.86$ and various values of $t_{step\,\max}$.

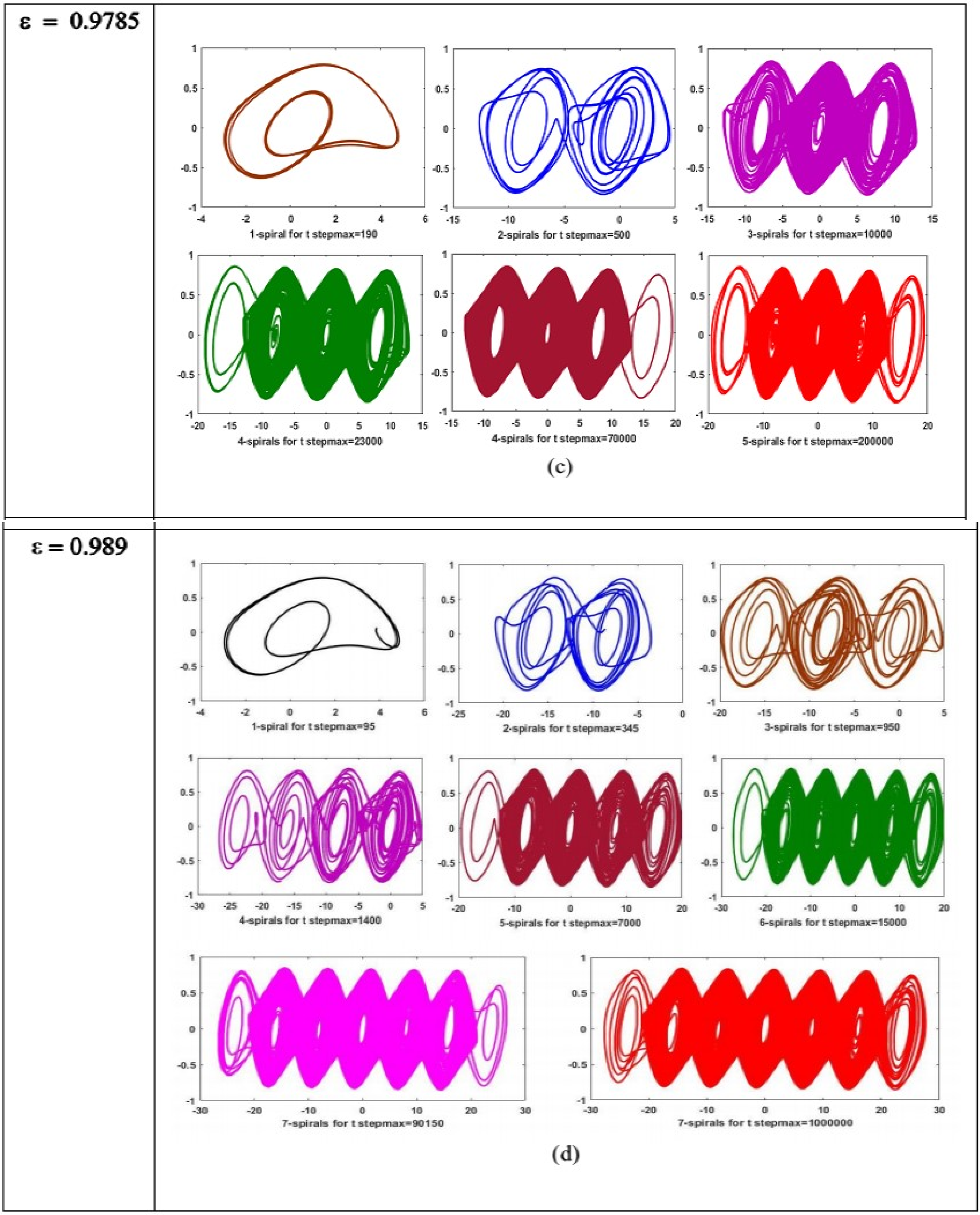


Figure 3.9: The increasing number of spirals for the same values of $\varepsilon = 0.9785$ and $\varepsilon = 0.989$ and various values of $t_{step\,\max}$.

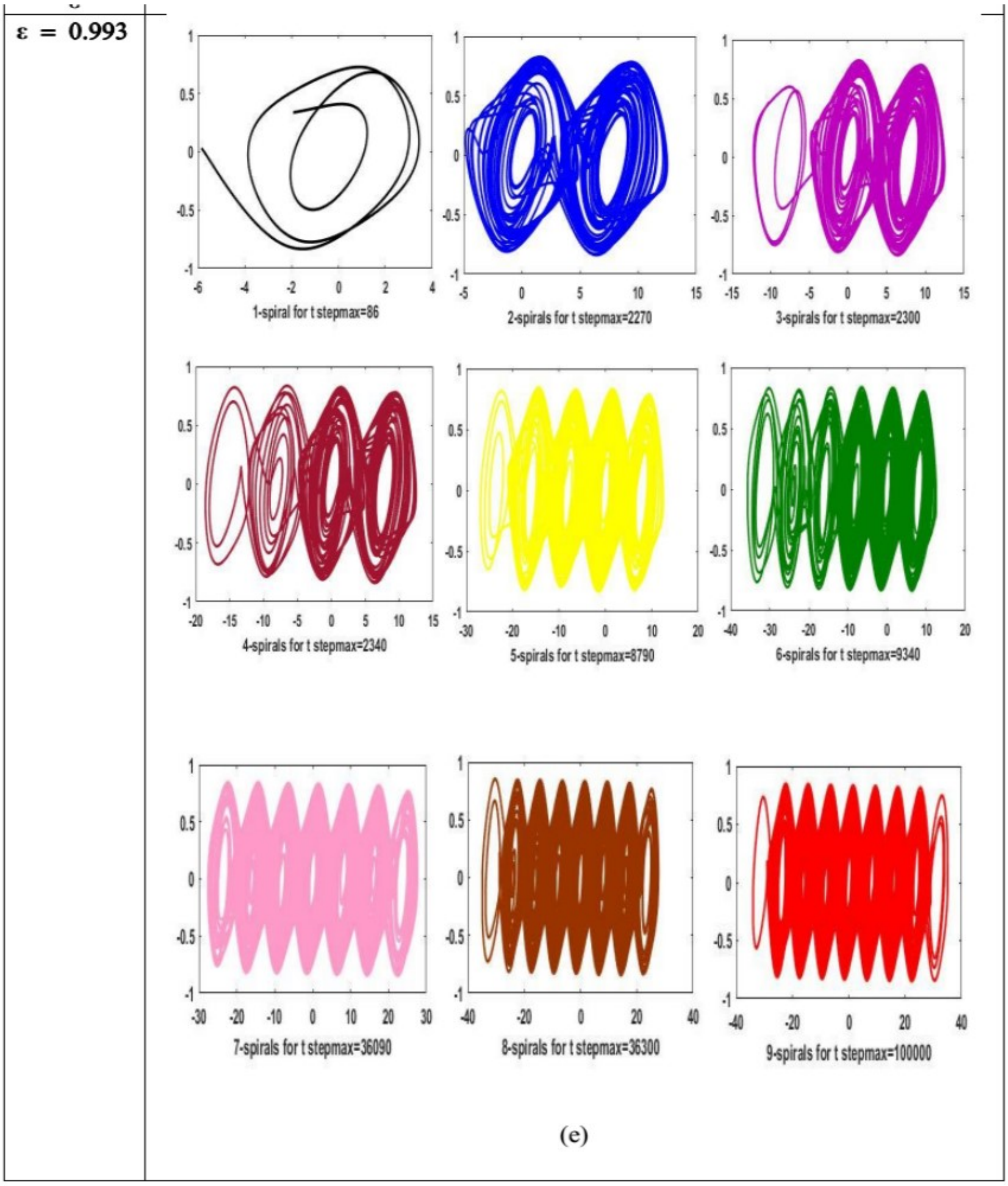
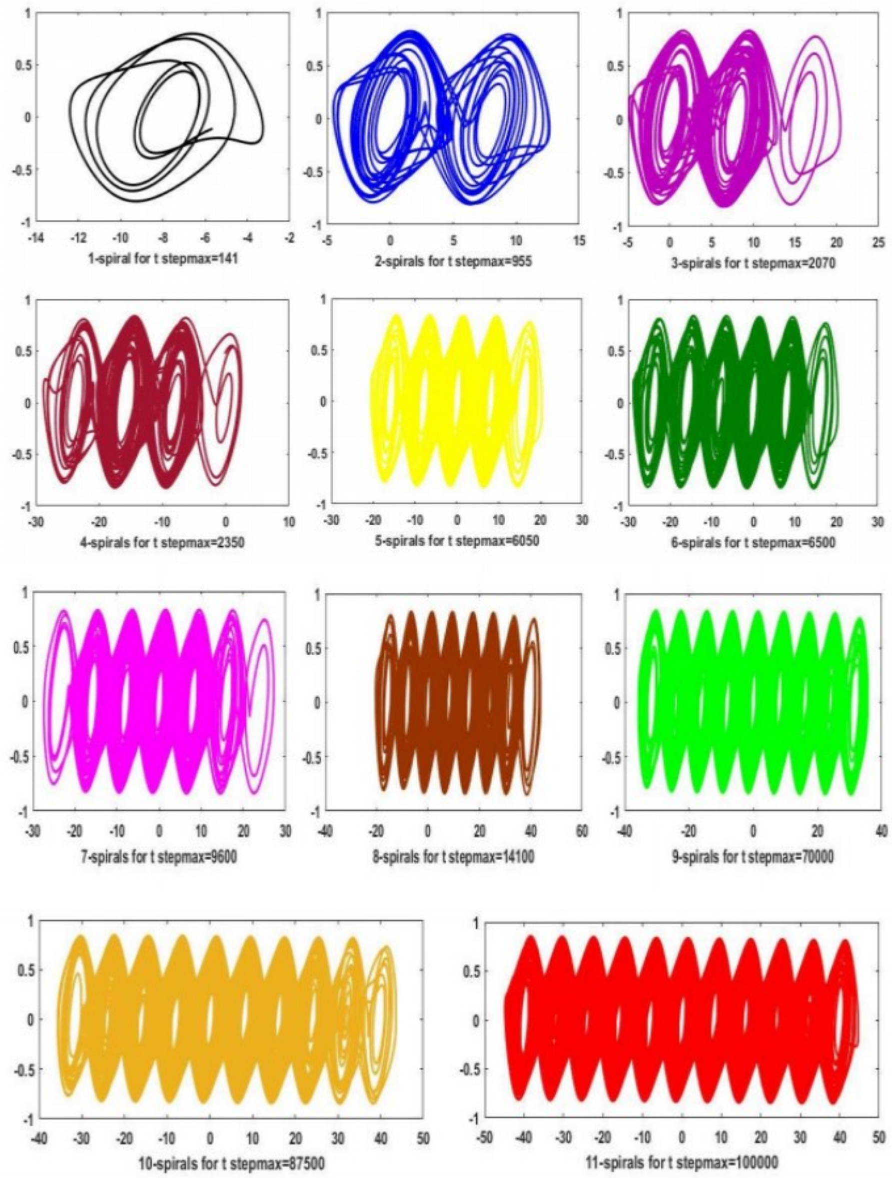


Figure 3.10: (The increasing number of spirals for the same value of $\varepsilon = 0.993$ and various values of $t_{step\,\max}$.

$\varepsilon = 0.9953$



(f)

Figure 3.11: The increasing number of spirals for the same value of $\varepsilon = 0.993$ and various values of $t_{step\,\max}$.

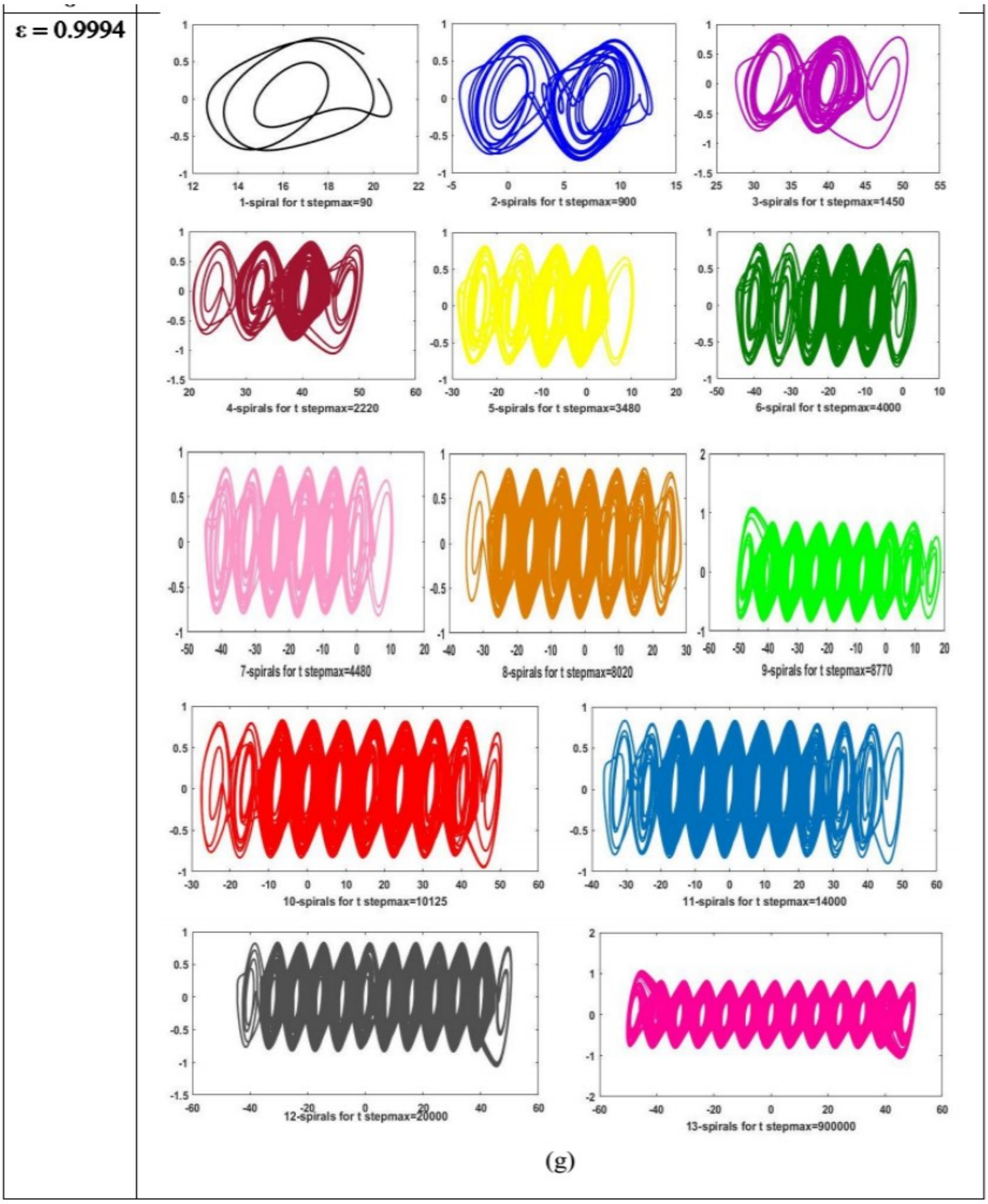


Figure 3.12: The increasing number of spirals for the same value of $\varepsilon = 0.9994$ and various values of $t_{step\,\max}$.

4 Stability, Bifurcations, and Analysis of Control and Synchronization in a Chaotic Attractor of Chua System

"***The only way to learn mathematics is to do mathematics***"

Paul.HALMOS

4.1 Introduction

In the second chapter of this part, we focus on hidden bifurcations of the multispiral Chua Chaotic attractor generated by the sine function. The general shape of the chaotic attractors can be described in terms of the number of spirals (also denoted multiscroll attractor) governed by an integer parameter c. Due to the integer nature of this parameter, it is not possible to observe bifurcations from n to $n+2$ spirals when this parameter is increased by one. However, using the method of hidden bifurcations, an additional real parameter ε allows us to observe such bifurcations. The highlighted routes of bifurcation display chaotic attractors with either an even number or an odd number of spirals. Moreover, this additional hidden parameter allows to find the bifurcation of the multispiral Chua attractor from a stable state to a chaotic state. Furthermore, the Routh-Hurwitz criteria are used to study the stability, control, and synchronization of the original equilibrium point of the Chua attractor. Joint work with T. Menacer and R. Lozi [BML23]

4.2 Stability Of The Origin Equilibrium Point E_0 With Respect To ε

In this section, we study the stability of the equilibrium point E_0 with respect to ε of the system (3.7) using Routh-Hurwitz's conditions [AEE06]. In [MLC16] Menacer et al. introduce the concept of hidden bifurcation in the system of Chua adding a new parameter epsilon, which controls the number of spirals. When the value of ε increases from 0 to 1 the number of scrolls decreases. Let $E(x_e, y_e, z_e)$ be an equilibrium solution of the general three-dimensional system:

$$\begin{cases} \dot{x}(t) & = & Q(x(t), y(t), z(t)), \\ \dot{y}(t) & = & R(x(t), y(t), z(t)), \\ \dot{z}(t) & = & V(x(t), y(t), z(t)). \end{cases} \tag{4.1}$$

The eigenvalues equation corresponding to this equilibrium point is given by the following polynomial:

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3. \tag{4.2}$$

Using the result of the Routh-Hurwitz conditions, where the necessary and sufficient condition for the equilibrium point E to be locally asymptotically stable is $a_1 > 0$, $a_3 > 0$ and $a_1 \times a_2 - a_3 > 0$.

In this section, the parameter c is constant, and the bifurcation is studied with respect to parameter ε, and the values of parameters are $\alpha = 11, \beta = 15, a = 2$, $b = 0.2$.

4.2.1 Stability of the Origin Equilibrium Point E_0

The origin $E_0(0, 0, 0)$ is an equilibrium point of System (3.7) independently to epsilon. We consider both cases $c = 11$ and $d = 0$, and $c = 12$ and $d = \pi$.

For $c = 11$ and $d = 0$, the Jacobian matrix evaluated at the equilibrium point $E_0(0, 0, 0)$ is:

$$J_{E_0} = \begin{pmatrix} -\alpha\chi + \alpha\varepsilon(\chi + \frac{\pi b}{2a}) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix} = \begin{pmatrix} -0.41756 + 11\varepsilon(0.03796 + \frac{0.628}{4}) & 11 & 0 \\ 1 & -1 & 1 \\ 0 & -15 & 0 \end{pmatrix}.$$

Its characteristic polynomial is:

$$P(\lambda) = \lambda^3 + (1.4176 - 2.1446\varepsilon)\lambda^2 + (4.4176 - 2.1446\varepsilon)\lambda + (6.2634 - 32.168\varepsilon).$$

According to Routh-Hurwitz conditions, the necessary and sufficient condition, for the equilibrium point E_0 to be stable is $0.00005139 < \varepsilon < 0.1947$.

Proof.

$$a_1 = 1.4176 - 2.1446\varepsilon > 0 \Longrightarrow \varepsilon < 0.66101,$$

$$a_3 = 6.2634 - 32.168\varepsilon > 0 \Longrightarrow \varepsilon < 0.19472,$$

$$a_1 \times a_2 - a_3 = 4.5993\varepsilon^2 + 19.655\varepsilon - 1.6102 \times 10^{-3} > 0 \Longrightarrow \varepsilon < -4.2733 \text{ or } \varepsilon > 5.1399 \times 10^{-5}.$$

◀

For $c = 12$ and $d = \pi$, the Jacobian matrix evaluated at the equilibrium point E_0 is:

$$J_{E_0} = \begin{pmatrix} -\alpha\chi + \alpha\varepsilon(\chi - \frac{\pi b}{2a}) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix} = \begin{pmatrix} -0.41756 + 11\varepsilon(0.03796 - \frac{0.628}{4}) & 11 & 0 \\ 1 & -1 & 1 \\ 0 & -15 & 0 \end{pmatrix}.$$

Its characteristic polynomial is:

$$P(\lambda) = \lambda^3 + (1.4176 + 1.3094\varepsilon)\lambda^2 + (1.3094\varepsilon + 4.4176)\lambda + (19.642\varepsilon + 6.2634).$$

According to Routh-Hurwitz conditions, the equilibrium point E_0 is unstable.

Proof.

$$\begin{aligned} a_1 &= 1.4176 + 1.3094\varepsilon > 0 \Longrightarrow \varepsilon > -1.0826, \\ a_3 &= 19.642\varepsilon + 6.2634 > 0 \Longrightarrow \varepsilon > -0.3188, \\ a_1 \times a_2 - a_3 &= 1.7145\varepsilon^2 - 12.001\varepsilon - 1.012 \times 10^{-3} > 0 \Longrightarrow \varepsilon < -8.4175 \times 10^{-5} \\ &\text{or } \varepsilon > 6.998. \end{aligned}$$

because there is a contradiction with $0 < \varepsilon < 1$. Therefore, the limit cycle is unstable. ◀

Special case $\varepsilon = 0$: the system (3.7) becomes linear:

$$\begin{cases} \dot{x}(t) &= -\alpha\chi x(t) + \alpha y(t), \\ \dot{y}(t) &= x(t) - y(t) + z(t), \\ \dot{z}(t) &= -\beta y(t). \end{cases} \tag{4.3}$$

The Jacobian matrix is:

$$J_{E_0} = \begin{pmatrix} -\alpha\chi + \alpha\varepsilon(\chi + \frac{\pi b}{2a}) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix} = \begin{pmatrix} -0.41756 + 11\varepsilon(0.03796 + \frac{0.628}{4}) & 11 & 0 \\ 1 & -1 & 1 \\ 0 & -15 & 0 \end{pmatrix}.$$

The characteristic polynomial is given by:

$$P(\lambda) = \lambda^3 + (1.4176 - 2.1446\varepsilon)\lambda^2 + (4.4176 - 2.1446\varepsilon)\lambda + (6.2634 - 32.168\varepsilon).$$

One has

$$a_1 = 1.4176 - 2.1446\varepsilon > 0 \Longrightarrow \varepsilon < 0.66101,$$
$$a_3 = 6.2634 - 32.168\varepsilon > 0 \Longrightarrow \varepsilon < 0.19472,$$
$$a_1 \times a_2 - a_3 = 4.5993\varepsilon^2 + 19.655\varepsilon - 1.6102 \times 10^{-3} > 0 \Longrightarrow \varepsilon < -4.2733 \text{ or } \varepsilon > 5.1399 \times 10^{-5}.$$

Therefore, the Routh-Hurwitz conditions are not verified, because $\varepsilon = 0$, and then the system (3.7) is unstable. **Fig. 4.1** displays the corresponding unstable limit cycle .

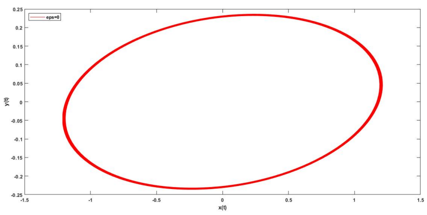


Figure 4.1: The attractor of the system (4.3) where $\varepsilon = 0$: limit cycle unstable.

4.2.2 Other Equilibrium Points:

The equilibrium point **E** of system (3.7) is obtained solving,

$$\begin{cases} \dot{x}(t) &= 0 \\ \dot{y}(t) &= 0 \\ \dot{z}(t) &= 0 \end{cases} \Leftrightarrow \begin{cases} -\alpha(\chi x(t) - y(t)) + \varepsilon g(x(t)) &= 0, \\ x(t) - y(t) + z(t) &= 0, \\ -\beta y(t) &= 0. \end{cases} \tag{4.4}$$

In addition to the origin equilibrium point $E_0(0,0,0)$, there are several other equilibrium points: $E_{k^+}(x_{eq}, 0, -x_{eq})$ and $E_{k^-}(-x_{eq}, 0, x_{eq})$.

The solution of (4.4) are :

- If $x \geq 2ac$ or $x \leq -2ac$

$$x_{eq} = \frac{2\varepsilon b\pi ac}{2a(\varepsilon\chi - \chi) - \varepsilon b\pi}.$$

For the values of parameters above, with $c = 11$; one finds:

$$x_{eq} = \pm\frac{27.6460\varepsilon}{0.1518(\varepsilon - 1) - 0.6283\varepsilon}.$$

- If $-2ac < x < 2ac$; one obtains:

$$-\alpha(\chi x(t) - y(t)) + \varepsilon g(x(t)) = 0, \tag{4.5}$$

that is

$$-\alpha(\chi x(t) - \varepsilon b \sin(\frac{\pi x(t)}{2a}) - \varepsilon \chi x(t)) = 0.$$

Case $\varepsilon = 1$: the system (3.7) is the original system (3.1). In this case, in addition to the origin $E_0(0,0,0)$ the other equilibrium points of this system are $(x_{eq}, 0, -x_{eq})$, with $x_{eq} = 2ak$ and $k = \pm 1, \pm 2, ..., \pm c$ [Tan+01].

Case $0 < \varepsilon < 1$: in addition to the origin $E_0(0,0,0)$ the other equilibrium points cannot be obtained using a closed formula. It is possible to compute them numerically by solving $\dot{x}(t) = 0$ in equation (4.5). The number of equilibrium points set are shown in **Table.** 4.1 and **Fig.** 4.2, **Fig.** 4.3 replacing the values of parameters above and $c = 11$ in the interval $[-44, 44]$

Table 4.1: Number of equilibrium points in the equation (4.5) for different values of ε and $c = 11$.

Values of ε	0.30	0.50	0.70	0.80	0.90	0.95	1
Number of equilibrium points	2	2	6	10	22	22	22

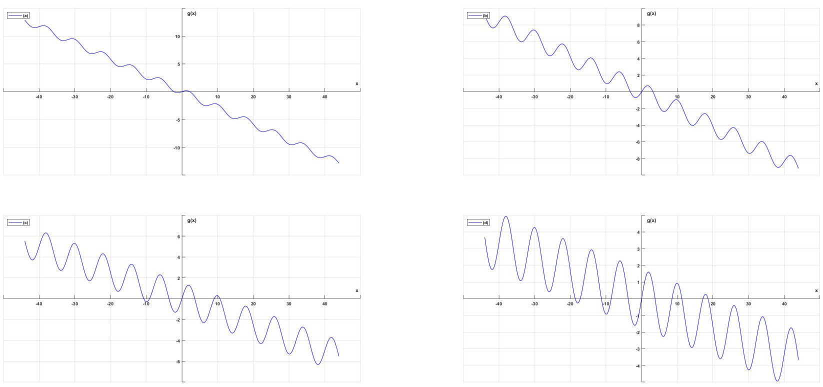


Figure 4.2: Number of equilibrium points in the equation (4.5) for different values of ε and $c = 11$. (a) for $\varepsilon = 0.30$, (b) for $\varepsilon = 0.50$, (c) for $\varepsilon = 0.70$, (d) for $\varepsilon = 0.80$.

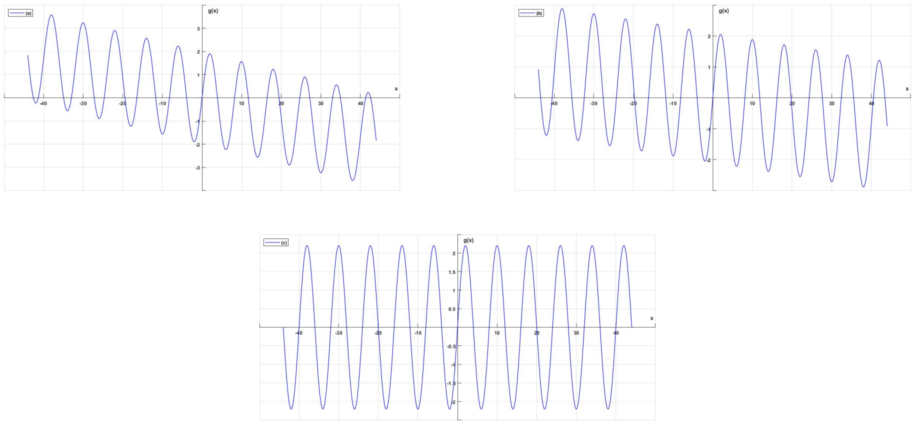


Figure 4.3: Number of equilibrium points in the equation (4.5) for different values of ε and $c = 11$. (a) for $\varepsilon = 0.90$, (b) for $\varepsilon = 0.95$, (c) for $\varepsilon = 1$.

4.3 Numerical Analysis of Bifurcations

4.3.1 Case $c = 11$

In this section, a numerical analysis of the bifurcations of this system is done for c = 11. In this case, one can observe only an even number of scrolls. The value of ε is increased from 0 to 1 . For each value of ε the same initial conditions $[0.001, 0, 0.001]$ are chosen. The bifurcation diagram with respect to ε is given in **Fig. 4.4**, **Fig. 4.5**

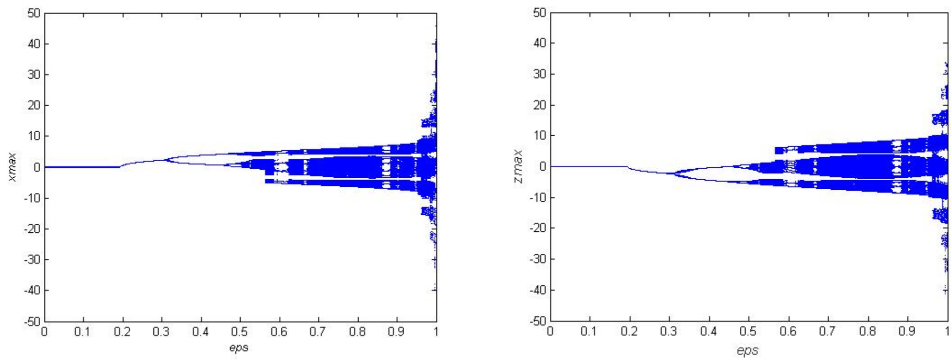


Figure 4.4: Bifurcation diagram with respect to ε of the component x and z, with c=11.

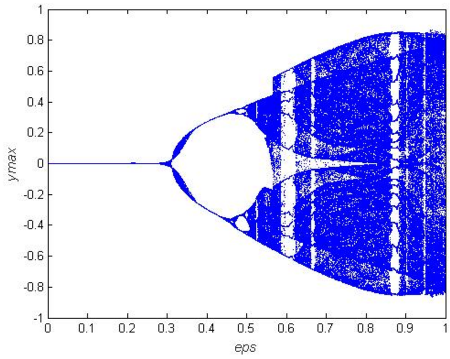


Figure 4.5: Bifurcation diagram with respect to ε of the component y, with c=11.

In order to highlight the symmetry of the bifurcation diagrams versus the components x and z, **Fig. 4.6** displays the superimposition of both **Fig. 4.4** and **Fig. 4.5**.

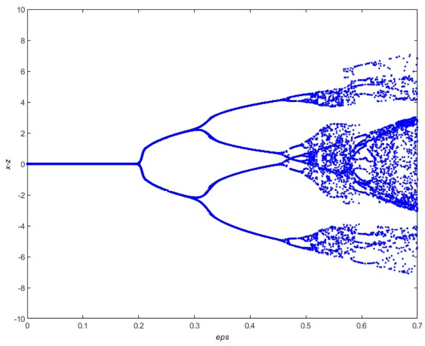


Figure 4.6: Bifurcation diagram with respect to ε of the superimposed components x and z with c=11.

In **Figs. 4.7** and **4.8** one displays some corresponding attractors.

- For $\varepsilon < 0.195$, the equilibrium point E_0 is a locally asymptotically stable focus.
- For $0.195 < \varepsilon \leq 0.43$ the fixed point E_0 becomes unstable, a period-one limit cycle appears, as shown in **Fig. 4.7a**.
- When $\varepsilon \approx 0.46$, a new bifurcation occurs, the period-one limit cycle becomes unstable and a period-two limit cycle appears (**Fig. 4.7b**).

- For $\varepsilon \approx 0.495$, a period-4 limit cycle appears through a new bifurcation, as shown in **Fig. 4.7c**, followed by a bifurcation to a period-8 limit cycles at $\varepsilon \approx 0.501$ (**Fig. 4.7d**). This doubling period bifurcation process continues up to the critical value $\varepsilon = 0.56$, where one chaotic attractor appears (see **Fig. 4.7e**). For $\varepsilon = 0.57$ a two-scrolls chaotic attractor appears (see **Fig. 4.7f**).

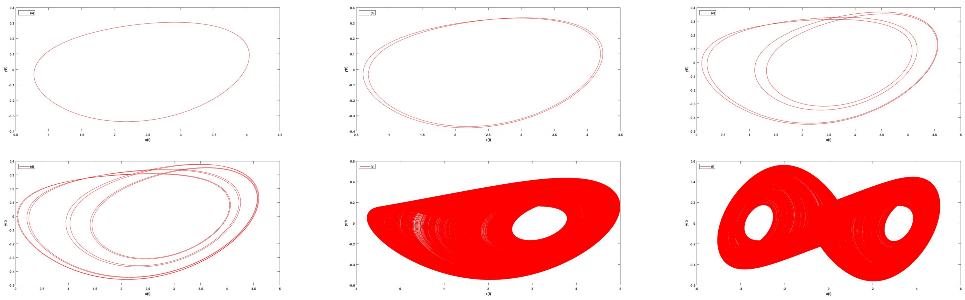


Figure 4.7: Phase portrait of Chua system for different values ε and $c = 11$(attractors). (a) a period-one limit cycle for $\varepsilon = 0.43$, (b) a period-two limit cycle for $\varepsilon = 0.46$, (c) a period-4 limit cycle for $\varepsilon = 0.495$, (d) a period-8 limit cycle for $\varepsilon = 0.501$, (e) a chaotic attractor chaotic attractor for $\varepsilon = 0.56$, (f) a 2-spirals for $\varepsilon = 0.57$.

Fig. 4.8 displays the sequence of bifurcations of the number of spirals of the chaotic attractors

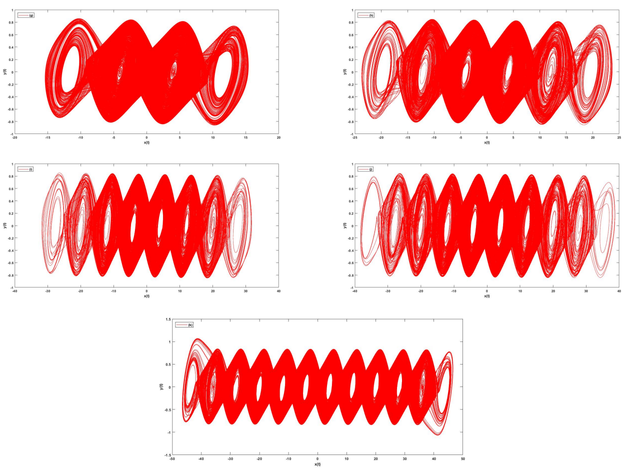


Figure 4.8: (Continued). Phase portrait of Chua system for different values ε and $c = 11$(attractors). (g) 4-spirals for $\varepsilon = 0.97$, (h) 6-spirals for $\varepsilon = 0.987$, (i) 8-spirals for $\varepsilon = 0.992$, (j) 10-spirals for $\varepsilon = 0.995$, (k) 12-spirals for $\varepsilon = 0.998$.

4.3.2 Case $c = 12$

In this case, one can observe only an odd number of scrolls. **Fig.** 4.9 displays the bifurcation diagram with respect to ε.

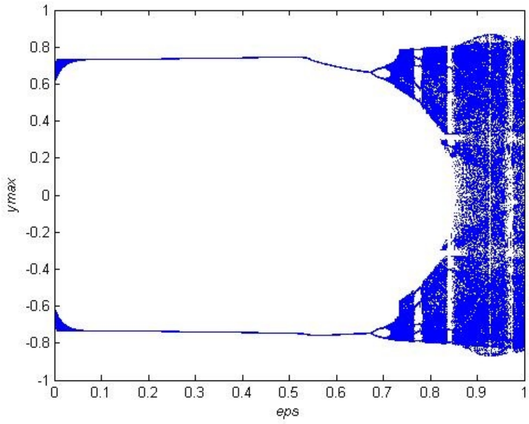


Figure 4.9: Bifurcation diagram with respect to ε of the component y, for $c = 12$.

In **Fig.** 4.10 and **Fig.** 4.11 one displays some corresponding attractors.

- For $\varepsilon < 0.55$, the equilibrium point E_0 is a locally asymptotically stable focus.
- When $\varepsilon \approx 0.55$, the system of Chua in the fixed point E_0 becomes unstable, and a period-one limit cycle appears for $0.55 < \varepsilon \leq 0.60$, as shown in **Fig.** 4.10**a**.
- When $\varepsilon \approx 0.68$, a new bifurcation occurs, and the period-one limit cycle becomes unstable and a period-two limit cycle appears (**Fig.** 4.10**b**).
- For $\varepsilon \approx 0.706$, a-period-4 limit cycle appears through a new bifurcation, as shown in **Fig.** 4.10**c**, followed by a bifurcation to a period-8 limit cycles at $\varepsilon \approx 0.7109$; (see **Fig.** 4.10**d**). This bifurcation process continues up to a critical value of $\varepsilon = 0.72$, where one chaotic attractor appears; (see **Fig.** 4.10**e**). At $\varepsilon = 0.80$ a 1-scroll chaotic attractor appears (see **Fig.** 4.10**f**).

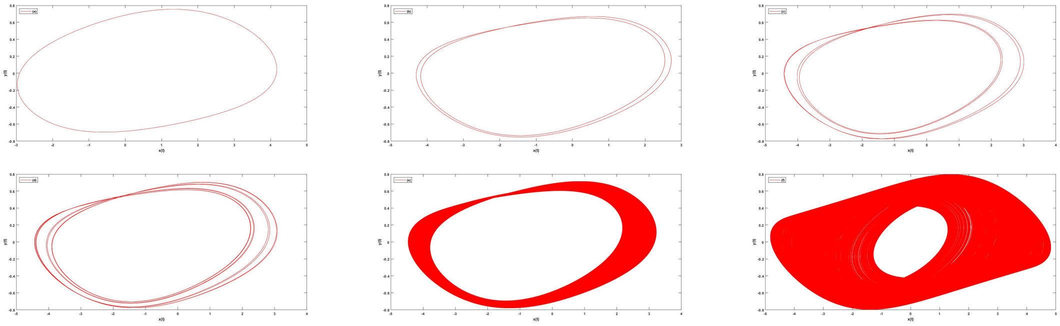


Figure 4.10: Phase portrait of Chua system for different values ε and $c = 12$(attractors). (a) a limit cycle for $\varepsilon = 0.60$, (b) a period-two limit cycle for $\varepsilon = 0.68$, (c) a period-4 limit cycle for $\varepsilon = 0.706$, (d) a period-8 limit cycle for $\varepsilon = 0.7109$, (e) one chaotic attractor for $\varepsilon = 0.72$, (f) 1-spirals for $\varepsilon = 0.80$

The **figure.** 4.11 displays the sequence of bifurcations of the number of spirals of the chaotic attractors with an odd number of scrolls.

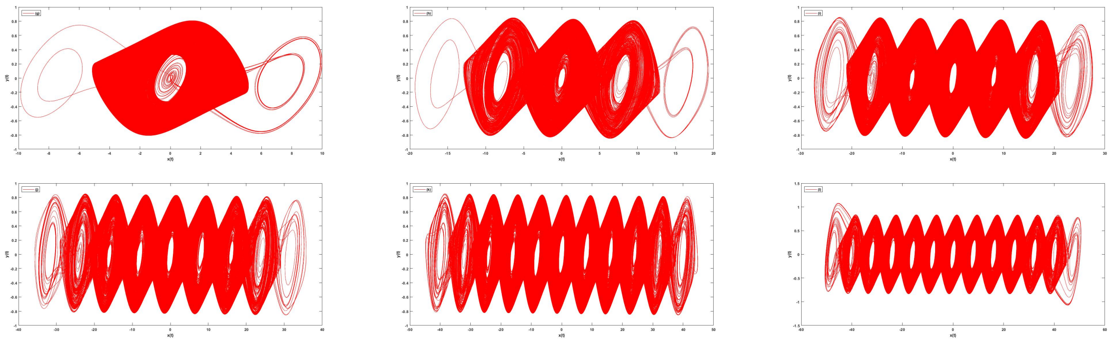


Figure 4.11: Phase portrait of Chua system for different values ε and $c = 21$(attractors). (g) 3-spirals for $\varepsilon = 0.86$, (h) 5-spirals for $\varepsilon = 0.978$, (i) 7-spirals for $\varepsilon = 0.989$, (j) 9-spirals for $\varepsilon = 0.993$, (k) 11-spirals for $\varepsilon = 0.9953$, (l) 13-spirals for $\varepsilon = 0.9994$

▶ **Remark 4.1.**

- When we increase the value of ε, the number of scrolls keeps increasing while staying in the chaos zone.
- We have two different notions of bifurcation, the first one the classic definition, the second definition by Menacer et al [MLC16].

◀

4.4 Chaos Control

When dealing with chaotic dynamical systems in general, particular interest is paid to our ability to control or stabilize these systems. By control, we refer to the addition of new adaptively updated terms to the chaotic s system in order to force its states towards zero asymptotically. One of the applications of this topic is in robotics where the control of the chaotic motion of a rigid body is considered.
This section suggests the aim of this section is to control chaotic Chua circuit systems.
The description of a chaotic system in three dimensions is as follows:

$$\dot{U} = M(U) \tag{4.6}$$

where $U = (x, y, z) \in \mathbb{R}^3$, the The controlled system is described by :

$$\dot{U} = M(U) - T(U - \bar{U}) \tag{4.7}$$

where $T = diag(T_1, T_2, T_3)$; $T1, T2, T3 \geq 0$ and $\bar{U}$ is an equilibrium point of the system (4.6).

Now, if one selects the appropriate feedback control gains T_1, T_2, T_3 which then make the eigenvalues of the linearized equation of the controlled system (4.7) satisfy one of the above-mentioned Routh-Hurwitz conditions [AEE06], therefore the trajectories of the controlled system (4.7) asymptotically approaches the unstable equilibrium point $\bar{U}$ in the sense that $\lim_{t\to\infty} \|U - \bar{U}\| = 0$ where is the Euclidean norm[35].

let us consider the controlled system of the system has the form:

$$\begin{cases} \dot{x}(t) &= -\alpha(\chi x(t) - y(t)) + \varepsilon g(x(t)) - T_1(x(t) - \bar{x}(t)), \\ \dot{y}(t) &= x(t) - y(t) + z(t) - T_2(y(t) - \bar{y}(t)), \\ \dot{z}(t) &= -\beta y(t) - T_2(z(t) - \bar{z}(t)), \end{cases} \tag{4.8}$$

Where $T_1, T_2, T_3 > 0$ are external control inputs that will drag the chaotic trajectory (x,y,z) of the modified Chua system (4.8) to $E = (\bar{x}, \bar{y}, \bar{z})$ one of the steady states E_0 and E is equilibrium point of the system (4.8).

We choose the feedback gains $T = diag(T_1, 0, 0)$.The control law take the following form:

$$\begin{cases} \dot{x}(t) &= -\alpha(\chi x(t) - y(t)) + \varepsilon g(x(t)) - T_1(x(t) - \bar{x}(t)), \\ \dot{y}(t) &= x(t) - y(t) + z(t), \\ \dot{z}(t) &= -\beta y(t). \end{cases} \tag{4.9}$$

4.4.1 Stabilizing the Equilibrium Point $E_0 = (0.0.0)$

▶ **Proposition 4.2.** If $\mathbf{T}_1 \in [\mathbf{T}_\varepsilon^0; \infty]$ and $\mathbf{T}_1 \in [\mathbf{T}_\varepsilon^1; \infty]$where :

$$\mathbf{T}_\varepsilon^0 = 2.1455\varepsilon - 0.41756,$$

$$T_\varepsilon^1 = 2.1455\varepsilon + 4.5824 + 0.5\sqrt{2.2281 \times 10^{-4}\varepsilon^2 - 1.5194 \times 10^{-3}\varepsilon + 83.998}.$$

The system (4.9) is **controlled**. ◀

Proof. We consider the system (4.9) in the cases $c = 11$ and $d = 0$, the Jacobian matrix for evaluated at the equilibrium point $E_0 = (0, 0, 0)$ is:

$$J_{E_0} = \begin{pmatrix} -\alpha\chi + \alpha\varepsilon(\chi + \frac{\pi b}{2a}) - T_1 & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}.$$

The characteristic equation of the controlled system (4.9) at $E_0 = (0.0.0)$ is given as: $P(\lambda) = \lambda^3 + \left(T_1 + \alpha\chi - \alpha\varepsilon\left(\chi + \frac{\pi}{2a}b\right) + 1\right)\lambda^2 + \left(\beta - \alpha + T_1 + \alpha\chi - \alpha\varepsilon\left(\chi + \frac{\pi}{2a}b\right)\right)\lambda + \left(\beta\left(T_1 + \alpha\chi - \alpha\varepsilon\left(\chi + \frac{\pi}{2a}b\right) + 1\right) - \beta\right).$

For the values of parameters fixed for: $\alpha = 11,\ \beta = 15,\ a = 2,\ \chi = 0.03796$ and $b = 0.2$, then :
$P(\lambda) = \lambda^3 + (T_1 - 11\varepsilon(0.05\pi + 0.03796) + 1.4176)\lambda^2 + (4 + T_1 + 11 * 0.03796 - 11\varepsilon\left(0.03796 + \frac{\pi}{4}0.2\right)\lambda) + 15\left(T_1 + 11 * 0.03796 - 11\varepsilon\left(0.03796 + \frac{\pi}{4}0.2\right) + 1\right) - 15.$

According to the Routh-Hurwitz conditions :

$a_1 = T_1 - 2.145\,4\varepsilon + 1.4176 > 0 \Longrightarrow T_1 \in [2.1454\varepsilon - 1.4176; +\infty],$

$a_3 = 15T_1 - 32.182\varepsilon + 6.263\,4 > 0 \Longrightarrow T_1 \in [2.1455\varepsilon - 0.41756; +\infty],$

from a_1 et a_3 we find: $T_1 \in [T_\varepsilon^0; +\infty]$ where $T_\varepsilon^0 = 2.1455\varepsilon - 0.41756,$
and
$a_1 \times a_2 - a_3 = 4.602\,9\varepsilon^2 - 4.2909\varepsilon T_1 + 19.663\varepsilon + T_1^2 - 9.1648T_1 - 1.0669 \times 10^{-3} > 0,$
$\Rightarrow T_1^1 < 2.1455\varepsilon + 4.5824 - 0.5\sqrt{2.2281 \times 10^{-4}\varepsilon^2 - 1.5194 \times 10^{-3}\varepsilon + 83.998} or$
$\mathrm{T}_1^2 > 2.1455\varepsilon + 4.5824 + 0.5\sqrt{2.2281 \times 10^{-4}\varepsilon^2 - 1.5194 \times 10^{-3}\varepsilon + 83.998}$

$\Rightarrow \mathbf{T}_1 \in [\mathbf{T}_\varepsilon^1; \infty]$;where $\mathbf{T}_\varepsilon^1 = 2.1455\varepsilon + 4.5824 + 0.5\sqrt{2.2281 \times 10^{-4}\varepsilon^2 - 1.5194 \times 10^{-3}\varepsilon + 83.998}.$

We conclude that

$$\mathbf{T}_1 \in [\mathbf{T}_\varepsilon^0; \infty] \text{ and } \mathbf{T}_1 \in [\mathbf{T}_\varepsilon^1; \infty].$$

◀

4.4.2 Numerical Results

We will show a numerical experiment to demonstrate the effectiveness of the proposed control scheme. The MATLAB's code standard solver for the fourth-order Runge–Kutta method is used to integrate the differential equations. The initial states are $x(0) = 0.001$, $y(0) = 0$ and $z(0) = 0.001$. The equilibrium $E_0 = (0, 0, 0)$ of the system (1.1) is stable :

- **For** $\varepsilon = 0.10$:

  we have:

$$\begin{aligned}
T_\varepsilon^0 &> 2.1455\varepsilon - 0.41756 \Rightarrow T_\varepsilon^0 > -0.20301 \\
\text{so } T_1 &\in [T_\varepsilon^0; \infty], \\
\text{and } a_1 \times a_2 - a_3 &= 4.6029\varepsilon^2 - 4.2909\varepsilon T_1 + 19.663\varepsilon + T_1^2 - 9.1648 T_1 - 1.0669 \times 10^{-3} > 0 \\
&\Rightarrow T_1 \in T_\varepsilon^1 \cup T_\varepsilon^2, \\
\text{where } T_\varepsilon^1 &= [-\infty; 0.21443] \text{ or } T_\varepsilon^2 = [9.3795; \infty];
\end{aligned}$$

  we conclude that : $\mathbf{T}_1 \in \mathbf{T}_\varepsilon^0 \cap \mathbf{T}_\varepsilon^1$.

- **For** $\varepsilon = 0.70$:

$$\begin{aligned}
T_\varepsilon^0 &= 1.0843 \Rightarrow T_1 \in [T_\varepsilon^0; \infty], \\
\text{and } a_1 \times a_2 - a_3 &= 4.6029\varepsilon^2 - 4.2909\varepsilon T_1 + 19.663\varepsilon + T_1^2 - 9.1648 T_1 - 1.0669 \times 10^{-3} > 0, \\
&\Rightarrow T_1 \in T_\varepsilon^1 \cup T_\varepsilon^2, \\
\text{where } T_\varepsilon^1 &= [-\infty; 1.5017] \text{ or } T_\varepsilon^2 = [10.667; \infty];
\end{aligned}$$

  we conclude that : $\mathbf{T}_1 \in \mathbf{T}_\varepsilon^0 \cap \mathbf{T}_\varepsilon^1$.

We choose $T_1 = 0.15$ for $\varepsilon = 0.10$ and $T_1 = 1.10$, for $\varepsilon = 0.70$ as shown in the figure. 4.12a and 4.12b respectively, where chaos is suppressed to E_0 with time. The control is activated at $t = 180$ and $t = 120$ respectively.

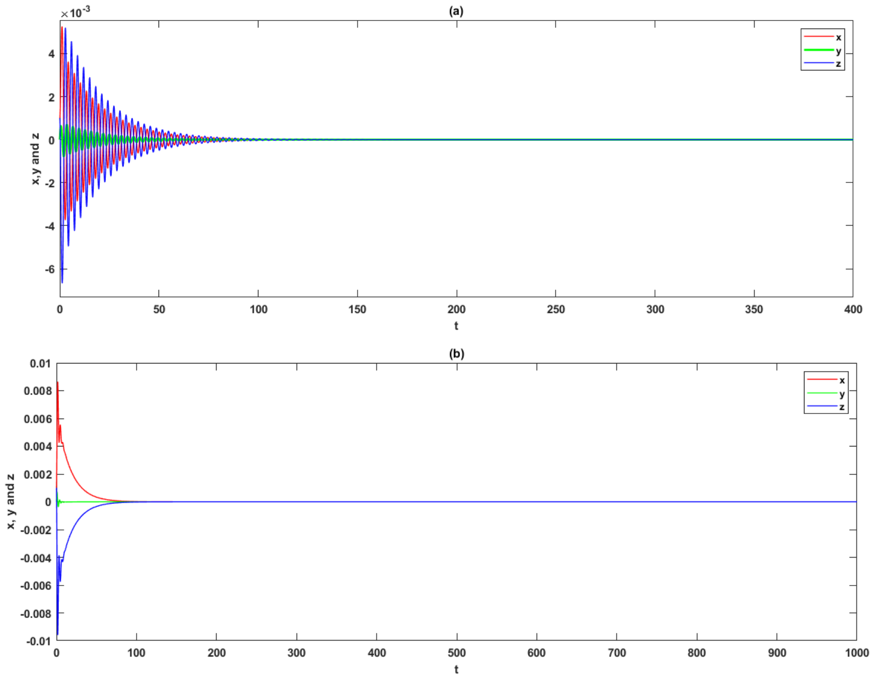


Figure 4.12: The trajectories of the controlled system (1.1). (a) The control $T_1 = 0.15$, $T_2 = T_3 = 0$ is activate at $t = 180$ for E_0. (b) The control $T_1 = 1.10$, $T_2 = T_3 = 0$ is activate at $t = 120$ for E_0.

4.5 Chaos Synchronization

▶ **Definition 4.3.** The fact of happening at the same time, or the act of making things happen at the same time: The words flash on a TV screen in synchronization with the music, and like the hummingbird migration depends on the synchronization between the end of the relatively warm winter in Mexico and the beginning of the temperate summer in the Elk Mountains. (Cambridge Dictionary) ◀

▶ **Definition 4.4.** In the realm of chaotic systems, synchronization refers to the remarkable ability of two or more such systems to evolve together over time, displaying similar or identical behavior, despite their intrinsic chaos. ◀

Types of Synchronization:

Different types of synchronization exist within chaotic systems, including complete synchronization, phase synchronization, and lag synchronization. Each type signifies a specific level of coordination and order between the systems.

In recent years, chaos synchronization has been widely explored and studied because of its potential applications, such as in secure communication, biological systems, and information science [MKH10] [KA] and[AVO17]. Therefore, a variety of approaches have been proposed for the synchronization of chaotic systems, such as complete synchronization, generalized synchronization, and projective synchronization. Each type signifies a specific level of coordination and order between the systems[Mes20] .

In recent years, research has shown that synchronization can also be extended to systems with fractional derivatives and this is what we attempt to do in future research with the fractional version of a system modeling the dynamics systems circuits.

Conclusions & Outlook

We conclude this thesis with some research perspectives that could be the subject of future research

- Study the adaptive control and synchronization of Fractional-order Chua's and Chen circuit systems with other functions, with a comparison of numerical results.

- Based on the results given in chapters two and three, we can establish a novel system of three-dimensional (3D $n \times m \times l$ grid scroll) parameters in the situation of multidirectional multi-scroll chaotic Chen attractors with saturated function.

- Study the synchronization for the same chaotic Chua system with multispiral attractors generated via tooth function and saturated function.

- Applied the same method for the numerical localization of hidden attractors in a Fractional-order chaotic system.

Appendix: Programs in Matlab for Hidden Bifurcation

Plot the orbit around the Chua chaotic attractor:

```
function dy = Chuanew(t,y)
a=2;
b=0.2;
c=12;
d=pi;
alpha=11;
beta=15;
k=0.03796;
eps=0.80;
if y(1)<=-2*a*c
f=b*pi*(y(1)+2*a*c)/(2*a);
end
if abs(y(1))< 2*a*c
f=-b1*sin((pi*y(1)/(2*a))+d);
end
if y(1)>= 2*a*c
f=b*pi*(y(1)-2*a*c)/(2*a);
end
dy = double(zeros(3,1));
dy(1) = alpa*(y(2)-k*y(1)-eps*f+eps*k*y(1));
```

```
dy(2) = y(1)-y(2)+y(3);
dy(3) = -beta*y(2);
end
```

****

```
clear all
close all
clc
hold on
options = odeset('AbsTol',1e-11,'RelTol',1e-6);
fsize=15;
N=size(Y);
nn=round(9*N(1)/10);
for i=1:N(1)-nn
y1(i)=Y(i+nn,1);
y2(i)=Y(i+nn,2);
y3(i)=Y(i+nn,3);
end
yy=Y(N(1),:)
figure(1)
plot(y1,y2,'r')
```

****

Plot the diagramm of bifurcation:

```
clc
clear
global eps
eps=0:0.001:1;
```

```
L=length(eps);
for i=1:L
eps=eps(i);
datb=Y(500:end,2); for j=2:(length(data)-1)
plot(eps,data(j),'b.',eps,datc(j),'b.');
hold on;
if j>=100 break;
end
end
end end
xlabel('eps')
ylabel('ymax)
```

Bibliography

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Abstract

Chaos is a typical phenomenon of nonlinear systems and is currently widely studied, because of its features and many potential applications.

The main aim of this thesis concerns principally two major subjects. the first is to deploy a method for uncovering a hidden bifurcation in a multispiral Chua system with a sine function that is based on the famous paper by Menacer et al. This method is based on the core idea of Leonov and Kuznetsov for searching hidden attractors while keeping ε *constant, a new bifurcation parameter is introduced. We end this part with the introduction of a novel method based on the duration of integration for unveiling hidden patterns of an even number of spirals.*

In the second part, We used the Routh-Hurwitz Criteria for studying the stability of the Chua system at equilibrium point E_0 *concerning* ε*. Furthermore, we made a theoretical and numerical study of bifurcation and chaos control on Multispiral Chua's system.*

Keywords: *Chaos, Hidden Attractor, Bifurcation, Equilibrium point, Stability, Chua Chaotic system, Integration duration, Control, Routh-Hurwitz criterion, Synchronization.*

Résumé

Le chaos est un phénomène typique des systèmes non linéaires, et actuellement largement étudié, en raison de ses caractéristiques et de ses nombreuses applications potentielles.

L'objectif principal de cette thèse concerne principalement deux sujets majeurs. Le premier est de déployer une méthode, pour découvrir une bifurcation cachée dans un système multispirale de Chua avec une fonction sinus, qui est basée sur le célèbre article de Menacer et al. Cette méthode est basée sur l'idée centrale de Leonov et Kuznetsov pour la recherche d'attracteurs cachés, tout en gardant ε constant, un nouveau paramètre de bifurcation est introduit. Nous terminons cette partie par la présentation d'une nouvelle méthode basée sur la durée d'intégration pour dévoiler les modèles cachés d'un nombre pair de spirales.

Dans la deuxième partie, nous avons utilisé les critères de Routh-Hurwitz pour étudier la stabilité du système de Chua, aux point d'équilibre E_0 par rapport à ε . De plus, nous avons fait une étude théorique et numérique de la bifurcation et du contrôle du chaos sur le système de Chua multispiral.

Mots-clés: *Chaos, Attracteur caché, Bifurcations, Point d'équilibre, Stabilité, Système chaotique de Chua, Durée d'intégration, Contrôle, Critères de Routh-Hurwitz, Synchronisation.*

ملخص

الفوضى: هي ظاهرة نموذجية للأنظمة غير الخطية، تتم دراستها حاليا على نطاق واسع بسبب ميزاتها والعديد من التطبيقات المحتملة.

الهدف الرئيسي من هذه الأطروحة يتعلق بشكل أساسي بموضوعين أساسين. يتناول الموضوع الأول: نشر طريقة للكشف عن التشعب الخفي في نظام تشوا متعدد اللوالب معرف بدالة جيبيه، هذه الطريقة تعتمد على ورقة البحث التي كتبها الباحث منصر وزملاؤه. تستند هذه الأخيرة الى الفكرة الأساسية للباحثين ليونوف وكوزنتسوف، للبحث عن عوامل الجذب المخفية بإبقاء المتغير ε ثابت فينتج عنه معامل تشعب جديد. ننهي هذا الجزء بتقديم طريقة جديدة تعتمد على مدة التكامل للكشف عن التشعبات والجواذب المخفية في عدد اللوالب الزوجية.

فيما يخص الموضوع الثاني، تم استخدام معايير روث-هرويترز لدراسة استقرار النظام الفوضوي تشوا عند نقطة التوازن بالنسبة الى ε. علاوة على ذلك، أجرينا دراسة نظرية وعددية للتحكم في التشعب والفوضى على نظام تشوا متعدد اللوالب.

الكلمات المفتاحية: الفوضى، جواذب مخفية، التشعبات، الاستقرار، النظام الفوضوي تشوا، مدة التكامل، التحكم، معيار روث-هورويتز، المزامنة.