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Adomian Decomposition Method for Population Balance Equations and the Study of Convergence

Phd thesis to achieve the Doctoral Degree in Mathematics

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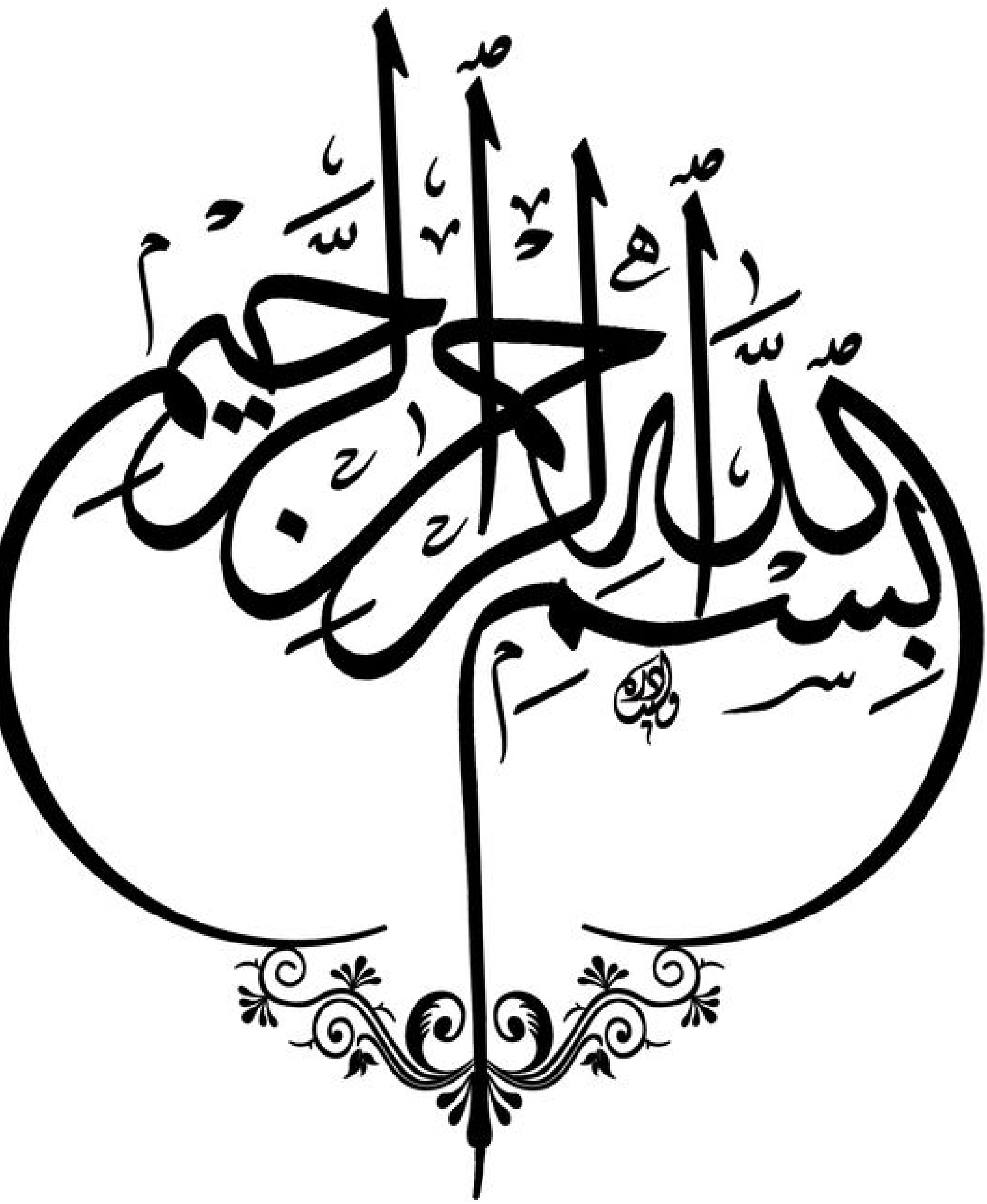
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Scientific Contributions

List of Papers

1. I. Achour and A. Bellagoun (January,13 2023). *Adomian Decomposition Method For Solving Spatially Inhomogenuous Population Balance Equation.* Advances in Mathematics: Scientific Journal 12 (**2023**), no.1, 115-136

Conference paper and communication

1. I. Achour and A. Bellagoun, *Adomian Decomposition Methods For Population Balance Equations .* The 1st International Conference on Pure and Applied Mathematics **IC-PAM'21**, May 26-27, 2021, **Ouargla**, Algeria .

2. I. Achour and A. Bellagoun, *Decomposition Method For Some Boundary Value Problems Of Nonhomogeneous Spatially Population Balance Equations.* The 1st International workshop on Applied Mathematics **1st-IWAM'2022**, December 6-8, 2022, **Constantine**, Algeria.

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4. I. Achour and A. Bellagoun, *A Semi-Analytical Technique For Solving Pure Coagulation Equation.* The 1st national applied mathematics seminar **1st-NAMS'21**, May 14-15, 2023, **Biskra**, Algeria .

5. I. Achour and A. Bellagoun, *The Continuous Pure Fragmentation Equations in continuous flow system With Diffusion.* The second national conference on pure and applied mathematics **2nd-NCMPA'23**, July 04, 2023, **Tebessa**, Algeria .

Contents

Resumé

La méthode de décomposition d'Adomian (ADM) a reçu beaucoup d'attention dans Ces dernières années en mathématiques appliquées en général et dans le domaine des solutions en série en particulier. Il s'agit d'une technique efficace pour la résolution analytique d'une large classe de systèmes dynamiques. L'équation de bilan de population (PBE) a été utilisée pour modéliser une variété de processus particulaires. Cependant, seuls quelques cas où des solutions analytiques pour le processus de rupture/coalescence existent, la plupart de ces solutions sont pour le système spatialement homogène. L'objectif principal de cette thèse est de dériver des solutions analytiques de PBE spatialement inhomogènes pour les processus de rupture/coalescence en utilisant la méthode de décomposition d'Adomian qui utilise un type spécifique de polynômes appelés polynômes d'Adomian pour décomposer la partie non linéaire d'une telle équation. Les résultats obtenus indiquent que l'ADM évite les problèmes de stabilité numérique qui caractérisent souvent les techniques numériques générales dans ce domaine.

Mots clés: Modèle de Bilan de population, Méthode decompositionnelle d'Adomian, Polynômes d'Adomian, Convergence, Equation intégro-differentielle .

Abstract

The Adomian decomposition method has received much attention in recent years in applied mathematics in general and in the area of series solutions in particular. It is an effective technique for the analytical solution of a wide class of dynamical systems. The population balance equation (PBE) has been used to model a variety of particulate Process. However, only a few cases where analytical solutions for the breakage/coalescence process exist, most of these solutions are for the spatially homogeneous system. The main objective of this thesis is to derive analytical solutions of spatially inhomogeneous PBE For breakage/ coalescence processes using the Adomian decomposition method which uses a specific kind of polynomials named "Adomian's polynomials" to decompose the nonlinear part of such equation. The results obtained indicate that the ADM avoids numerical stability problems that often characterize general numerical techniques in this area.

Key words: Population Balance Model, Adomian Decomposition Method, Adomian Polynomials, Convergence, Integro-differential Equation.

الملخص

حظيت طريقة التحلل الادومية باهتمام كبير في السنوات الأخيرة في الرياضيات التطبيقية بشكل عام وفي مجال حلول المتسلسلة بشكل خاص. إنها تقنية فعالة للحلول التحليلية لفئة واسعة من الأنظمة الديناميكية. تم استخدام معادلة التوازن السكاني لنمذجة مجموعة متنوعة من العمليات الجزيئية. ومع ذلك، هناك حالات قليلة فقط حيث توجد حلول تحليلية لعملية الكسر/التحام، ومعظم هذه الحلول مخصصة للنظام المتجانس مكانيًا. الهدف الرئيسي من هذه الأطروحة هو استخلاص حلول تحليلية لمعادلة التوازن السكاني غير المتجانسة مكانياً لعمليات الكسر/الالتحام باستخدام طريقة التحلل الادوميان والتي تستخدم نوعاً محدداً من كثيرات الحدود تسمى "متعددات حدود أدوميان" لتحليل الجزء غير الخطي من هذه المعادلة. تشير النتائج التي تم الحصول عليها إلى أن طريقة التحلل الأدومية، تتجنب مشاكل الاستقرار العددي التي غالبا ما تميز التقنيات الرقمية العامة في هذا المجال.

الكلمات المفتاحية: نموذج التوازن السكاني، طريقة التحلل لأدوميان، متعددات الحدود لأدوميان، التقارب، المعادلة التكاملية التفاضلية.

Introduction

Population Balances may be viewed as either an ancient subject with origins in the Boltzmann equation from more than a century ago or as a topic that is relatively contemporary given the diversity of applications that engineers have lately used population balances. The PBE determines the temporal and spatial evolution of particle distribution due to the interactions within the population of particles on the one hand and the interaction of particles and the continuous phase in which they are embedded on the other hand [69]. It is a hyperbolic integro-partial differential equation characterized by a nonlinear source term. This source term accounts for various mechanisms by which particles of a specific state can either form or disappear from the system. These mechanisms are discrete and relatively instantaneous compared to the system scale, such as particle breakage, aggregation, growth, and nucleation [69].

In the framework of PBEs, the state of each individual particle is characterized by a particle state vector containing external coordinates, such as the position of a particle in physical space, and internal coordinates representing the particle properties, such as particle size, volume, etc. If x_e represents the external and x_i the internal coordinates, then the particle state vector x is given by $x = (x_e, x_i)$. A population of particles is characterized by its particle property distribution, which is described mathematically by a number density function $f(t, x)$ and is a function of time t and the state vector x. This function represents the number of particles per volume of particle state space. It is understood that this deterministic approach is only reasonable if large populations are considered. It is further assumed that the number density function is sufficiently smooth to be differentiated with respect to its arguments. The actual number of particles in a certain area of the particle state space is determined by the integral of the number density function over this area.

Several numerical techniques, such as weighted residual method, moments' method, orthogonal collocation, finite element collocation and pivot techniques, have been proposed in the literature and reviewed by [25, 53, 55–57, 71, 73]. In [71], a comprehensive review of the numerical methods available for solving the PBE was discussed until the mid-1980s. In a series of papers, Kumar and Ramkrishna [55–57]presented critical comments on previous numerical techniques for solving

PBE until the mid-1990s. These authors discovered the internal consistency problem using direct discretization methods based on finite difference schemes. In this regard, they presented the fixed pivot and movable pivot methods to overcome this problem. Recently, Santos et al [73] used the generalized double moment quadrature (MDQMG) method to solve the PBE with only moments where the distribution was recovered using parallelized algorithms to reduce the computational time. On the other hand, Attarakih [25] presented cumulative MDQM (MDQMC) to overcome the distribution reconstruction which is lost using the application of MDQMG.

George Adomian, an American Applied Mathematician (1922-1996), introduced a potent decomposition methodology for the practical solution of linear or nonlinear, deterministic or stochastic operator equations, including ordinary differential equations, partial differential equations, integral equations, etc. The technique has since come to be known as the Adomian decomposition method, or simply the ADM. It is a substantial, potent technique that offers an effective way to solve differential equations analytically AND numerically, simulating real-world physical applications. Recently, a significant amount of study has been put into applying this approach to a variety of partial differential equations, integral differential equations, and linear and nonlinear ordinary differential equations.

Adomian [12–14, 16] and others have successfully applied the ADM to algebraic equations, ordinary, partial, delay, and non-integer order or fractional differential equations [23, 32, 74] for a wide class of nonlinearities, including polynomial, exponential, trigonometric, hyperbolic, composite, negative power, radical, and even decimal power nonlinearities. The ADM solves nonlinear differential equations for any analytic nonlinearity. The ADM allows one to solve nonlinear differential equations without having to appeal to the decidedly questionable practices of perturbation or linearization.

The limitation of such an approach is its dependence on particle breakup and coalescence kernels, and therefore each problem must be treated independently. However, this method still provides analytical solutions in many cases, which are useful for simplified analytical systems and bench-type problems for examining numerical techniques.

A notable contribution was made by Hasseine, Bellagoun, and H.J. Bart [45] those who applied for the first time a semi-analytical technique the ADM to simulate the operation of a reactor in continuous and batch mode using the resolution of the PBE. This method is exempt from crucial problems of numerical discretization and stability which often characterize common numerical techniques in this field. The proposed approach is widely used in applied sciences and engineering to solve problems involving differential, integral, integro-differential, delay differential, and systems of such equations [4, 33–36, 45].

Unlike prior publications which consider the spatially homogeneous case of PBE. The goal of this study is to employ this novel methodology, the ADM to solve the population balance equations in the spatially inhomogeneous case.

Thesis outline

This thesis is in the form of

The first chapter is devoted to some definitions and theorems, in addition to some important spaces in the next chapters.

The second chapter the population balance equation model is presented. also included a review concerning the existence and uniqueness of the PBE's solution.

The third chapter describes the method of Adomain, and the study of its convergence is reviewed.

The fourth chapter presents the results obtained by using the ADM in continuous and batch systems for different cases.

Finally, a general conclusion summarizes all the important results obtained in this work.

Dedication

To

My Parents,
My Brother Ahmed and my Sister Salma,
My friends Sara and Halima,
I dedicate this thesis.

Imane Achour

Acknowledgements

I thank first and foremost Allah for giving me the courage, the patience, and the will to achieve this thesis.

I wish to express my dearest gratitude to my supervisor Prof. **Abdelghani BELLAGOUN** for his continuous support, ideas, and patience in guiding me during the preparation of my thesis. I also acknowledge the opportunity he provided me to grow in this field.

I extend my sincere thanks also to my thesis committee members: Prof. **TIDJANI Mancer**, Prof. **BERBICHE Mohamed**, Prof. **KHELIL Nacer**, from university of Biskra, and Prof. **GUEDJIBA Said** from university of Batna 2, for agreeing to examine this thesis.

Finally, I would like to thank my family, my friends, and all the people whose support and assistance was instrumental in the completion of this work.

Imane Achour
Biskra, Algeria, 2024 .

List of Abbreviations

ADM	Adomian decomposition method
PBMS	Population balance models
PBE	Population balance equation
IVP	Initial value problem
ODE	Ordinary differential equation
PDE	Partial differential equation

List of Symbols

$\mathcal{G}$	general nonlinear operator
$\mathcal{N}$	nonlinear operator
$\mathcal{L}$	linear operator to be inverted
$\mathcal{R}$	linear remainder
g	given function
f	unknown function
f_m	Series solution components
$\mathcal{A}_m$	Adomian polynomial
ξ	Analytic parameter
φ_m	m-term approximation of series solution by ADM
$f_i(t)$	The densities of particles of discrete size i at time t
$\omega_{i,j}$	Coagulation kernel in the discret model
$f(t,v)$	Particle mass density funtion
$\omega(v,v')$	Coagulation kernel
$\varphi(v,v')$	Multiple fragmentation kernel
$F(v,v')$	Binary fragmentation kernel
$\Gamma(v)$	Selection function
$\beta(v,v')$	Breakage function (or daughter distribution function)
$G(t,v)$	Growth velocity
$f_{nuc}(t)$	The number density of nuclei

$\Gamma_{nuc}(t)$	Nucleation rate

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Part I

Preliminary Theory

1

Preliminary Concepts

The first chapter introduces briefly some basic concepts and fundamental theorems concerning operators, Semigroup, and multilinear maps. Some definitions will also be discussed in this chapter and are very important in the next chapters as Sobolev spaces $W^{s,p}[E]$ and L_p spaces. The main references of this chapter are [5], [66], [8].

Contents

1.1 Operators Notion

1.1.1 Linear operators

Definition 1.1.1

Let X, Y be $\mathbb{K}$-vector spaces, an operator $\mathcal{L} : X \longrightarrow Y$ is said to be linear if

$$\mathcal{L}(\lambda_1 u + \lambda_2 v) = \lambda_1 \mathcal{L}(u) + \lambda_2 \mathcal{L}(v), \qquad \forall u, v \in X, \forall \lambda_1, \lambda_2 \in \mathbb{K}(\mathbb{K} = \mathbb{R}, \mathbb{C})$$

For a linear operator $\mathcal{L}$, we generally write $\mathcal{L}(u)$ as $\mathcal{L}u$.

Let X and Y be normed vector spaces and $\mathcal{L} : X \to Y$ is a linear operator. The following theorem characterized the continuity of a linear operator.

Theorem 1.1.2

The following properties are equivalent:

1. $\mathcal{L}$ is continuous.
2. $\mathcal{L}$ is continuous at 0.
3. There exist a constant c such that

$$\|\mathcal{L}x\|_Y \leq c\|x\|_X, \qquad \forall x \in X \tag{1.1.1}$$

Let X and Y be normed vector spaces.

Definition 1.1.3

A linear operator $\mathcal{L} : X \to Y$ which is continuous is said to be bounded.

Proposition 1.1.4

If the operator $\mathcal{L} : X \longrightarrow Y$ is linear and continuous, we have

$$C = \sup_{u \neq 0, u \in X} \left(\frac{\|\mathcal{L}x\|_Y}{\|u\|_X} \right) = \sup_{\|u\|_X \leq 1} \|\mathcal{L}u\|_Y = \sup_{\|u\|_X = 1} \|\mathcal{L}u\|_Y < +\infty$$

And

$$C = \inf\{\alpha \geq 0, \|\mathcal{L}u\|_Y \leq \alpha\|u\|_X\}$$

the number C is called the norm of the operator $\mathcal{L}$.

1.1.2 Integral operator

1.1.2.1 The Urysohn integral operator

The Urysohn operator and its continuity properties are defined in this section. This operator is, at present, widely used in applications.

Definition 1.1.5

Let $K(s,t,u)$, ($s,t \in \Omega$, $-\infty < u < +\infty$, Ω a measurable subset of $\mathbb{R}^n$) given function of three variables. Then, the operator $\mathcal{U}$ defined as

$$\mathcal{U}(u)(s) = \int\limits_{\Omega} K(s,t,u(t))\,dt$$

is called the Urysohn operator.

The following theorems establish sufficient conditions for the Urysohn operator $\mathcal{U}$ to be continuous and compact on the space C and L^p, respectively, using the fact that in a compact set, the continuous functions are uniformly continuous.

Theorem 1.1.6 (*Ladyzhenskii [54]*)

Suppose that the function

$$K(s,t,u), \qquad (s,t \in \Omega, -\infty < u < +\infty)$$

satisfies the following conditions.

1. $K(s,t,u)$ is continuous with respect to u for almost all $s,t \in \Omega \times \Omega$ and measurable with respect to t for all $s \in \Omega, -\infty < u < +\infty$.
2. for every positive number α

$$\int\limits_{\Omega} \sup_{|u|\leq\alpha} |K(s,t,u)|\,dt < \infty$$

$$\lim_{\|h\|\to 0} \int\limits_{\Omega} \sup_{|u|\leq\alpha} |K(s+h,t,u) - K(s,t,u)|\,dt = 0$$

Then, the Urysohn operator $\mathcal{U}$ acts on C and is continuous and compact.

Theorem 1.1.7 (*Krasnoselskii and Ladyzenskii*)

Let the function $K(s,t,u)$ $(s,t \in \Omega \times \Omega,\ -\infty < u < +\infty$, Ω a bounded closed subset of $\mathbb{R}^n$) be continuous with respect to u and satisfy the inequality

$$|K(s,t,u)| \leq R(s,t)(a + b|u|^{\frac{p}{q}})$$

for all $s,t \in \Omega \times \Omega$, $-\infty < u < +\infty$ with

$$\int_\Omega \int_\Omega |R(s,t)|^p \, ds \, dt < +\infty, \qquad a,b > 0$$

Then, the Urysohn operator $\mathcal{U}$ is a compact and continuous operator on L^p.

1.1.2.2 The Nemytskii operator

Let S be a nonempty set, X and Y be Banach spaces over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and a nonempty open set $U \subset E$, let $\mathbb{G}$, $\mathbb{F}$, and $\mathbb{H}$ be vector spaces[1] of functions acting from S into E, from U into F , and from S into F , respectively.

Given the functions

$$\psi\colon \begin{array}{l} S \times U \longrightarrow F \\ (s,u) \longmapsto \psi(s,u) \end{array}, \qquad g\colon \begin{array}{l} S \longrightarrow U \\ s \longmapsto g(s) \end{array}$$

The Nemytskii operator or a superposition operator N_ψ is defined by

$$N_\psi(g)(s) := (N_\psi g)(s) := \psi(s, g(s)), \qquad s \in S$$

Definition 1.1.8

For a set $V \subset \mathbb{G}$ such that each $g \in V$ has values in U, we say that N_ψ acts from V into $\mathbb{H}$ if $N_\psi \in \mathbb{H}$ for each $g \in V$.

[1] Usually, but not always, $\mathbb{G}$, $\mathbb{F}$, and $\mathbb{H}$ will be normed spaces

1.1.2.3 The Hammerstein integral operator

Theorem 1.1.9

Suppose that the function $\psi(s,u)$ is continuous as a map from $S \times \mathbb{R}$ to $\mathbb{R}$. Then, the Nemytskii operator N_ψ acts on C and is continuous and bounded.

Definition 1.1.10

Let S be a compact subset of $\mathbb{R}$, $k(s,t)$ a kernel defined on $S \times S$, and let $\psi(s,u)$ be as before. Then, the Hammerstein operator $\mathcal{H}$ is defined as

$$\mathcal{H}(u)(s) = \int\limits_S k(s,t)\psi(t,u(t))\,dt$$

If $\mathcal{K}$ is the linear integral operator defined by:

$$\mathcal{K}(u)(s) = \int\limits_S k(s,t)u(t)\,dt$$

then $\mathcal{H}$ can be written in the form $\mathcal{H} = \mathcal{K}N_\psi$.

1.1.3 Differential operators

Let V be an open set of $\mathbb{R}^n$. A differential operator on V is a finite linear combination of derivatives of arbitrary order with coefficients in $C^\infty(V)$. It is said to be of order m if the derivations of higher order do not appear there. In other words, a differential operator of order m on V if it is of the form :

$$L = \sum_{|\beta|\leq m} a_\beta(u)D^\beta \tag{1.1.2}$$

Where $a_\beta \in C^\infty(V)$ are the coefficients of $\mathcal{L}$. And

$$D^\beta = D_1^{\beta_1}D_2^{\beta_2}\dots D_n^{\beta_n}$$

$$D^j = -i\frac{\partial}{\partial x_j}$$

$\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{N}^n$ is a multi-index and $|\beta| = \beta_1 + \beta_2 \cdots + \beta_n$ is its modulus.

Example : *The spaces $C[0,1]$ and $C^1[0,1]$ are associated with their standard norms*

$$\|v\|_{C[0,1]} = \max_{0\leq x\leq 1} |v(x)|$$

And

$$\|v\|_{C^1[0,1]} = \|v\|_{C[0,1]} + \|v'\|_{C[0,1]} \tag{1.1.3}$$

The operator

$$L_1 = \frac{\partial}{\partial x} : C^1[0,1] \subset C[0,1] \longrightarrow C[0,1]$$

is a non-continuous differential operator using the infinite norm of $C[0,1]$. However, it is continuous using the norm (1.1.3).

1.1.4 Compact operators

Definition 1.1.11

Let E and F be two Banach spaces. A linear operator $\mathcal{L}$ from E to F is said to be compact if and only if $\mathcal{L}(B_E)$ is relatively compact, or $\mathcal{L}$ is compact if and only if for any bounded subset M of E, $\mathcal{L}(M)$ is relatively compact

Definition 1.1.12

Let E and F be two normed spaces, the linear operator $\mathcal{L}$ from E into F is an operator of finite rank if, the range of $\mathcal{L}$ of finite dimension.

Definition 1.1.13

Let E be an infinite-dimensional separable Hilbert space. If $(e_n)_{n\in\mathbb{N}}$ is a Hilbert basis of E. we say that an operator $\mathcal{L} \in \mathcal{L}(E)$ is a Hilbert Schmidt operator if the numerical series is $\sum\limits_{n\in\mathbb{N}} \|\mathcal{L}e_n\|^2$ convergent.

Example :

1. *Every continuous operator of finite rank is a compact operator.*
2. *Any Hilbert Schmidt operator is a compact operator.*
3. *The integral operator $\mathcal{L} : C[a,b] \longrightarrow C[a,b]$ with continuous kernel is a compact operator.*
4. *A linear combination $\mathcal{L} = \alpha\mathcal{L}_1 + \beta\mathcal{L}_2$ of compact operators is a compact operator.*

5. *The product $\mathcal{L}_1\mathcal{L}_2$ of two bounded operators $\mathcal{L}_1$ and $\mathcal{L}_2$ is compact if one of the operators $\mathcal{L}_1$ or $\mathcal{L}_2$ is compact.*

1.1.5 Invertible operators

Let E and F be two normed spaces, and let $\mathcal{L} \in \mathcal{L}(E, F)$

Definition 1.1.14

We say that $\mathcal{L}$ is invertible if there exists $B \in \mathcal{L}(F, E)$ such that $\mathcal{L}B = Id_F$ (right invertible) and $B\mathcal{L} = Id_E$ (left invertible), such an operator (when it exists) is unique. it is called the inverse operator of $\mathcal{L}$, and it is denoted by $\mathcal{L}^{-1}$.

Theorem 1.1.15 (*Banach inverse theorem*)

If $\mathcal{L}$ is a continuous linear operator from a Banach space E onto a Banach space F for which the inverse operator $\mathcal{L}^{-1}$ exists, then $\mathcal{L}^{-1}$ is continuous.

Theorem 1.1.16 (*Neumann Series*)

Let $\mathcal{L} \in \mathcal{L}(E)$ such that $\|\mathcal{L}\| < 1$, then $Id_E - \mathcal{L}$ invertible and

$$\Big(Id_E - \mathcal{L}\Big)^{-1} = \sum_{n=0}^{+\infty} \mathcal{L}^n \tag{1.1.4}$$

Remark : *If $\mathcal{L} \in \mathcal{L}(E)$ such that $\|Id_E - \mathcal{L}\| < 1$, then $\mathcal{L}$ invertible and*

$$\Big(\mathcal{L}\Big)^{-1} = \sum_{n=0}^{+\infty} \Big(Id_E - \mathcal{L}\Big)^n$$

1.1.6 Contractive operators

Definition 1.1.17

Let $(X, \|.\|)$ be a normed space and $\mathcal{T} : X \to X$ be a mapping such that $x \in X$ is called a fixed point of $\mathcal{T}$ if $\mathcal{T}(x) = x$.

Definition 1.1.18

$\mathcal{T}$ is called a contraction mapping on $(X, \|.\|)$ if there exists a real number $k \in (0,1)$, such that

$$\|\mathcal{T}(u) - \mathcal{T}(v)\| \leq k\|u - v\|, \forall u, v \in X \tag{1.1.5}$$

1.2 Operators and Banach spaces

Definition 1.2.1

Let E be a normed space, a sequence $\{x_n\}_n \subseteq E$ is called a Cauchy sequence if:

$$\lim_{n,m \longrightarrow +\infty} \|x_n - x_m\| = 0$$

Obviously, any convergent sequence is a Cauchy sequence.

Definition 1.2.2

A normed space is said to be complete if any Cauchy sequence of E converges to an element in E, a complete normed space is called Banach space.

Example : *Let E be a normed space, and F be a Banach space. Then, $\mathcal{L}(E, F)$ is a Banach space.*

Let E and F be Banach spaces.

Definition 1.2.3

Let $D(A)$ be a vector subspace of E. The set $\{(x, Ax); x \in D(A)\} \subset E \times F$ is called the graph of the operator A. It will be denoted $Gr(A)$

Proposition 1.2.4

1. If $D(A) = E$, one verifies that $Gr(A)$ is a vector subspace of $E \times F$.
2. If the operator A is continuous, then the vector subspace $Gr(A)$ is closed.

Theorem 1.2.5 (*Closed graph theorem*)

If the graph of a linear operator $\mathcal{L} : E \longrightarrow F$ is closed in $E \times F$ then the operator $\mathcal{L}$ is continuous.

Theorem 1.2.6 (*Banach fixed-Point Theorem*)

Let $\mathcal{T} : E \to E$ be a contraction mapping on a complete normed space $(E, \|.\|)$. Then, $\mathcal{T}$ has a unique fixed point.

1.3 Semigroup and Exponential Functions

Definition 1.3.1

Let E be a Banach space, a map $\left(\mathbb{R}^+ \ni t \mapsto T(t)\right) \in \mathcal{L}(E)$ is called a one-parameter operator semigroup (or an operator semigroup, or just a semigroup for short), if

$$T(0) = I, \text{and} \qquad T(t+s) = T(t)T(s), \qquad \text{for all} \qquad t, s \geq 0 \tag{1.3.1}$$

Definition 1.3.2

Let E be a Banach space, and $A \in \mathcal{L}(E)$, for each $t \geq 0$ define the operator e^{tA} by

$$e^{tA} := \sum_{k=0}^{+\infty} \frac{t^k A^k}{k!} \tag{1.3.2}$$

Observe that the operator defined by (1.3.2) estimated by

$$\left\|e^{tA}\right\| \leq \sum_{k=0}^{+\infty} \frac{t^k \|A\|^k}{k!} = e^{t\|A\|} \tag{1.3.3}$$

For all $t \geq 0$. Therefore, the series $\sum_{k=0}^{+\infty} \frac{t^k A^k}{k!}$ is absolutely convergent. Due to the Banach space, it is convergent. Thus, the operator e^{tA} is well defined and bounded.

Proposition 1.3.3

For $A \in \mathcal{L}(E)$, the following properties hold for its exponential function $T(t) := e^{tA}$

1. The functional equation

$$T(0) = I, \qquad T(t+s) = T(t)T(s) \tag{1.3.4}$$

   is valid for all $t, s \geq 0$
2. The function $\mathbb{R}^+ \ni t \mapsto T(t)$ is continuous.
3. The function $\mathbb{R}^+ \ni t \mapsto T(t)$ is differentiable and satisfies the differential equation

$$\begin{aligned} \dot{T}(t) &= AT(t) \\ T(0) &= I \end{aligned} \tag{1.3.5}$$

Proof : Essentially, all the statements follow as in the scalar case, once we justify the appropriate operations for the operators

1. By using the Cauchy formula for the product of infinite series, we have

$$\sum_{k=0}^{+\infty} \frac{(t+s)^k A^k}{k!} = \sum_{k=0}^{+\infty} \frac{A^k}{k!} \sum_{m=0}^{k} \frac{k!}{(k-m)!m!} t^{k-m} s^m \tag{1.3.6}$$

$$= \sum_{k=0}^{+\infty} \sum_{m=0}^{k} \frac{A^{k-m} t^{k-m}}{(k-m)!} \cdot \frac{s^m A^m}{m!} \tag{1.3.7}$$

$$= \left(\sum_{k=0}^{+\infty} \frac{t^k A^k}{k!} \right) \cdot \left(\sum_{k=0}^{+\infty} \frac{s^k A^k}{k!} \right) \tag{1.3.8}$$

2. Since equation (1.3.4) implies

$$e^{(t+h)A} - e^{tA} = e^{tA}(e^{hA} - I)$$

it suffices to prove continuity at 0, which follows from

$$\left\| e^{hA} - I \right\| = \left\| \sum_{k=1}^{+\infty} \frac{h^k A^k}{k!} \right\| \leq \sum_{k=1}^{+\infty} \frac{|h|^k \|A\|^k}{k!} = e^{|h| \cdot \|A\|} - 1 \tag{1.3.9}$$

3. By a similar argument as above, it suffices to see that

$$\left\| \frac{e^{hA} - I}{h} - A \right\| = \left\| \sum_{k=2}^{+\infty} \frac{h^{k-1} A^k}{k!} \right\|$$

$$\leq \sum_{k=2}^{+\infty} \frac{|h|^{k-1} \|A\|^k}{k!}$$

$$= \frac{e^{|h| \cdot \|A\|} - 1}{|h|} - \|A\| \longrightarrow 0$$

as $h \longrightarrow 0$

■

The most important property of continuous semigroups is that they are nothing but exponential functions.

Proposition 1.3.4

Let $(T(t))_{t \geq 0}$ be a semigroup that is continuous. Then there is an operator $A \in \mathcal{L}(E)$ such that $T(t) = e^{tA}$.

Proof : Since the function $t \mapsto T(t)$ is continuous and $T(0) = I$, we see that

$$\left\| I - \frac{1}{t_0} \int_0^{t_0} T(s)\, ds \right\| < 1$$

for sufficiently small $t_0 > 0$. So, by the Neumann series (1.1.4), the operator $\frac{1}{t_0} \int_0^{t_0} T(s)\, ds$ is invertible, and hence

$$J(t_0) := \int_0^{t_0} T(s)\, ds$$

is invertible, too. It follows that

$$\begin{aligned} T(t) &= J(t_0)^{-1} J(t_0) T(t) \\ &= J(t_0)^{-1} \int_0^{t_0} T(t+s)\, ds \\ &= J(t_0)^{-1} \int_t^{t+t_0} T(s)\, ds \\ &= J(t_0)^{-1} \Big(J(t+t_0) - J(t) \Big) \end{aligned}$$

holds for all $t \geq 0$.

Since J is the integral of a continuous function, it is differentiable and so is the function $t \mapsto T(t)$. For simplicity, the notation $\dot{T}(0) := A$ is used. Then $A \in \mathcal{L}(E)$ and the functional equation implies that

$$\dot{T}(t) = \lim_{h \longrightarrow 0} \frac{T(t+h) - T(t)}{h} = \lim_{h \longrightarrow 0} \frac{T(h) - I}{h} T(t) = AT(t)$$

for all $t \geq 0$. Hence, T satisfies a linear differential equation of the form

$$\dot{T}(t) = AT(t)$$

with $T(0) = I$. But $S(t) = e^{tA}$ also satisfies the same differential equation. Fix $t > 0$ and consider the function $[0, t] \ni s \mapsto T(s)S(t-s) =: u(s)$. Then u is differentiable and its derivative is given by the product rule

$$\begin{aligned}\frac{\partial}{\partial s}u(s) &= \left(\frac{\partial}{\partial s}T(t-s)\right)S(s) + T(t-s)\left(\frac{\partial}{\partial s}S(s)\right)\\ &= -AT(t-s)S(s) + T(t-s)AS(s)\\ &= 0\end{aligned}$$

and as a result, $T(t) = u(t) = u(0) = S(t)$. ■

1.4 The Class $\mathcal{H}(E_1, E_0)$

Let E_1 and E_0 be Banach spaces. A densely injected Banach couple is a pair of Banach spaces (E_0, E_1) such that

$$E_1 \overset{d}{\hookrightarrow} E_0$$

We denote by $\mathcal{H}(E_1, E_0)$ the set of all $A \in \mathcal{L}(E_1, E_0)$ such that $-A$, considered as a linear operator in E_0 with domain E_1, is the infinitesimal generator of a strongly continuous semigroup $\{e^{-tA}, t \geq 0\}$ on E_0, that is, in $\mathcal{L}(E_0)$.

In order to derive uniform estimates for these semigroups and related operators, it is important to possess quantitative descriptions of $\mathcal{H}(E_1, E_0)$. For this purpose, given $\varkappa \geq 1$ and $\varrho > 0$, and write

$$A \in \mathcal{H}(E_1, E_0, \varkappa, \varrho)$$

if and only if $A \in \mathcal{L}(E_1, E_0)$ with $\varrho + A$ being an isomorphism from E_1 onto E_0 and

$$\varkappa^{-1} \leq \frac{\|(\lambda + A)f\|_0}{|\lambda|\|f\|_0 + \|f\|_1} \leq \varkappa, \qquad Re\lambda \geq \varrho, \qquad f \in \dot{E}_1$$

Where $\|\cdot\|_j$ is the norm in E_j, $j = 1, 2$. Furthermore,

$$\mathcal{H}(E_1, E_0) := \bigcup_{\varkappa \geq 1, \varrho > 0} \mathcal{H}(E_1, E_0, \varkappa, \varrho)$$

Then $\mathcal{H}(E_1, E_0)$ is an open in $\mathcal{L}(E_1, E_0)$.

1.5 Interpolation functor

The pair (E_0, E_1) is said to be an interpolation couple if there exists a locally convex space X such that $E_j \hookrightarrow X$, $j = 0, 1$. In the case $E_0 \cap E_1$ and $E_0 + E_1$ are well-defined Banach spaces. Observe that $E_0 \cap E_1 \doteq E_1$ and $E_0 + E_1 \doteq E_0$ if $E_1 \hookrightarrow E_0$ so X can be chosen to be E_0.

If (E_0, E_1) is an interpolation couple and

$$E_0 \cap E_1 \hookrightarrow E \hookrightarrow E_0 + E_1$$

then E is said to be an intermediate space with respect to (E_0, E_1).

Let B be the category of ($\mathbb{K}$-)Banach spaces. Thus the objects of B are the $\mathbb{K}$-Banach spaces, the morphisms of B are the bounded linear operators and the composition is the usual composition of maps. Denoting by B_1 the category of interpolation couples, that is, the objects of B_1 are the interpolation couples, the morphisms of B_1 are the elements $A \in \mathcal{L}(E_0 + E_1, F_0 + F_1)$ satisfying $A \in \mathcal{L}(E_j, F_j)$, $j = 0, 1.$ where (E_0, E_1) and (F_0, F_1) are interpolation couples, and the composition is the natural composition of maps. We write $A : (E_0, E_1) \longrightarrow (F_0, F_1)$ if (E_0, E_1) and (F_0, F_1) are interpolation couples and A is a morphism of B_1.

Let (E_0, E_1) and (F_0, F_1) be interpolation couples. Then E and F are said to be interpolation spaces with respect to (E_0, E_1) and (F_0, F_1) if E and F are intermediate spaces with respect to (E_0, E_1) and (F_0, F_1) , respectively, and $A \in \mathcal{L}(E, F)$ whenever$A : (E_0, E_1) \longrightarrow (F_0, F_1)$. Moreover, E and F are said to be interpolation spaces of exponent θ, where $0 < \theta < 1$, with respect to (E_0, E_1) and (F_0, F_1) if there exists $c(\theta) \geq 0$ such that

$$\|A\|_{\mathcal{L}(E,E)} \leq c(\theta) \|A\|^{1-\theta}_{\mathcal{L}(E_0,F_0)} \|A\|^{\theta}_{\mathcal{L}(E_1,F_1)}$$

for $A : (E_0, E_1) \longrightarrow (F_0, F_1)$. If $c(\theta) = 1$ then E and F are exact interpolation spaces of exponent θ with respect to (E_0, E_1) and (F_0, F_1).

A covariant functor $\mathfrak{B}$ from B_1 into B is said to be an [exact] interpolation functor [of exponentθ] if, given interpolation couples (E_0, E_1) and (F_0, F_1), it follows that $\mathfrak{B}(E_0, E_1)$ and $\mathfrak{B}(F_0, F_1)$ are [exact] interpolation spaces [of exponent θ] with respect to (E_0, E_1) and (F_0, F_1) and if

$$\mathfrak{B}(A) = A \in \mathcal{L}(\mathfrak{B}(E_0, E_1), \mathfrak{B}(F_0, F_1)), \qquad A : (E_0, E_1) \longrightarrow (F_0, F_1)$$

Example : *Real Interpolation Functors*

Let (E_0, E_1) be an interpolation couple. Given $x \in E_0 + E_1$, define the K-functional by

$$K(t,x) := K(t,x,E_0,E_1) := \inf\{\|x_0\|_{E_0} + t\|x_1\|_{E_1}; \quad x = x_0 + x_1\}$$

for $t > 0$. Put

$$\|x\|_{\theta,q} := \left\|t^{-\theta}K(t,x)\right\|_{L_q(\dot{\mathbb{R}}^+,dt\backslash t)}, \qquad 0 < \theta < 1, \qquad 1 \le q \le \infty$$

and

$$(E_0,E_1)_{\theta,q} := \left(\{xE_0 + E_1 \in; \|x\|_{\theta,q} < \infty\}, \|\ \cdot\ \|_{\theta,q}\right)$$

for $0 < \theta < 1, \qquad 1 \le a \le \infty$. Let

$$\mathfrak{B}_{\theta,q}(E_0,E_1) := (E_0,E_1)_{\theta,q}, \qquad \mathfrak{B}_{\theta,q}(A) := A$$

for $A : (E_0,E_1) \longrightarrow (F_0,F_1)$. Then given any $q \in [1,\infty]$ and $\theta \in (0,1)$, it follows that $\mathfrak{B}_{\theta,q}$ is an exact interpolation functor of exponent θ. Henceforth, we denote it by

$$(\ \cdot\ ,\ \cdot\ )_{\theta,q}$$

and call it the real interpolation functor of exponent θ and parameter q.

Definition 1.5.1 (*Admissible Interpolation Functors*)

Let $0 < \theta < 1$. An admissible interpolation functor, denoted by $(\ \cdot\ ,\ \cdot\ )_\theta$, is an interpolation functor of exponent θ for the category of densely injected Banach couples such that

$$E_1 \qquad \text{is dense in} \qquad (E_0,E_1)_\theta$$

whenever (E_0,E_1) is such a couple.

Observe that the real interpolation functors $(\ \cdot\ ,\ \cdot\ )_{\theta,p}$, $1 \le p < \infty$ is admissible. As an abbreviation

$$E_\theta := (E_0,E_1)_\theta$$

and denote the norm in E_θ by $\|\ \cdot\ \|_\theta$.

1.6 Some useful spaces

In this section, we review the definitions and properties of some useful spaces that will appear in the next chapter. These spaces are necessary for the study of the existence and uniqueness of PBE solutions.

1.6.1 Lebesgue spaces

Let Ω be a domain in $\mathbb{R}^n$, $n \in \mathbb{N}$ and let p be a positive real number. $L_p(\Omega; E)$ (or simply $L_p[E]$) is the class of all measurable $E-$valued functions u defined on Ω for which

$$\left(\int_{\Omega} |u(x)|^p \, dx \right)^{\frac{1}{p}} < \infty \tag{1.6.1}$$

The elements of $L_p[E]$ are thus equivalence classes of measurable functions satisfying 1.6.1. The functional $\|.\|_p$ defined by

$$\|u\|_p = \left(\int_{\Omega} |u(x)|^p \, dx \right)^{\frac{1}{p}} \tag{1.6.2}$$

is a norm in $L_p[E]$.

A function f measurable on Ω is said to essentially bounded on Ω if there is a constant K such that

$$|f(x)| \leq K, \qquad a.e. \quad x \in \Omega$$

The greatest lower bound of such constants K is called the essential supremum of $|f|$ on Ω, and is denoted by $ess \sup_{x \in \Omega} |f(x)|$.

The vector space of all functions f that are essentially bounded on Ω (functions being once again identified if they are equal a.e. on Ω) is denoted by $L_\infty(\Omega, E)$ and it is endowed with the norm

$$\|f\|_\infty = ess \sup_{x \in \Omega} |f(x)|$$

1.6.2 Spaces of Continuous Functions

Let Ω be a domain in $\mathbb{R}^n$, E be a Banach space, and m be a nonnegative integer. Let $C^m(\Omega, E)$ denote the vector space consisting of all functions $f : \Omega \longrightarrow E$ which, together with all their partial derivatives $\partial^m f$ of orders $|\alpha| \leq m$, are continuous on Ω.
As an abbreviation $C^0(\Omega, E) = C(\Omega, E)$ and $C^\infty(\Omega, E) = \bigcap_{m=0}^{\infty} C^m(\Omega, E)$. The subspaces $C_0(\Omega, E)$ and $C_0^\infty(\Omega, E)$ consist of all those functions in $C(\Omega, E)$ and $C^\infty(\Omega, E)$, respectively, that have compact support in Ω.

1.6.3 Spaces of Bounded, (Uniformly) Continuous functions

The space that consists of functions $f \in C^m(\Omega, E)$ for which $\partial^m f$ is bounded on Ω for $|\alpha| \leq m$, is denoted by $BC^m(\Omega, E)$. It is a Banach space endowed with the norm

$$f \mapsto \|f\|_{m,\infty} := \max_{|\alpha| \leq m} \|\partial^m f\|_\infty \tag{1.6.3}$$

If $f \in C(\Omega, E)$ is bounded and uniformly continuous on Ω, then it possesses a unique, bounded, continuous extension to the closure $\bar{\Omega}$ of Ω. The vector space $BUC^m(\Omega, E)$ consists of all those functions $f \in BC^m(\Omega, E)$ for which $\partial^m f$ is bounded and uniformly continuous on Ω. It is a closed subspace of $BC^m(\Omega, E)$, and therefore also a Banach space with the same norm (1.6.3).

1.6.4 Spaces of Hölder continuous functions

Let $0 < \sigma < 1$, we define the space $BUC^{m,\sigma}(\Omega, E)$ (or simply $BUC^{m,\sigma}[E]$) to be the subspace of $BUC^m(\Omega, E)$ consisting of all those functions f for which, for $|\alpha| \leq m$, $\partial^m f$ satisfies in Ω a Hölder condition of exponent σ, that is, there exists a constant K such that

$$|\partial^m f(x) - \partial^m f(y)| \leq |x - y|^\sigma, \qquad x, y \in \Omega$$

It is a Banach space endowed with the norm

$$\|f\|_{\sigma,\infty} := \|f\|_{m,\infty} + \max_{|\alpha| \leq m} [\partial^m f]_{\sigma,\infty}$$

where $[\ \cdot\ ]_{\sigma,\infty}$ is the Hölder seminorm defined by

$$[f]_{\sigma,\infty} := \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\sigma}}, \qquad 0<\sigma<1$$

1.6.5 The space of uniformly Lipschitz continuous functions

The set of all maps from E_1 into E_0, which are uniformly Lipschitz continuous on bounded subsets of E_1 is denoted by $C_b^{1-}(E_1,E_0)$. It is a locally convex space endowed with the family of seminorms

$$f \mapsto \sup_{x\in B} \|f(x)\|_0 + \sup_{x,x\in B, x\neq y} \frac{\|f(x)-f(y)\|_0}{\|x-y\|_1} \tag{1.6.4}$$

where B runs through the family of all bounded subsets of E_1. As an easy consequence of the mean value theorem

$$C_b^1(E_1,E_0) \hookrightarrow C_b^{1-}(E_1,E_0) \tag{1.6.5}$$

where $C_b^1(E_1,E_0)$ and $C^1(E_1,E_0)$ endowed with the topology of uniform convergence of the functions and their first derivatives on bounded subsets of E_1.

1.6.6 Sobolev spaces

The Sobolev space $W^{m,p}[E]$ consists of functions $f\in L_p[E]$ such that for every multi-index α with $|\alpha|\leq m, m\in\mathbb{N}$, the weak (or distributional) derivative $D^\alpha f$ exists and $D^\alpha f\in L_p[E]$. Thus

$$W^{m,p}[E] = \left\{ f\in L_p[E] : D^\alpha f \in L_p[E], \qquad |\alpha|\leq m \right\}$$

If $f\in W^{m,p}[E]$, we define its norm

$$\|f\|_{W^{m,p}} = \begin{cases} \left(\sum\limits_{|\alpha|\leq m} \int\limits_{\Omega} |D^\alpha f|^p \, dx \right)^{1/p}, & 1\leq p< +\infty \\ \sum\limits_{|\alpha|\leq m} ess\sup_{\Omega} |D^\alpha f|, & p=+\infty \end{cases}$$

1.6.7 Sobolev-Slobodeckij spaces

The Sobolev spaces of fractional order (also called Sobolev–Slobodeckij spaces) and some properties are reviewed in this section. Let Ω be an open set of $\mathbb{R}^n$, and let $s \in]0, 1[$ and $p \in [1, +\infty[$ we define the fractional Sobolev space $W^{s,p}[E]$ by :

$$W^{s,p}[E] = \left\{ f \in L_p[E]; \qquad \frac{|f(x) - f(y)|}{|x-y|^{\frac{n}{p}+s}} \in L_p(\Omega \times \Omega) \right\}$$

It is a Banach space endowed by the norm:

$$f \mapsto \|f\|_{s,p} := \left(\|f\|_{m,p}^p + \sum_{|\alpha|=m} [\partial^\alpha f]_{s,p}^p \right)^{\frac{1}{p}}$$

with

$$[f]_{s,p} = \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}}$$

Remark :

1. *$W^{-s,1}[E]$ is the space consists of all E-valued distributions f on $\mathbb{R}^n$ such that there exist $f_\alpha \in W^{m-s,1}[E]$ for $|\alpha| \leq m$, satisfying*

$$f = \sum_{|m| \leq m} \partial^\alpha f_\alpha \tag{1.6.6}$$

2. *It is a Banach space with the norm*

$$f \mapsto \|f\|_{-s,1} := \inf \left(\sum_{|m| \leq m} \|f_\alpha\|_{m-s,1} \right) \tag{1.6.7}$$

   where the infimum is taken over all representation (1.6.6).

3. *It follows that*

$$W^{s,1}[E] \overset{d}{\hookrightarrow} W^{t,1}[E], \qquad -\infty < t < s < \infty \tag{1.6.8}$$

   where $\hookrightarrow$ denotes "continuous injection" and "d" stands for "dense".

4. *Moreover,*

$$\partial^\alpha \in \mathcal{L}\big(W^{s+|\alpha|,1}[E], W^{s,1}[E]\big), \qquad \alpha \in \mathbb{N}^n, \qquad s \in \mathbb{R} \tag{1.6.9}$$

1.7 Multilinear Maps

Let $E, E_1, E_2, \dots, E_m$ be Banach spaces. A map M

$$M : E_1 \times E_2 \times \cdots \times E_m \longrightarrow E_0$$

is called **Multilinear** [2] if it is linear in each variable, that is, if all maps

$$M(e_1, e_2, \dots, e_{i-1}, ., e_{i+1}, \dots, e_m) : E_i \longrightarrow E_0$$

are linear.
The Banach space $\mathcal{L}(E_1, E_2, \dots, E_m; E_0)$ is the space of all continuous $m-$linear maps from $E_1 \times E_2 \times \cdots \times E_m$ into E_0, And

$$\mathcal{L}^m(E; E_0) := \mathcal{L}(E_1, E_2, \dots, E_m; E_0) \qquad if \qquad E_1 = E_2 = \cdots = E_m = E$$

Moreover, $\mathcal{L}(E; E_0) := \mathcal{L}^1(E; E_0)$ and $\mathcal{L}(E) := \mathcal{L}(E; E)$. Elements of $\mathcal{L}(E_1, E_2, \dots, E_m; E_0)$ are sometimes simply denoted by

$$(e_1, e_2, \dots, e_m) \mapsto e_1 \bullet e_2 \bullet \cdots \bullet e_m \tag{1.7.1}$$

which are referred to as multiplications. For $u_i \in E_i^{\mathbb{R}^n}$, the point-wise product induced by (1.7.1) is defined by

$$u_1 \bullet u_2 \bullet \cdots \bullet u_m(z) := u_1(z) \bullet u_2(z) \bullet \cdots \bullet u_m(z), \qquad z \in \mathbb{R}^n \tag{1.7.2}$$

Let $\mathfrak{B}_i[E_j]$ be Banach spaces of E_j-valued functions on $\mathbb{R}^n$ for $0 \le i \le m$. Then we write

$$\mathfrak{B}_1[E_1] \bullet \cdots \bullet \mathfrak{B}_m[E_m] \hookrightarrow \mathfrak{B}_0[E_0]$$

if the point-wise product (1.7.2) defines a continuous m-linear map

$$\mathfrak{B}_1[E_1] \times \cdots \times \mathfrak{B}_m[E_m] \hookrightarrow \mathfrak{B}_0[E_0], \qquad (u_1, \dots, u_m) \mapsto u_1 \bullet u_2 \bullet \cdots \bullet u_m$$

[2] Also called $m-$linear

The point-wise multiplication induced by (1.7.1).

Some properties of point-wise multiplication are mentioned in The following lemma:

Lemma 1.7.1

(i) Suppose $E_1 \times E_2 \to E_0$, $(e_1, e_2) \mapsto e_1 \bullet e_2$ is a multiplication.
Then

$$BUC^s[E] \bullet W^{t,p}[E_2] \hookrightarrow W^{t,p}[E_0], \qquad 0 \leq t < s < \infty, \qquad 1 \leq p < \infty$$

(ii) Suppose $E_1 \times E_2 \times E_3 \to E_0$, $(e_1, e_2, e_3) \mapsto e_1 \bullet e_2 \bullet e_3$ is a multiplication.
Then

$$BUC^r[E] \bullet W^{s,p}[E_2] \bullet W^{s,p}[E_3] \hookrightarrow W^{t,p}[E_0]$$

provided $n \leq t + n < 2s < 2n$ and $t < r$, and

$$BUC^r[E] \bullet W^{s,p}[E_2] \bullet W^{s,p}[E_3] \hookrightarrow W^{s,p}[E_0]$$

if $n < s < r$.

Proof : See [18]. ■

2

Modelling of Population Balances

This chapter is devoted to the formulation of the population balance equation model. Also, included a review concerning the existence and uniqueness of the PBE's solution. In addition, a study of the convergence of ADM applied to PBE for fragmentation and Aggregation is presented.

Contents

2.1 Introduction

Population balance models (PBMs) are similar to the well-known mass and energy balance models. They are encountered in numerous scientific and engineering disciplines. They can be used to describe the time evolution of one or more property distributions of an individual's population. In 1964, **Hulburt** and **Katz** [50] introduced them to the field of chemical engineering as well as **Randolph** and **Larson** [72]. In the late seventies, **Ramkrishna** [70] reviewed them.

Population balance equations (PBEs) describe a balance law for the number of individuals in a population, such as crystals, droplets, bacteria,... etc. The diversity of phenomena responsible for the change in the population of individuals is what makes PBEs more interesting than mass balance equations. Fluid flow induces the inflow and outflow of particles from a given control volume. In addition to them, there are several other mechanisms that are responsible for the change in particle population in the same control volume.

Due to the aforementioned phenomena, the description of the dynamic behavior of the particulate processes essentially involves specifying the temporal change of the particle property distribution. This distribution is a part of the system state. Hence, particulate processes are inherently distributed parameter systems. PBMs are usually used to model this class of systems.

Apart from particle-particle interactions, the dispersed phase usually also interacts with its environment, e.g., the continuous phase in crystallization. The state of the continuous phase may influence the rate of growth, birth and death processes and thus affects the particle population. In the other direction, the dispersed particle phase generally affects the continuous phase, e.g., by mass transfer from liquid to solid due to growth in crystallization or by heat transfer due to the heat transfer of crystallization. Therefore, in general, a model for a particulate system consists of a population balance equation, which describes the dispersed phase, coupled with a mass (or mole) balance, and an energy balance, which represents the continuous phase. A typical dispersed two-phase system is shown in Figure 2.1 **Motz et al.** [64].

Historically, In 1917, **Smoluchowski** introduced under the basic assumption that binary collisions occur simultaneously an infinite set of nonlinear ordinary differential equations known **the discrete Smoluchowski equation** [76, 77]

$$\frac{\partial f_i(t)}{\partial t} = \frac{1}{2}\sum_{j=1}^{i-1} \omega_{i-j,j} f_{i-j}(t) f_j(t) - f_i(t)\sum_{j=1}^{+\infty} \omega_{i,j} f_j(t), \quad i = 1, 2, \ldots \tag{2.1.1}$$

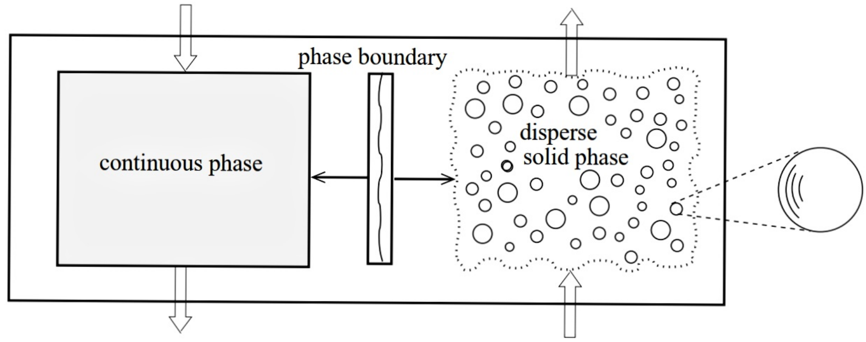

Figure 2.1: An illustration of a dispersed two-phase system

With initial conditions

$$f_i(0) = f_i^{(0)} \geq 0 \tag{2.1.2}$$

Where $\omega_{i,j}$ are to be non-negative and symmetric, i.e. $\omega_{i,j} \geq 0$ and $\omega_{i,j} = \omega_{j,i}, \quad \forall i,j \geq 1$, these functions $\omega_{i,j}$ are called coagulation kernel, it describes the intensity interaction between particles of mass i and j and is assumed to be known function. The unknown function $f_i(t)$ is the concentration of particles with mass i, $i \geq 1$ at time t.

Figure 2.2: Marian Smoluchowski (1872 -1917) was a Polish physicist.

A number of modifications have been made to the discret Smoluchowski model. In 1928, **Müller** [65] rewrote equation (2.1.1) as an integro-differential equation to represent coagulation. As a result, it is called the **continuous Smoluchowski equation** because the size variable is allowed to be any positive real number. Then, the original model (2.1.1) becomes

$$\frac{\partial f(t,v)}{\partial t} = \frac{1}{2}\int_0^v \omega(v-v',v')f(t,v-v')f(t,v')\,dv' \\ -f(t,v)\int_0^{+\infty} \omega(v,v')f(t,v')\,dv', \quad (t,v) \in (0,+\infty)^2 \tag{2.1.3}$$

$$f(0,v) = f_0(v), \quad v \in (0,+\infty) \tag{2.1.4}$$

✓ $f(t,v)$: is the particle mass density function.

✓ $f(t,v)dv$: represents the average number of particles per unit volume at time t whose masses lie between v and $v+dv$.

✓ $\omega(v,v')$: is the coagulation kernel.

✓ $\omega(v,v')f(t,v')f(t,v)\,dv\,dv'\,dt$: the number of coalescences between particles of mass v to $v+dv$ and those of mass v' to $v'+dv'$ during the time interval $(t,t+dt)$.

In 1957, **Melzak** developed an extension of the continuous Smoluchowski model when particles breakdown. By Combining binary coagulation with multiple fragmentation, he proposed the following equation [62]:

$$\begin{aligned}\frac{\partial f(t,v)}{\partial t} &= \frac{1}{2}\int_0^v \omega(v-v',v')f(t,v-v')f(t,v')\,dv' - f(t,v)\int_0^{+\infty}\omega(v,v')f(t,v')\,dv' \\ &+ \int_v^{+\infty}\varphi(v',v)f(t,v')\,dv' - \frac{f(t,v)}{v}\int_0^v v'\varphi(v,v')\,dv', \quad (t,v)\in(0,+\infty)^2\end{aligned} \tag{2.1.5}$$

✓ $\varphi(v,v')\geq 0$: is the multiple breakage kernel

✓ $f(t,v)\varphi(v,v')\,dv\,dv'\,dt$: is the average number of particles of mass v' to $v'+dv'$ created from the breakdown of particles of mass v to $v+dv$, during the time interval $(t,t+dt)$

✓ If $v'>v$, then $\varphi(v,v')=0$.

✓ The third integral describes the formation of particles of mass v from the breakdown of particles of mass v' $(v\leq v'<+\infty)$.

✓ The fourth integral indicates the disappearance of particles of mass v as a result of their breakdown into particles of mass v' $(0\leq v'\leq v)$.

In 1960, The case of binary fragmentation was considered by **Friedlander** [43]. Therefore, the coagulation-fragmentation equation was given by:

$$\begin{aligned}\frac{\partial f(t,v)}{\partial t} &= \frac{1}{2}\int_0^v \omega(v-v',v')f(t,v-v')f(t,v')\,dv' - f(t,v)\int_0^{+\infty}\omega(v,v')f(t,v')\,dv' \\ &+ \int_0^{+\infty}F(v,v')f(t,v+v')\,dv' - \frac{f(t,v)}{2}\int_0^v F(v-v',v')\,dv', \quad (t,v)\in(0,+\infty)^2\end{aligned} \tag{2.1.6}$$

The above equation can be found from **Melzak's Model** (2.1.5) by taking into consideration that each particle can only break into two new particles, then $\varphi(v, v') = \varphi(v, v - v')$ and equation (2.1.5) may be rewritten as (2.1.6) in which the fragmentation kernel F becomes $F(v - v', v') = \varphi(v, v')$. It is important to mention that F is a symmetric function, i.e.

$$F(v, v') = \varphi(v + v', v') = \varphi(v + v', v) = F(v', v)$$

in contrast with the breakdown function φ. All functions concerned are non-negative.

In 1991, **Ziff** [82] gave another form of the multiple breakage equation by taking

$$\varphi(v', v) = \beta(v, v')\Gamma(v'), \qquad \text{and} \qquad \Gamma(v) = \int_0^v \frac{v'}{v}\varphi(v, v')\, dv' \tag{2.1.7}$$

where β and Γ are described in the next section.

Due to the importance of solving the PBEs, several techniques have been proposed to solve them. **Attarakih et al.** [24], **Kopriwa et al.** [52], and **Su et al.** [79] reviewed several solution methods for solving PBE. Numerical techniques can be classified into the following categories: moment method, stochastic, high-order, zero-order methods, and analytical methods. To facilitate their classification, they are schematically represented in Figure 2.3.

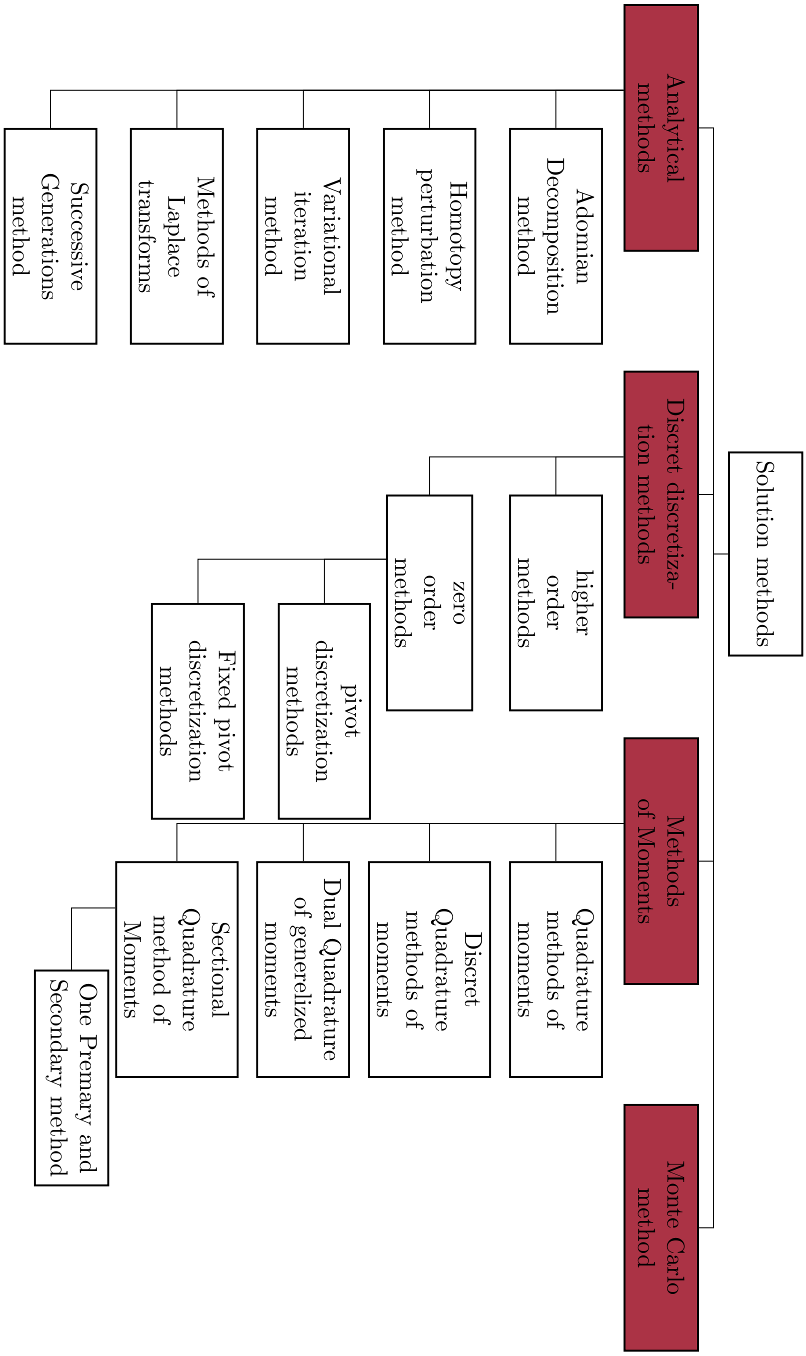

Figure 2.3: Analytical and numerical methods for solving the population balance equation.

2.2 Particulate Processes

In the following, we give a brief description of different particulate processes and their corresponding population balance models.

2.2.1 Nucleation and Growth processes

Nucleation process is the process of new particle formation in a supersaturated solution. During this process, the population of small particles increases. The nuclei are usually considered the smallest possible particles in the system. In practical applications, such as crystallization, nucleation is assumed to take place at the minimum particle size due to problems in particle size measurement in this range. Furthermore, in this size range, it is not possible to distinguish between nuclei of different sizes due to the insufficient resolution of measuring devices. Through growth and agglomeration, these particles become visible.

The particles grow when molecular matter is added to the surface of a particle. During the growth process, the total number of particles remains the same, but the total volume (mass) of particles increases. The size of a particle increases continuously in this process.

Growth and nucleation processes are very common in a wide range of particulate processes. The crystallization process is one example of such a process. The population balance equation in this case has the form

$$\frac{\partial f(t,v)}{\partial t} = -\frac{\partial \big[G(t,v)f(t,v)\big]}{\partial v} + \mathcal{R}_{nuc}(t,v), \quad (t,v) \in (0,+\infty)^2 \tag{2.2.1}$$

Where G represents the growth velocity and the nucleation term is defined as

$$\mathcal{R}_{nuc}(t,v) = f_{nuc}(t,v)\Gamma_{nuc}(t) \tag{2.2.2}$$

In the reaction term due to the nucleation process $\mathcal{R}_{nuc}$, $f_{nuc} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{+} \longrightarrow \mathbb{R}_{\geq 0}$ represents the number density of nuclei while $\Gamma_{nuc}(t) \in \mathbb{R}_{\geq 0}$ is the nucleation rate.

This is a hyperbolic equation with a source term. If the nucleation term on the right-hand side is zero, then the above equation is a homogenous hyperbolic equation for modeling a pure growth.

2.2.2 Aggregation process

Aggregation is a nonlinear phenomenon that appears in a large class of applications, e.g., in physics (aggregation of colloidal particles), meteorology (merging of drops in atmospheric clouds, aerosol transport, minerals), chemistry (reacting polymers, soot formation, pharmaceutical industries, fertilizers).

In an aggregation process, two or more particles combine to form a large particle. The total number of particles decreases in an aggregation process while mass remains conserved.

The PBE for the aggregation process is stated in its general form as

$$\frac{\partial f(t,v)}{\partial t} = c(t,v,f), \quad (t,v) \in (0,+\infty)^2 \tag{2.2.3}$$

Where

$$c(t,v,f) = \underbrace{\frac{1}{2}\int_0^v \omega(v-v',v')f(t,v-v')f(t,v')\,dv'}_{\substack{\text{Birth due}\\ \text{to Aggregation}}} - \underbrace{f(t,v)\int_0^{+\infty} \omega(v,v')f(t,v')\,dv'}_{\substack{\text{Death due}\\ \text{to Aggregation}}} \tag{2.2.4}$$

The aggregation kernel $\omega(v,v') \geq 0$ gives the rate at which particles of size v aggregate with particles of size v'. If binary aggregation is the case, then the kernel ω is symmetric, i.e.,

$$\omega(v,v') = \omega(v',v), \quad (v',v) \in (0,+\infty)^2 \tag{2.2.5}$$

Birth term: expresses the fact that a particle of size v can only come into existence if two particles of volumes $v-v'$ and v' aggregate. The factor $\frac{1}{2}$ guarantees that each combination is counted only once.

Death term: This term says that a particle of size v disappears from "level v" if it aggregates with a cluster of any volume.

Remark : *Set $\mathbb{V}$ and K_{coag} as*

$$\mathbb{V} := L_1(\mathcal{V},(1+v)dv) = L_1(\mathcal{V},dv) \cap L_1(\mathcal{V},vdv)$$

K_{coag} : the closed linear subspace of $L_\infty(\mathcal{V}^2,d^2v)$ consisting of all ω satisfying

$$\omega(v,v') = \omega(v',v), \qquad a.e. \quad v,v' \in \mathcal{V}$$

Given $\omega \in K_{coag}$ *and for* $f, g \in \mathbb{V}$*, we put*

$$c_\omega(f,g)(v) = \frac{1}{2}\int_0^v \omega(v-v', v')f(v-v')g(v')\,dv'$$
$$-f(v)\int_0^{+\infty} \omega(v,v')g(v')\,dv', \quad a.e. \quad v \in \mathcal{V}$$

where $\mathcal{V}$ *represents the support of* dv*.*

1. *It is easily verified that*

$$\Big((\omega, f, g) \mapsto c_\omega(f,g)\Big) \in \mathcal{L}(K_{coag}, \mathbb{V}, \mathbb{V}; \mathbb{V}) \tag{2.2.6}$$

2. *Also* $c_\omega(f,g)$ *satisfying*

$$\int_{\mathcal{V}} c_\omega(f,f)\,dv = -\frac{1}{2}\int_{\mathcal{V}^2} \omega f \otimes f\, d^2v \tag{2.2.7}$$

$$\int_{\mathcal{V}} c_\omega(f,f)v\,dv = 0 \tag{2.2.8}$$

   for $f \in \mathbb{V}$*.*

3. *If* (ω, f, g) *maps* $\mathbb{R}^n$ *into* $K_{coag} \times \mathbb{V} \times \mathbb{V}$*, then*

$$c_\omega(f,g)(z) := c_{\omega(z)}(f(z), g(z)), \qquad z \in \mathbb{R}^n \tag{2.2.9}$$

2.2.3 Breakage process

Breakage is the process by which particles of larger sizes break into two or more fragments. Unlike the aforementioned aggregation process, The total number of particles in a breakage process increases while the total volume (mass) remains conserved.

Population balances for breakage are widely known in high shear granulation, crystallization, atmospheric science and many other particle related engineering problems. The general form of population balance equation for breakage process is given as

$$\frac{\partial f(t,v)}{\partial t} = b(t,v,f), \qquad (t,v) \in (0,+\infty)^2 \tag{2.2.10}$$

Where

$$b(t,v,f) = -\underbrace{\Gamma(v)f(t,v)}_{\substack{\text{Death due}\\ \text{to Breakage}}} + \underbrace{\int_{v}^{+\infty} \beta(v,v')\Gamma(v)f(t,v')\,dv'}_{\substack{\text{Birth due}\\ \text{to Breakage}}} \tag{2.2.11}$$

It is assumed that only a single size variable, such as particle mass, is required to differentiate between the reacting particles, with $f(t,v)$ regarded as a density function, denoting the density of particles of size $v > 0$ at time t.

In the Eq (2.2.11) $\Gamma(v)$ represents the overall rate of fragmentation of a v-sized particle. The coefficient $\beta(v,v')$, often called the fragmentation kernel or daughter distribution function, plays a key role in the model. More precisely, it is the distribution function of the sizes of the daughter particles. Roughly speaking, $\beta(v,v')$ gives the number of v-size particles produced by the fragmentation of a v'-size particle. In most investigations into (2.2.11), this daughter distribution function β is assumed to be non-negative and measurable, with $\beta(v,v') = 0$ for $v > v'$ and it has the following important properties:

$$\int_0^v \beta(v,v')\,dv' = \overline{N}(v), \qquad \int_0^v v'\beta(v,v')\,dv' = v \tag{2.2.12}$$

For each $v > 0$, $\overline{N}(v)$ represents the number of fragments obtained from the breakage of a particle of size v. While the second integral ensures the property that the total mass created from the breakage of a particle of size v is again v.

Birth term: accounts for the production of particles of size v by the breakup of particles of larger volumes.

Death term: takes care of the disappearance of v-particles by their fragmentation into smaller ones.

Remark : *Set* $\mathcal{V}_\Delta^2 := \{(v,v') \in \mathcal{V}^2; 0 \le v' \le v\}$ *and define*

$$(\varphi \mapsto \Phi_\varphi) \in \mathcal{L}\Big(L_\infty(\mathcal{V}_\Delta^2, d^2v), L_{1,loc}(\mathcal{V}, dv)\Big)$$

by

$$\Phi_\varphi(v) := \frac{1}{v}\int_0^v \varphi(v,v')v'\,dv', \qquad a.e. \quad v \in \mathcal{V}$$

Then

$$K_{frag} := \{\varphi \in L_\infty(\mathcal{V}^2_\Delta, d^2v); \Phi_\varphi \in L_\infty(\mathcal{V}, dv)\}$$

is a Banach space with the norm $\varphi \mapsto \|\varphi\|_\infty + \|\Phi_\varphi\|_\infty$. *For each* $\varphi \in K_{frag}$

$$f_\varphi(g)(v) := \int_v^\infty \varphi(v', v)g(v')\,dv' - \Phi_\varphi(v)g(v), \qquad g \in \mathbb{V}, \qquad a.e. \qquad v \in \mathcal{V}$$

1. *It is obvious that*
$$\Big((\varphi, g) \mapsto f_\varphi(g)\Big) \in \mathcal{L}(K_{frag}, \mathbb{V}; \mathbb{V}) \tag{2.2.13}$$

2. *It is also easy to see that*
$$\int_{\mathcal{V}} f_\varphi(g)\,dv = \int_{\mathcal{V}} \int_0^v (1 - v'/v)\varphi(v, v')\,dv' g(v)\,dv \tag{2.2.14}$$
and
$$\int_{\mathcal{V}} f_\varphi(g) v\,dv = 0 \tag{2.2.15}$$
for $\varphi \in K_{frag}$ *and* $g \in \mathbb{V}$.

3. *If* (φ, g) *maps* $\mathbb{R}^n$ *into* $K_{frag} \times \mathbb{V}$, *then*
$$f_\varphi(g)(z) := f_{\varphi(z)}(g(z)), \qquad z \in \mathbb{R}^n \tag{2.2.16}$$

The aforementioned particulate processes are schematically represented in Figure 2.4.

2.3 Formulation of PBEs

THis section deals with the mathematical foundation of coagulation-fragmentation processes, taking into account the movement of the particles due to diffusion and superimposed transport processes. Formally, the equations under consideration take the form of an initial value problem of the reaction-diffusion type:

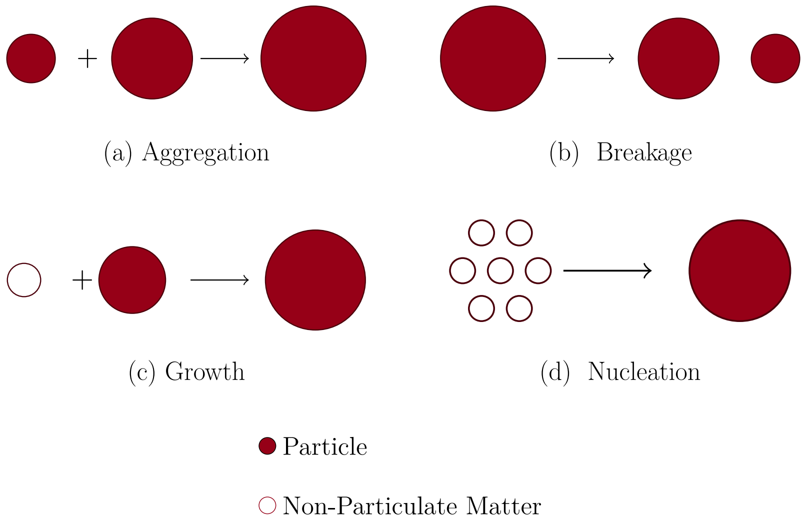

Figure 2.4: Different particle formation mechanisms

$$\partial_t f + \mathcal{A}(t,v,z)f = r(t,v,z,f), \qquad z \in \mathbb{R}^n, \qquad t > 0 \tag{2.3.1}$$

$$f(0,v,z) = f^0(v,z), \qquad z \in \mathbb{R}^n \tag{2.3.2}$$

The PBE uses a density function defined in phase space in order to balance a population of particles that may evolve through the influence of particle-particle and particle-continuous phase interactions. The particle phase space constitutes the internal and external particle coordinates.

- **External coordinates** z: refer to the spatial distribution of the particles ($z \in \mathbb{R}^n$, here $n = 1, 2$ or 3).

- **Internal coordinates** v: is one or more property of the particles in the population, such example is particle size (i.e., volume, diameter, and mass). Other examples of particle properties are chemical composition, energy content, age, chemical activity, etc.

In (2.3.1) $\mathcal{A}$ are diffusion-convection operators, r is the reaction term that describes kinetic behavior of the process, and f is the particle-size distribution function

$$f(t,v,z) \geq 0 \tag{2.3.3}$$

And

$$\int_Z \int_{v_0}^{v_1} f(t,v,z)\, dv\, dz \tag{2.3.4}$$

is the total number of particles with volumes belonging to the interval $[v_0, v_1] \subset \mathbb{R}_+$ and being at time t contained in the space region $Z \subset \mathbb{R}^n$. The measure dv is either Lebesgue's measure on $\mathbb{R}_+$ or the counting measure on $\mathbb{N}^*$={1,2,3,...}. The reaction term consists of three terms:

$$r(t,v,z,f) = c(t,v,z,f) + b(t,v,z,f) + h(t,v,z) \tag{2.3.5}$$

accounting for coagulation, fragmentation, and particle input, respectively, where the terms expressing coagulation and fragmentation were defined in the previous section.

$$\begin{aligned} c(t,v,z,f) = \frac{1}{2}\int_0^v & \omega(t,v-v',v',z) f(t,v-v',z) f(t,v',z)\, dv' \\ & - f(t,v,z)\int_0^{+\infty} \omega(t,v,v',z) f(t,v',z)\, dv' \end{aligned} \tag{2.3.6}$$

And

$$b(t,v,z,f) = \int_v^{+\infty} \varphi(t,v',v,z) f(t,v',z)\, dv' - \Phi_\varphi(t,v,z) f(t,v,z) \tag{2.3.7}$$

where the fragmentation kernel satisfies

$$0 \le \varphi(t,v',v,z), \qquad 0 < v' \le v < \infty, \qquad v, v' \in \mathcal{V} \tag{2.3.8}$$

and

$$\Phi_\varphi(t,v,z) := \frac{1}{v}\int_0^v v' \varphi(t,v,v',z)\, dv', \qquad v \in \mathcal{V} \tag{2.3.9}$$

Lastly, the source term satisfies

$$h(t,v,z) \ge 0, \qquad v \in \mathcal{V} \tag{2.3.10}$$

and accounts for creation of particles of size v at time t and position z due to particle input, as example.

The operator $\mathcal{A}$ defined as

$$\begin{aligned} \mathcal{A}(t,v,z)f := -div_z\Big(d(t,v,z) grad_z f + \vec{a}(t,v,z) f \Big) \\ + \vec{b}(t,v,z) grad_z f + a_0(t,v,z) f \end{aligned} \tag{2.3.11}$$

Here,

- $d(t,v,z)$: is the diffusion matrix.
- $\vec{a}$, $\vec{b}$: are drift vectors. the first describes the particle transport due to external forces such as gravitational, electrical or thermal fields. While the second is produced by temperature gradients in the gas or fluid in which the particles are being suspended.
- a_0 : is the absorption rate.

are sufficiently smooth functions of (t,z) and measurable with respect to v
The following lemma establishes the continuity properties of the maps (2.2.9), (2.2.16).

Lemma 2.3.1

(i) If $0 \le \tau < r < n$ and $\tau + n < 2\sigma < 2n$, then

$$\Big((\omega, f, g) \mapsto c_\omega(f,g)\Big) \in \mathcal{L}(BUC^r[K_{coag}], W^{\sigma,1}[\mathbb{V}], W^{\sigma,1}[\mathbb{V}]; W^{\tau,1}[\mathbb{V}])$$

If $n < \tau < r < \infty$, then

$$\Big((\omega, f, g) \mapsto c_\omega(f,g)\Big) \in \mathcal{L}(BUC^r[K_{coag}], W^{\tau,1}[\mathbb{V}], W^{\tau,1}[\mathbb{V}]; W^{\tau,1}[\mathbb{V}])$$

(ii) If $0 \le \tau < r < \infty$. Then

$$\Big((\varphi, g) \mapsto f_\varphi(g)\Big) \in \mathcal{L}(BUC^r[K_{frag}], W^{\tau,1}[\mathbb{V}]; W^{\tau,1}[\mathbb{V}])$$

Proof : Using (2.2.9) and (2.2.16) the maps

$$(\omega, f, g) \mapsto c_\omega(f,g), \qquad \text{and} \qquad (\varphi, g) \mapsto f_\varphi(g)$$

are point-wise multiplication induced by (2.2.6) and (2.2.13) respectively. Hence the assertion is a consequence of Lemma 1.7.1. ■

Throughout the rest of this chapter I denotes a closed subinterval of $\mathbb{R}^+$ containing 0 and more than one point. For each subinterval I' of I we put $\dot{I}' := I' \backslash \{0\}$. Moreover, $\tau^+ := \tau \vee 0$ for $\tau \in \mathbb{R}$.

Corollary 2.3.2

Suppose that $\tau \in (-1, r) \backslash \mathbb{N}$ with $r > 0$ and that

$$\tau^+ + n < 2\sigma < 2n, \qquad \text{if} \qquad \tau < n$$

Whearas

$$\sigma := \tau, \qquad \text{if} \qquad \tau > n$$

Also suppose that

$$\Big(t \mapsto \big(\omega(t), \varphi(t)\big)\Big) \in C^{\rho}\big(I, BUC^{r}[K_{coag} \times K_{frag}]\big), \qquad \text{for some} \qquad \rho \in \mathbb{R}^{+}$$

Then,

$$\Big(t \mapsto \big(c_{\omega(t)}, f_{\varphi(t)}\big)\Big) \in C^{\rho}\Big(I, \mathcal{L}^{2}(W^{\sigma,1}[\mathbb{V}], W^{\tau,1}[\mathbb{V}]) \times \mathcal{L}(W^{\sigma,1}[\mathbb{V}], W^{\tau,1}[\mathbb{V}])\Big)$$

In the following, setting

$$\chi(t,z) := \chi(t)(z) \qquad \text{and} \qquad \chi(t,v,v',z) := \chi(t,z)(v,v')$$

for $\chi \in \{\omega, \varphi\}$, $(t,z) \in I \times \mathbb{R}^{n}$, and $(v,v') \in \mathcal{V} \times \mathcal{V}$. we also put

$$c(t,v,z,f) := c_{\omega(t,z)}(f,f)(v), \qquad b(t,v,z,f) := f_{\varphi(t,z)}(f)(v)$$

for $(t,z) \in I \times \mathbb{R}^{n}$, $v \in \mathcal{V}$, and $f \in \mathbb{V}$. Finally, $C(t,\ \cdot\ )$ and $B(t,\ \cdot\ )$ denote the Nemyteskii operators induced by $c(t,\ \cdot\ ,\ \cdot\ ,\ \cdot\ )$ and $b(\ \cdot\ ,\ \cdot\ ,\ \cdot\ )$ respectively, That is,

$$C(t,f)(z) := c(t,\ \cdot\ ,z,f(z)), \qquad B(t,f)(z) := b(t,\ \cdot\ ,z,f(z))$$

for $f : \mathbb{R}^{n} \to \mathbb{V}$ and $(t,z) \in I \times \mathbb{R}^{n}$. Then it follows that, given the hypotheses of Corollary 2.3.2,

$$\Big(t \mapsto C(t,\ \cdot\ ) + B(t,\ \cdot\ )\Big) \in C^{\rho}\Big(I, C_{b}^{\infty}(W^{\sigma,1}[\mathbb{V}], W^{\tau,1}[\mathbb{V}])\Big) \tag{2.3.12}$$

where $C_{b}^{\infty}(E_1, E_0)$ is the vector space $C^{\infty}(E_1, E_0)$ endowed with the topology of uniform convergence of all derivatives on bounded subsets of E_1.

2.4 The Diffusion-Convection Semigroup

Set for $k \in \dot{\mathbb{N}}$

$$L_{\infty}^{k} := L_{\infty}(\mathcal{V}, dv; \mathbb{R}^{k})$$

And

$$L_{\infty,sym}^{n\times n} := L_\infty(\mathcal{V}, dv; \mathbb{R}_{sym}^{n\times n})$$

Where $\mathbb{R}_{sym}^{n\times n}$ is the space of all symmetric $(n \times n)$-matrices. then,

$$\mathcal{A}[e]f := -\boldsymbol{\nabla} \cdot (d\boldsymbol{\nabla} \cdot f + \vec{a}f) + \vec{b}.\boldsymbol{\nabla} \cdot f + a_0 f, \qquad e := (d, \vec{a}, \vec{b}, a_0) \in \mathbb{E}^\sigma$$

where for a given $\sigma \in \mathbb{R}^+$,

$$\mathbb{E}^\sigma := BUC^\sigma[L_{\infty,sym}^{n\times n}] \times BUC^\sigma[L_\infty^n] \times BUC^\sigma[L_\infty^n] \times BUC^{(\sigma-1)^+}[L_\infty^1]$$

In the following, we write $e(z, v)$ for $e(z)(v)$ for $z \in \mathbb{R}^n$ and $v \in \mathcal{V}$

Lemma 2.4.1

If $-1 \leq s < \sigma - 1 < \infty$. Then

$$\Big(e \mapsto \mathcal{A}[e]\Big) \in \mathcal{L}\Big(\mathbb{E}^\sigma, \mathcal{L}(W^{s+2,1}[\mathbb{V}], W^{s,1}[\mathbb{V}])\Big)$$

Proof : First observe that

$$L_\infty(\mathcal{V}, dv) \times \mathbb{V} \longrightarrow \mathbb{V}, \qquad (a, f) \mapsto af$$

with $af(v) := a(v)f(v)$ for a.e. $v \in \mathcal{V}$, is a multiplication. Now the assertion is an easy consequence of (1.6.8), (1.6.9), and Lemma 1.7.1. ■

Now after these preparations, we can formulate the following basic generation result:

Theorem 2.4.2

Suppose that $s \in (-1, \infty)\backslash\mathbb{N}$ with $\sigma > s + 1$, and $\alpha, M > 0$. Then there exist $\varkappa \geq 1$ and $\varrho > 0$ such that

$$\mathcal{A}[e] \in \mathcal{H}\Big(W^{s+2,1}[\mathbb{V}], W^{s,1}[\mathbb{V}]; \varkappa, \varrho\Big)$$

whenever $e = (d, \vec{a}, \vec{b}, a_0) \in \mathbb{E}^\sigma$ satisfies $\|e\|_{\mathbb{E}^\sigma} \leq M$ and

$$d(v, z)\zeta \cdot \zeta \geq \alpha|\zeta|^2, \qquad z \in \mathbb{R}^n, \qquad a.e. \quad v \in \mathcal{V}, \quad \zeta \in \mathbb{R}^n \tag{2.4.1}$$

Proof : See [18]. ■

Suppose that $t \mapsto e(t) : I \to \mathbb{E}^\sigma$. The putting

$$(d, \vec{a}, \vec{b}, a_0)(t, z) := e(t)(z), \qquad z \in \mathbb{R}^n, \qquad t \in I$$

and

$$\mathcal{A}(t) := \mathcal{A}[e(t)], \qquad t \in I$$

Using this notation the proof of the following theorem is an easy consequence of Lemma 2.4.1 and Theorem 2.4.2.

Theorem 2.4.3

Suppose that $s \in (-1, r)\backslash\mathbb{N}$ with $r > 0$ and

$$\Big(t \mapsto e(t)\Big) \in C^\rho(I, \mathbb{E}^{1+r}), \qquad \text{for some} \qquad \rho \in \mathbb{R}^+ \tag{2.4.2}$$

Also suppose that there exists $\alpha > 0$ such that

$$d(t,v,z)\zeta \cdot \zeta \geq \alpha|\zeta|^2, \qquad (t,z) \in I \times \mathbb{R}^n, \qquad a.e. \quad v \in \mathcal{V}, \quad \zeta \in \mathbb{R}^n \tag{2.4.3}$$

Then

$$\Big(t \mapsto \mathcal{A}(t)\Big) \in C^\rho\Big(I, \mathcal{H}\Big(W^{s+2,1}[\mathbb{V}], W^{s,1}[\mathbb{V}]\Big)\Big) \tag{2.4.4}$$

2.5 Existence and Uniqueness of PBEs' solutions

Melzak was the first to study kinetic equations [62, 63] under the assumptions

Only binary coagulation occurs [1]: which means that the coagulation kernel is symmetric with respect to the second and third arguments, i.e.

$$0 \leq \omega(t,v,v') = \omega(t,v',v), \qquad v',v \in \mathcal{V} \tag{2.5.1}$$

Multiple fragmentation occurs :

$$0 \leq \varphi(t,v,v'), \qquad 0 < v' \leq v < \infty, \qquad v',v \in \mathcal{V} \tag{2.5.2}$$

Boundedness of coagulation and fragmentation rates : there exists a positive constant M such that

$$\omega(t,v,v') \leq M, \qquad \varphi(t,v,v') \leq M \tag{2.5.3}$$

[1] triple and higher collisions assumed to be rare.

for all possible arguments (t, v, v') of ω and φ, respectively.

Boundedness of Φ_φ : which means that the volume rate of change in the fragmentation process is bounded as well

$$\Phi_\varphi(t, v) \leq M, \qquad (t, v) \in \mathbb{R}_+ \times \mathcal{V} \tag{2.5.4}$$

No source term : i.e.

$$h(t, v) = 0, \qquad (t, v) \in \mathbb{R}_+ \times \mathcal{V} \tag{2.5.5}$$

He proved the existence of a unique positive global solution by means of series expansions. **Melzak** 's ideas have been extended by **Marcus** [37] to include a transport term in one space dimension, which depends on v only, that is,

$$\mathcal{A}(t, v, z) := b(v)\partial_z f$$

Aizenman and **Bak** [17] have initiated a different approach. They consider the autonomous kinetic coagulation-fragmentation equations with bounded coagulation and fragmentation rates without the assumption about Φ_φ. They establish the existence of a unique nonnegative volume-preserving solution using **semigroup techniques**.

McLaughlin, **Lamb**, and **McBride** [58–61] have extended this semigroup approach to cover certain kinds of unbounded kernels. Further results can be found in the papers by **Dubovski** and **Stewart** [41].

A completely different approach was given by **Amann** [18] where he viewed problem (2.3.1)-(2.3.2) as a single semilinear evolution equation

$$f + \mathcal{A}(t)f = R(t, f), \qquad t > 0 \tag{2.5.6}$$

$$f(0) = f^0 \tag{2.5.7}$$

in the Banach space $W^{s,1}[E]$, where f is a Banach-space-valued function of $(t, z) \in \mathbb{R}^+ \times \mathbb{R}^n$. In other words, he interpreted (2.3.1) as a vector-valued evolution equation which is easily handleable using Fourier multiplier theorems for operator-valued symbols and Banach space-valued distributions [19]. Besides the physical conditions (2.5.1)-(2.5.5) he proposed the following mild regularity hypotheses.

H_1 : Assuming that $\omega(t, \cdot, \cdot, z)$, $\varphi(t, \cdot, \cdot, z)$, and $\Phi_\varphi(t, \cdot, z)$ are measurable for $(t, z) \in \mathbb{R}^+ \times \mathbb{R}^n$ and sufficiently smooth with respect to (t, z).

H_2 : Assuming that the diffusion matrix $d(t, v, z)$ symmetric and positive definite, uniformly with respect to $(t, v, z) \in \mathbb{R}_+ \times \mathcal{V} \times \mathbb{R}^n$.

H_3 : Assuming that the diffusion matrix $d(t, v, z)$, the drift vectors $\vec{a}$ and $\vec{b}$ and the absorption rate a_0 are sufficiently smooth functions of (t, z) and measurable with respect to v.

In problem (2.5.6), the operator $\mathcal{A}$ satisfies (2.4.4) and $R(t, f) := C(t, f) + B(t, f) + h(t)$ with

$$\Big(t \mapsto R(t,\ \cdot\ )\Big) \in C^\rho\Big(I, C_b^\infty(W^{\sigma,1}[\mathbb{V}], W^{\tau,1}[\mathbb{V}])\Big) \tag{2.5.8}$$

by supposing that

$$\Big(t \mapsto h(t)\Big) \in C^\rho\Big(I, W^{\tau,1}[\mathbb{V}]\Big) \tag{2.5.9}$$

for some $\tau \in (s, r)$.

The following theorem establishes the well-posedness of system (2.3.1)-(2.3.2)

Theorem 2.5.1

Suppose that $r, \rho > 0$ and

$$(-2 + \frac{n}{2}) \vee (-1) < s < r, \qquad s \notin \mathbb{N} \tag{2.5.10}$$

Also suppose that

$$\Big(t \mapsto (e, (\omega, \varphi), h)(t)\Big) \in C^\rho\Big(I, \mathbb{E}^{r+1} \times BUC^r[K_{coag} \times K_{frag}] \times W^\tau[\mathbb{V}]\Big)$$

for some $\tau > s$, such that (2.4.3) is satisfied for some $\alpha > 0$. Finally, assume that

$$(s^+ + n)/2 < \sigma < n \wedge (s + 2), \qquad s < n, \tag{2.5.11}$$

and

$$s < \sigma < s + 2, \qquad s > n \tag{2.5.12}$$

with $\sigma \notin \mathbb{N}$.

Then, given any $f^0 \in W^{\sigma,1}[\mathbb{V}]$, the coagulation-fragmentation system (2.3.1)-(2.3.2), that is, problem (2.5.6)-(2.5.7), has a unique maximal solution

$$f(\ \cdot\ , f^0) \in C\Big(I(f^0), W^{\sigma,1}[\mathbb{V}]\Big) \cap C\Big(\dot{I}(f^0), W^{s+2,1}[\mathbb{V}]\Big) \cap C^1\Big(\dot{I}(f^0), W^{s,1}[\mathbb{V}]\Big) \tag{2.5.13}$$

where the maximal interval of existence, $I(f^0)$, is open in I.

The solution, $f(\ \cdot\ , f^0, e, \omega, \varphi, h) := f(\ \cdot\ , f^0)$ depends continuously on the data in the following sense: given $T \in I(f^0)$, there exists a neihborhood U of $(f^0, (e, (\omega, \varphi), h))$ in

$$W^{\sigma,1}[\mathbb{V}] \times BUC^{\rho}\Big(I, \mathbb{E}^{r+1} \times BUC^{\rho}[K_{coag} \times K_{frag}] \times W^{\tau,1}[\mathbb{V}]\Big)$$

such that $f(\,\cdot\,, \tilde{f}^0, \tilde{e}, \tilde{\omega}, \tilde{\varphi}, \tilde{h})$ exists on $[0, T]$ and

$$f(\,\cdot\,, \tilde{f}^0, \tilde{e}, \tilde{\omega}, \tilde{\varphi}, \tilde{h}) \to f(\,\cdot\,, f^0, e, \omega, \varphi, h) \qquad \text{in} \qquad C([0, T], W^{\sigma,1}[\mathbb{V}])$$

as $(\tilde{f}^0, \tilde{e}, \tilde{\omega}, \tilde{\varphi}, \tilde{h}) \to (f^0, e, \omega, \varphi, h)$ in U.

Proof : First note that (2.5.10) implies $s > -1$ if $n = 1, 2$ and $s > -1\backslash 2$ if $n = 3$. Moreover, (2.5.10) guarantees that the condition (2.5.11) is not void.

By making τ smaller , if necessary, we can assume that

$$(\tau^+ + n)/2 < \sigma < n \wedge (s + 2), \qquad \text{if} \qquad s < n,$$

and that $\tau < \sigma$ if $s > n$. Then fixing σ_1 such that

$$\tau \vee (\tau^+ + n)/2 < \sigma_1 < \sigma < n \wedge (s + 2), \qquad \text{if} \qquad s < n,$$

and

$$s < \tau < \sigma_1 < \sigma, \qquad s > n$$

Also, we can assume that $\tau, \sigma_1 \notin \mathbb{Z}$. Setting $E_1 := W^{s+2,1}[\mathbb{V}]$ and $E_0 := W^{s,1}[\mathbb{V}]$. Also setting $E_\theta := (E_0, E_1)_{\theta,1}$ for $0 < \theta < 1$. Then it follows from (2.5.11) and ([19], formula (5.7)) that

$$E_\theta \doteq W^{s+2\theta,1}[\mathbb{V}], \qquad s + 2\theta \notin \mathbb{Z} \tag{2.5.14}$$

Put $\alpha := (\sigma - s)\backslash 2$, $\beta := (\sigma_1 - s)\backslash 2$, $\gamma := (\tau - s)\backslash 2$. Then Theorem 2.4.3 and assertions (2.5.8) and (1.6.5) imply that the problem (2.5.6)-(2.5.7) satisfies the hypotheses of Theorem B.0.1 (see Appendix B, with g replaced by R). This completes the proof. ■

The following proposition shows that $f(t, f^0)$ is independent of the choice of s and σ, provided $t > 0$, and that problem (2.5.6)-(2.5.7) enjoys a smoothing property.

Proposition 2.5.2

Presuppose the hypotheses of Theorem 2.5.1 and fix $\bar{\sigma}$ in $(n/2, n \wedge 2)$. Then, given $f^0 \in$

$W^{\bar{\sigma},1}[\mathbb{V}]$), problem (2.3.1)-(2.3.2) has a unique maximal solution

$$f(\,\cdot\,,f^0)\in C\Big(I(f^0),W^{\bar{\sigma},1}[\mathbb{V}]\Big)\cap C\Big(\dot{I}(f^0),W^{s+2,1}[\mathbb{V}]\Big)\cap C^1\Big(\dot{I}(f^0),W^{s,1}[\mathbb{V}]\Big)$$

and $I(f^0)$ is independant of s satisfying (2.5.10).

Proof : See [18]. ■

As a notation $C_0^m[E] := C_0^m(\mathbb{R}^n, E)$ is the closed subspace of $BUC^m[E]$ consisting of all f such that ∂^α vanishes at infinity for $|\alpha| \leq m$. Furthermore,

$$C_0^\infty[E] := \bigcap_{m\geq 0} C_0^m[E]$$

equipped with the natural projective limit toplogy. Similar definitions apply to $\mathfrak{B}^\infty$ for $\mathfrak{B} \in \{BUC, W^1\}$ and to $\mathbb{E}^\infty$.

Corollary 2.5.3

Suppose that $\rho > 0$ and

$$\Big(t\mapsto (e,(\omega,\varphi),h)(t)\Big)\in C^\rho\Big(I,\mathbb{E}^\infty\times BUC^\infty[K_{coag}\times K_{frag}]\times W^{\infty,1}[\mathbb{V}]\Big)$$

such that (2.4.3) is satisfied. Then, if $f^0 \in W^{\sigma,1}[\mathbb{V}]$ for some $\sigma \in (n/2, n\wedge 2)$, the unique maximal solution of (2.5.6)-(2.5.7) belongs to $C^1(\dot{I}(f^0), C_0^\infty[\mathbb{V}])$.

Proof : This follows from the preceding proposition and the Sobolev embedding

$$W^{s,1}[E]\overset{d}{\hookrightarrow} C_0^m[E],\qquad s>m+n,\qquad m\in\mathbb{N} \tag{2.5.15}$$

which is also valid in the case of an arbitrary Banach space E.

■

Remark :

1. *It should be observed that this corollary applies, in particular, if all data are independant of* $z \in \mathbb{R}^n$.

2. *It can also be shown that the solution is more regular in the time variable than stated here, Roughly speaking,* $\dot{f}$ *is* ρ*-Hölder continuous with respect to* $t > 0$.

2.6 Positivity

Each one of the spaces $L_p(M, \mu; E)$, $\quad 1 \le p \le \infty$, $BUC^s[E]$, $W^{s,1}[E]$, $s \in \mathbb{R}^+$, is an ordered Banach space, provided E is an ordered Banach space. In particular, $\mathbb{V}$ is an ordered Banach space with respect to the natural order induced by the positive cone $\mathbb{V}^+ = L_1^+(\mathcal{V}, (1+v)dv)$, and all function spaces considered below are given their natural orders.

An approximation result for positive cones is shown by the next Lemma.

Lemma 2.6.1

$\mathcal{D}^+(\mathbb{R}^n) \otimes C_c^+(\mathcal{V})$ is dense in $W^{s,1}[\mathbb{V}]^+$ for $s \in \mathbb{R}^+$.

A bounded linear operator A on an ordered Banach space E is positive (in symbols: $A \ge 0$) if $A(E^+) \subset E^+$. A closed linear operator B in E is resolvent positive if there exists $\lambda_0 \ge 0$ such that $[\lambda_0, \infty)$ belongs to the resolvent set $\rho(-B)$ of $-B$ and $(\lambda + B)^{-1} \ge 0$ for $\lambda \ge \lambda_0$.

Proposition 2.6.2

Suppose that $s \in (0, r)\backslash\mathbb{N}$, and let $e \in \mathbb{E}^{1+r}$ satisfy (2.4.1). Then $\mathcal{A}[e]$ is resolvent positive on $W^{s,1}[\mathbb{V}]$.

Proof : Theorem 2.4.2 implies that $\mathcal{A} := \mathcal{A}[e]$ is a closed linear operator in $W^{s,1}[\mathbb{V}]$ with $[\eta, \infty) \subset \rho(-\mathcal{A})$ for some $\eta > 0$.

(i) Suppose that $s > n$ and put $\mathbb{V}_\infty := L_\infty(\mathcal{V}, dv)$. Then the proof of the Theorem 2.4.2 applies to give

$$\mathcal{A} \in \mathcal{H}\Big(W^{s+2,1}[\mathbb{V}_\infty], W^{s,1}[\mathbb{V}_\infty]\Big)$$

Hence there exists $\eta_\infty > 0$ such that $[\eta_\infty, \infty) \subset \rho(-\mathcal{A}_\infty)$, where $\mathcal{A}_\infty$ denotes $\mathcal{A}$, but considered as a linear operator in $W^{s,1}[\mathbb{V}_\infty]$. Put

$$\lambda_0 := \eta \vee \eta_\infty \vee \Big(\|a_0\|_{BUC[L^1_{loc}]} + \|\nabla \cdot \vec{a}\|_{BUC[L^1_{loc}]} \Big)$$

Fix $\lambda \ge \lambda_0$ and $g \in \mathcal{D}^+(\mathbb{R}^n) \otimes C_c^+(\mathcal{V})$, and put $f = (\lambda + \mathcal{A}_\infty)^{-1} g$. Then $f \in W^{s+2,1}[\mathbb{V}_\infty]$ and

$$(\lambda + \mathcal{A}) f(v, z) = g(v, z), \qquad z \in \mathbb{R}^n, \qquad a.e. \quad v \in \mathcal{V}$$

Note that (2.5.14) implies $f \in C_0^2[\mathbb{V}_\infty]$. Thus it follows that, for a.e. $v \in \mathcal{V}$, the function

$f(v,\ \cdot\ )$ belongs to $C_0^2[\mathbb{R}^n]$ and satisfies the elliptic differential inequality

$$-d(v,\ \cdot\ ): \quad \nabla^2 f(v,\ \cdot\ ) + \vec{c}(v,\ \cdot\ )\cdot\nabla f(v,\ \cdot\ ) + d_1(v,\ \cdot\ )f(v,\ \cdot\ ) \geq 0 \tag{2.6.1}$$

on $\mathbb{R}^n$ where

- $d_1 := \lambda + a_0 - \nabla\cdot\vec{a} \geq 0$.
- $\nabla^2 f$: denotes the Hessian of f.
- $A : B$ is the trace of the matrix product $AB^\top$

The coefficients of (2.6.1) are uniformely bounded on $\mathbb{R}^n$. Since $f(v,\ \cdot\ )$ vanishes at infinity, the classical maximum principle implies that $f(v,\ \cdot\ )$ is nonnegative. This being true for *a.e.* $v \in \mathcal{V}$, so $f \in W^{s+2,1}[\mathbb{V}_\infty]^+$.

Since $\mathcal{D}^+(\mathbb{R}^n)\otimes C_c^+(\mathcal{V}) \subset W^{s,1}[\mathbb{V}]^+$ and $\lambda > \eta$, Theorem 2.4.2 guarantees that $f \in W^{s+2,1}[\mathbb{V}]$ as well. Consequently,

$$(\lambda+\mathcal{A})^{-1}(\mathcal{D}^+(\mathbb{R}^n)\otimes C_c^+(\mathcal{V})) \subset W^{s,1}[\mathbb{V}]^+, \qquad \lambda \geq \lambda_0$$

the continuity of $(\lambda+\mathcal{A})^{-1}$ on $W^{s,1}[\mathbb{V}]^+$ is deduced from Lemma 2.6.1, and the closedness of the positive cone that $\mathcal{A}$ is resolvent positive on $W^{s,1}[\mathbb{V}]$.

(ii) Suppose that $s < n$. Fix $n < t < r_1 < \infty$ with $t \notin \mathbb{N}$ and suppose that $e \in \mathbb{E}^{1+r_1}$. It follows from **(i)** that $\mathcal{A}$ is resolvent positive on $W^{t,1}[\mathbb{V}]$. Also Lemma 2.6.1 implies that $W^{t,1}[\mathbb{V}]^+$ is dense in $W^{s,1}[\mathbb{V}]^+$. Thus, since $(\lambda+\mathcal{A})^{-1}$ exists and is continuous on $W^{s,1}[\mathbb{V}]$ for sufficiently large λ, once more by approximation, $\mathcal{A}$ is resolvent positive on $W^{s,1}[\mathbb{V}]$.

(iii) Finally, if $r < r_1$, fix $r_0 \in (s, r)$ and suppose that $e \in \mathbb{E}^{1+r}$. Then it is well-known that there exists a sequence (e_j) in $\mathbb{E}^{1+r_1}$ converging in $\mathbb{E}^{1+r_0}$ towards e. hence from Lemma 2.6.1 and the continuity of the inversion map $B \to B^{-1}$, we deduce that

$$(\lambda+\mathcal{A}[e_j])^{-1} \to (\lambda+\mathcal{A})^{-1}, \qquad (j \to +\infty)$$

in $\mathcal{L}(W^{s,1}[\mathbb{V}])$ for sufficiently large λ, if e_j assumed to be satisfied (2.4.1) for all $j \in \mathbb{N}$ with α replaced by some smaller positive number. Thus, the resolvent positivity follows in this case also.

∎

After these preparations, we can prove the main result of this section, namely that the solution

$f(\,\cdot\,, f^0)$ of (2.5.6)-(2.5.7) is positive whenever $f^0 \geq 0$ and (ω, φ) and h are positive [2].

Theorem 2.6.3

Let the assumptions of the Theorem 2.5.1 be satisfied and suppose that $(\omega, \varphi) \geq 0$ and $h \geq 0$. Then, $f^0 \geq 0$ implies $f(\,\cdot\,, f^0) \geq 0$.

Proof :

(i) Suppose that $s > n$. Then Theorem 2.5.1 and (2.5.15) imply

$$f := f(\,\cdot\,, f^0) \in C(I(f^0), C_0(\mathbb{V}))$$

Fix $T \in \dot{I}(f^0)$ and put $\eta_0 := \|\omega\|_\infty \max_{0 \leq t \leq T} \|f(t)\|_{C_0(\mathbb{V})}$. Then

$$\left| \int_0^{+\infty} \omega(t, v, v', z) f(t, v', z)\, dv' \right| \leq \eta_0 \tag{2.6.2}$$

for $(t, z) \in [0, T] \times \mathbb{R}^n$ and a.e. $v \in \mathcal{V}$. Set

$$p_\omega(g, w)(v) := \frac{1}{2} \int_0^v \omega(v - v', v') g(v - v') w(v')\, dv'$$

and

$$q_\omega(g, w)(v) := g(v) \int_0^{+\infty} \omega(v, v') w(v')\, dv' \tag{2.6.3}$$

for $v \in \mathcal{V}$ and $g, w \in \mathbb{V}$. Also put $\eta := \eta_0 + \|\Phi_\varphi\|_\infty$ and

$$G(t, g) := p_{\omega(t)}(g, g) - q_{\omega(t)}(g, g) + \eta g + B(t, g) + h(t)$$

for $0 \leq t \leq T$ and $g \in \mathbb{V}$. Then

$$G(t, f(t)) = R(t, f(t)) + \eta f(t), \qquad 0 \leq t \leq T$$

and (2.6.2) and the structure of B imply

$$G(t, g(t)) \geq 0, \qquad g \in C([0, T], C^+[\mathbb{V}]), \qquad 0 \leq t \leq T \tag{2.6.4}$$

[2] An element x of an ordered vector space is positive if and only if $x \geq 0$.

Lastly, set $\mathcal{A}_\eta := \eta + \mathcal{A}$. Then f is the unique solution of the IVP

$$\dot{g} + \mathcal{A}_\eta(t)g = G(t,g), \qquad 0 \le t \le T, \qquad g(0) = f^0 \tag{2.6.5}$$

in $W^{s,1}[\mathbb{V}]$. Denote by U the parabolic evolution operator for $\mathcal{A}_\eta$, whose existence is guaranteed by ([20], Corollary 4.4.2). Put

$$V(g)(t) := \int_0^t U(t,\tau)G(\tau, g(\tau))\,d\tau, \qquad g \in W^{\sigma,1}[\mathbb{V}], \qquad 0 \le t \le T$$

Then (2.6.5) implies that f solves the nonlinear Volterra integral equation

$$f = U(\,\cdot\,,0)f^0 + V(f) \tag{2.6.6}$$

in $W^{\sigma,1}[\mathbb{V}]$. If T is sufficiently small then equation (2.6.6) can be solved by the method of successive approximations, that is, the sequence (f_n), determined by $f_0 := f^0$ and

$$f_{n+1} = U(\,\cdot\,,0)f^0 + V(f_n), \qquad n \in \mathbb{N}$$

converges in $C([0,T], W^{\sigma,1}[\mathbb{V}])$ towards f. Since $\mathcal{A}_\eta$ is resolvent positive by Proposition 2.6.2, it follows from ([20], Theorem 6.4.1 and 6.4.2) that U is positive. Thus (2.5.14) and (2.6.4) require that $f_n \ge 0$ for $n \in \mathbb{N}$. consequently, $f \ge 0$.

These considerations show that there exists $T \in \dot{I}(f^0)$ such that $f^0 \ge 0$ implies $f(t,f^0) \ge 0$ for $0 \le t \le T$. Set

$$T^* := \max\left\{T \in \dot{I}(f^0); f(t,f^0) \ge 0\right\}$$

If $T^* < \sup I(f^0)$ then apply the above reasoning to the IVP

$$\dot{g} + \mathcal{A}(t+T^*)g = R(t+T^*, g), \qquad t \in I(f^0) - T^*, \qquad g(0) = f(T^*, f^0)$$

to find that $f(t,f^0) \ge 0$ on $[0, T^* + T^{**}]$ for some $T^{**} > 0$. Since this contradicts the choice of T^*, we see that $T^* = \sup I(f^0)$, that is, $f(\,\cdot\,, f^0) \ge 0$.

(ii) Suppose that $s < n$ and $r = \infty$. Fix $s_1, \sigma_1 \notin \mathbb{N}$ with

$$n < s_1 < \sigma_1 < s_1 + 2$$

and suppose that $f^0 \in W^{\sigma_1,1}[\mathbb{V}]^+$. Then, it follows from **(i)** that $f(\,\cdot\,, f^0) \ge 0$ in $W^{\sigma_1,1}[\mathbb{V}]$, hence in $W^{\sigma,1}[\mathbb{V}]$ by (1.6.8). Since $W^{\sigma_1,1}[\mathbb{V}]^+$ is dense in $W^{\sigma,1}[\mathbb{V}]^+$, the continuous dependence of $f(\,\cdot\,,f^0)$ on $f^0 \in W^{\sigma,1}[\mathbb{V}]$, is guaranteed by Theorem 2.5.1 implies $f(\,\cdot\,,f^0) \geq 0$ in $W^{\sigma,1}[\mathbb{V}]$ for $f^0 \in W^{\sigma,1}[\mathbb{V}]^+$.

(iii) Lastly, suppose that $s < n$ and $r > s$. Then, as in step **(ii)** of the proof of Proposition 2.6.2, we approximate $\Big(f^0, e, (\omega,\varphi), h\Big)$ by smooth functions and derive the positivity of $f(\,\cdot\,,f^0)$ from its continuous dependence on the data and from **(ii)**.

■

2.7 Conservation of Volume

The following assumptions are made throughout this section

$$\left.\begin{array}{l} r,\rho,\tau > 0, \\ \Big(t \mapsto (e,(\omega,\varphi),h)(t)\Big) \in C^{\rho}\Big(I, \mathbb{E}^{r+1} \times BUC^r[K^+_{coag} \times K^+_{frag}] \times W^{\tau,1}[\mathbb{V}]^+\Big) \\ \text{with (2.4.3) being satisfied. Moreover,} \quad \vec{b} = 0, \quad n/2 < \sigma < n, \quad f^0 \in W^{\sigma}[\mathbb{V}]^+ \end{array}\right\} \tag{2.7.1}$$

Fixing $s \in (0, \tau \wedge (2\sigma - n) \wedge r)$ and denoting by $f(\,\cdot\,,f^0)$ the unique maximal solution of the coagulation-fragmentation system (2.3.1)-(2.3.2). Theorem (2.5.1) implies that f is well-defined and satisfies (2.5.13). Thus

$$f \in C(\dot{I}(f^0), W^{2,1}[\mathbb{V}]) \cap C^1(\dot{I}(f^0), L_1[\mathbb{V}]) \tag{2.7.2}$$

and the Theorem (2.6.3) guarantees that $f \geq 0$.

Lemma 2.7.1

If $g \in W^{2,1}[\mathbb{V}]$ then

$$\int\limits_{\mathbb{R}^n}\int\limits_{\mathcal{V}} \mathcal{A}(t) g v^i \, dv\, dz = \int\limits_{\mathbb{R}^n}\int\limits_{\mathcal{V}} a_0 g v^i \, dv\, dz, \qquad i = 0, 1$$

for $t \in I$

Proof : See [18]. ■

For $t \in I(f^0)$, denoting

$V(t)$: is the total particle volume time t.

$A_0(t)$: is the total absorbed particle volume at time t.

$H(t)$: is the total particle input at time t.

which are defined as,

$$V(t) := \int_{\mathbb{R}^n} \int_{\mathcal{V}} f(t) v \, dv \, dz, \qquad A_0(t) := \int_{\mathbb{R}^n} \int_{\mathcal{V}} a_0(t) f(t) v \, dv \, dz,$$

and

$$H(t) := \int_{\mathbb{R}^n} \int_{\mathcal{V}} h(t) v \, dv \, dz,$$

The following theorem shows that if neither absorption nor particle input takes place, then the total particle volume is conserved.

Theorem 2.7.2

$$V(t) = V(0) + \int_0^t (H(\tau) - A_0(\tau)) \, d\tau, \qquad t \in I(f^0)$$

Proof : By integrating

$$\dot{f}(t) + \mathcal{A}(t) f(t) = R(t, f(t)) \tag{2.7.3}$$

over $\mathbb{R}^n \times \mathcal{V}$ with respect to the measure $dz \otimes v dv$ and taking (2.2.8) and (2.2.15) into account

$$\dot{V}(t) = H(t) - A_0(t) \tag{2.7.4}$$

using (2.7.2) and Lemma 2.7.1. Now the assertion follows by integrating (2.7.4) from t_0 to t, where $0 < t_0 < t$, and letting t_0 tend to 0. ■

Corollary 2.7.3

Put $\beta^{\pm} := \left\| a_0^{\pm} \right\|_{\infty}$. Then

$$e^{-\beta^+ t} V(0) + \int_0^t e^{-\beta^+ (t-\tau)} H(\tau) \, d\tau \leq V(t) \leq e^{\beta^- t} V(0) + \int_0^t e^{\beta^- (t-\tau)} H(\tau) \, d\tau$$

for $t \in I(f^0)$.

Proof : Note that

$$-\beta^- V(t) \leq A_0(t) \leq \beta^+ V(t), \qquad t \in I(f^0)$$

Thus (2.7.3) entails the differential inequalities

$$H(t) - \beta^+ V(t) \leq \dot{V}(t) \leq H(t) + \beta^- V(t), \qquad t \in I(f^0)$$

which implies the assertion. ■

Finally, in the next section the problem of global existence is discussed, that is, whether or not $I(f^0) = I$.

2.8 Global Existence

Theorem 2.8.1

Let assumption (2.7.1) be satisfied. If one of the following assumptions is satisfied:

(i) There is no coagulation, that is, $\omega = 0$;

(ii) $n = 1$;

(iii) $\mathcal{A}$ is independent of $v \in \mathcal{V}$.

Then $f := f(\,\cdot\,, f^0)$ exists globally.

Proof :

(i) is obvious since in this case (2.5.6)-(2.5.7) is a linear evolution equation.

(ii) Set $\mathbb{V}_i := L_1(\mathcal{V}, v^i dv)$ for 0,1 $i = 0, 1$. Then, by integrating (2.7.3), deducing from Lemma (2.7.1), the positivity of f, and (2.2.7) and (2.2.14) that

$$\begin{aligned}\|f(t)\|^{\cdot}_{L_1[\mathbb{V}_0]} &= \int_{\mathbb{R}^n}\int_{\mathcal{V}} \dot{f}(t)\, dv\, dz \\ &\leq \|a_0\|_\infty \|f(t)\|_{L_1[\mathbb{V}_0]} + \frac{\|\varphi\|_\infty}{2} V(t) + \|h(t)\|_{L_1[\mathbb{V}_0]}\end{aligned}$$

for $t \in \dot{I}(f^0)$. Also, deduce from Corollary (2.7.3) that there exist $\varpi > 0$ and $\varsigma \in C^+(I)$ such that $\vartheta := \|f(\,\cdot\,)\|_{L_1[\mathbb{V}_0]}$ satisfies the differential inequality

$$\dot{\vartheta} \leq \varpi\vartheta + \varsigma(t), \qquad t \in \dot{I}(f^0)$$

Since $\vartheta \in C(I(f^0)) \cap C^1(\dot{I}(f^0))$, it follows that

$$\|f(t)\|_{L_1[\mathbb{V}_0]} \leq c(T), \qquad t \in I(f^0) \cap [0, T], \qquad T \in I$$

Thus, by taking $V(t) = \|f(t)\|_{L_1[\mathbb{V}_0]}$ into account and applying Corollary 2.7.3 once more. Then

$$\|f(t)\|_{L_1[\mathbb{V}]} \leq c(T), \qquad t \in I(f^0) \cap [0,T], \qquad T \in I \tag{2.8.1}$$

Deducing from (2.2.6) and (2.2.13) that

$$\|C(t, f(t))\|_1 \leq c\|f(t)\|_\infty \|f(t)\|_1$$

and

$$\|B(t, f(t))\|_1 \leq c\|f(t)\|_\infty$$

for $t \in I(f^0)$ (where $\| \cdot \|_{\lambda,q}$ is the norm in $W^{\lambda,q}[\mathbb{V}]$ and $\| \cdot \|_q := \| \cdot \|_{0,q}$). Hence we infer from (2.8.1) that

$$\|R(t, f(t))\|_1 \leq c(T)(\|f(t)\|_\infty + 1), \qquad t \in I(f^0) \cap [0,T], \qquad T > 0 \tag{2.8.2}$$

Fix $\bar{s} \in (-1, 0)$ and $\bar{\sigma} \in \mathbb{R}^+ \backslash \mathbb{N}$ with $1 < \bar{\sigma} < \bar{s} + 2$. Then (2.5.15), the injection $L_1[\mathbb{V}] \hookrightarrow W^{\bar{s},1}[\mathbb{V}]$, and (2.8.2) imply

$$\|R(t, f(t))\|_{\bar{s},1} \leq c(T)(\|f(t)\|_{\bar{\sigma},1} + 1), \qquad t \in I(f^0) \cap [0,T], \qquad T \in I \tag{2.8.3}$$

Theorem 2.5.1 guarantees that f is a solution on $I(f^0)$ of the linear IVP

$$\dot{g} + \mathcal{A}(t)g = R(t, g(t)), \qquad t \in \dot{I}(f^0), \qquad g(0) = f^0$$

where $R(\,\cdot\,, g(\,\cdot\,)) \in C(I(f^0), W^{\bar{\tau},1}[\mathbb{V}])$ with $\bar{s} < \bar{\tau} < 0$. Consequently, f satisfies in $W^{\bar{s},1}[\mathbb{V}]$ the integral equation

$$f(t) = U(t,0)f^0 + \int_0^t U(t,\tau)R(\tau, f(\tau))\, d\tau, \qquad t \in I(f^0) \tag{2.8.4}$$

Hence it follows from ([20], Lemma 5.1.3) and

$$\|f(t)\|_{\bar{\sigma},1} \leq c(T)\left(t^{\sigma-\bar{\sigma}}\|f^0\|_{\sigma,1} + \int_0^t (t-\tau)^{\frac{\bar{s}-\bar{\sigma}}{2}}(\|f(t)\|_{\bar{\sigma},1} + 1)\, d\tau\right)$$

for $t \in I(f^0) \cap [0,T]$ and $T \in I$. Thus the singular gronwall inequality ([20], Corollary 3.3.2) requires that, given $t_0 \in \dot{I}(f^0)$

$$\|f(t)\|_{\sigma,1} \leq c\|f(t)\|_{\bar{\sigma},1} \leq c(T), \qquad t \in I(f^0) \cap [t_0, T] \tag{2.8.5}$$

for every $T \in I$ with $T > t_0$. Now the assertion is a consequence of the last part of Theorem B.0.1, since $E_\alpha = W^{\sigma,1}[\mathbb{V}]$ by the proof of Theorem 2.5.1.

(iii) By integrating (2.7.3) over $\mathcal{V}$ with respect to the measure dv and using (2.2.7), (2.2.14), and the positivity of f it follows that $\bar{f} := \int_{\mathcal{V}} f\, dv$ satisfies the parabolic differential inequality

$$\partial_t \bar{f} + \mathcal{A}(t)\bar{f} \leq \bar{h}(t), \qquad t \in \dot{I}(f^0), \qquad \bar{f}(0) = \int_{\mathcal{V}} f^0\, dv$$

on $\mathbb{R}^n$, where $\bar{h}(t) := \int_{\mathcal{V}} h(t)\, dv$. If $\bar{f} \in C_0^2(\mathbb{R}^n)$ then the maximum principle implies

$$\|f(t)\|_{L_\infty(\mathbb{R}^n,\mathbb{V}_0)} = \left\|\bar{f}(t)\right\|_{L_\infty(\mathbb{R}^n)} \leq c(T), \qquad t \in I(f^0) \cap [0, T]$$

In the general case this estimate obtained by an approximation argument similar to the one used in the proof of Theorem 2.6.3. Hence Corollary 2.7.3 and (2.8.2) imply

$$\|R(t, f(t))\|_{\bar{s},1} \leq c(T), \qquad t \in I(f^0) \cap [0, T], \qquad T \in I$$

Thus (2.8.4) and ([20], Corollary II.3.2.2) guarantee that (2.8.5) is true in this case also.

■

2.9 Convergence Analysis of ADM

R. Singh, **J. Saha** and **J. Kumar** [75] discussed the convergence of the series solution for the fragmentation and aggregation population balance equation separately in a spatially homogeneous physical system. They followed the approach discussed in [42]

2.9.1 For Fragmentation equation

Consider the PBE for fragmentation

$$\frac{\partial f(t,v)}{\partial t} = -\Gamma(v)f(t,v) + \int_{v}^{+\infty} \beta(v,v')\Gamma(v')f(t,v')\,dv' \tag{2.9.1}$$

supplemented by the initial condition

$$f(0,v) = f_0(v) \tag{2.9.2}$$

Let $\mathbb{X} = ([0,T] \times L^1[0,\infty), \|.\|)$ be a Banach space with the norm defined as

$$\|f\| = \sup_{t\in[0,t_0]} \int_{0}^{+\infty} e^{\lambda v}|f(t,v)|\,dv, \qquad \lambda > 0 \tag{2.9.3}$$

Operating with $\mathcal{L}^{-1}$ [3] on both sides of (2.9.1), and using the initial condition

$$f(t,v) = f_0(v) + \mathcal{L}^{-1}\Big(-\Gamma(v)f(t,v) + \int_{v}^{+\infty} \beta(v,v')\Gamma(v')f(t,v')\,dv'\Big) \tag{2.9.4}$$

Eq. (2.9.4) written in operator form as

$$f = \mathcal{T}f \tag{2.9.5}$$

With $\mathcal{T} : \mathbb{X} \longrightarrow \mathbb{X}$ is a linear operator given by

$$\mathcal{T}f(t,v) = f_0(v) + \mathcal{L}^{-1}\Big(-\Gamma(v)f(t,v) + \int_{v}^{+\infty} \beta(v,v')\Gamma(v')f(t,v')\,dv'\Big) \tag{2.9.6}$$

The idea here is to show that $\mathcal{T}$ is contractive; for this reason, rewrite (2.9.6) in the following equivalent form:

$$\frac{\partial}{\partial t}\Big(f(t,v)e^{\mathcal{F}(t,v)}\Big) = e^{\mathcal{F}(t,v)}\Big(\int_{v}^{+\infty} \beta(v,v')\Gamma(v')f(t,v')\,dv'\Big) \tag{2.9.7}$$

[3]In (2.9.1), $\mathcal{L} = \frac{\partial}{\partial t}$ is linear partial differential operator, The operator $\mathcal{L}^{-1}$ regarded as the inverse operator of $\mathcal{L}$, is defined by $\mathcal{L}^{-1}[.] := \int_0^t [.]\,dt$

By setting $e^{\mathcal{F}(t,v)} = t\Gamma(v)$. Thus,

$$\tilde{\mathcal{T}} f(t,v) = f_0(v) e^{-\mathcal{F}(t,v)} + \int_0^t \left[e^{-\mathcal{F}(s,v)-\mathcal{F}(t,v)} \left(\int_v^{+\infty} \beta(v,v')\Gamma(v') f(s,v')\, dv' \right) \right] ds \tag{2.9.8}$$

It is sufficient to show that $\tilde{\mathcal{T}}$ is contractive since $\tilde{\mathcal{T}}$ and $\mathcal{T}$ are equivalent.

Theorem 2.9.1

Let the linear operator $\tilde{\mathcal{T}}$ defined by (2.9.8) be contractive, that is,

$$\|\tilde{\mathcal{T}} f - \tilde{\mathcal{T}} f^*\| \le \delta \|f - f^*\|, \qquad \forall f, f^* \in \mathbb{X}$$

With

1. $\beta(v,v') = c\frac{v^{r-1}}{(v')^r}$, $\qquad r = 1, 2, \ldots$ and $c > 0$ is a constant satisfying

$$\int_0^v v' \beta(v,v')\, dv' = v \tag{2.9.9}$$

2. $\Gamma(v) \le v^k$, where $k = 1, 2, \ldots$
3. λ is chosen in such a way that $(e^{\lambda v} - 1) < 1$
4. $\delta := \frac{(k!)t_0}{\lambda^{k+1}} c < 1$ for some suitable t_0.

Proof : See [75]. ∎

Theorem 2.9.2 (*Theorem of convergence*)

1. Assume that all the conditions of Theorem 2.9.1 hold.
2. Let $f_0, f_1, f_2, \ldots$ be the components of the solution f by the recursive scheme

$$\begin{cases} f_0(t,v) & = f_0(v) \\ f_i(t,v) & = \mathcal{L}^{-1} \left(\int_v^{+\infty} \beta(v,v')\Gamma(v') f_{i-1}(t,v')\, dv' \; - \Gamma(v) f_{i-1}(t,v) \right), \quad i \ge 1 \end{cases} \tag{2.9.10}$$

3. Let $\psi_n = \sum_{i=0}^{n} f_i$ be the n-terms series solution defined by $f = \sum_{i=0}^{+\infty} f_i$

Then, the series solution ψ_n converges whenever $\delta := \frac{(k!)t_0}{\lambda^{k+1}} c < 1$ and $\|f_1\| < \infty$

Proof : From (2.9.10), we have

$$\begin{aligned}
\psi_n(t,v) &= \sum_{i=0}^{n} f_i(t,v) \\
&= f_0(v) + \sum_{i=0}^{n} \mathcal{L}^{-1}\left(\int_{v}^{+\infty} \beta(v,v')\Gamma(v') f_{i-1}(t,v')\, dv' - \Gamma(v) f_{i-1}(t,v) \right) \\
&= f_0(v) + \mathcal{L}^{-1}\left[\int_{v}^{+\infty} \beta(v,v')\Gamma(v') \left(\sum_{i=0}^{n-1} f_i(t,v') \right) dv' \right. \\
&\left. -\Gamma(v) \left(\sum_{i=0}^{n-1} f_i(t,v) \right) \right] \\
&= f_0(v) + \mathcal{L}^{-1}\left[\int_{v}^{+\infty} \beta(v,v')\Gamma(v') \psi_{n-1}(t,v')\, dv' - \Gamma(v)\psi_{n-1}(t,v) \right]
\end{aligned}$$

which is equivalent to the following operator equation form as

$$\psi_n = \mathcal{T}\psi_{n-1} \tag{2.9.11}$$

By following the steps of Theorem 2.9.1, we obtain

$$\begin{aligned}
\|\psi_{m+1} - \psi_m\| &\leq \delta \|\psi_m - \psi_{m-1}\| \\
&\leq \delta^2 \|\psi_{m-1} - \psi_{m-2}\| \\
&\vdots \\
&\leq \delta^m \|\psi_1 - \psi_0\|
\end{aligned}$$

Using the triangle inequality with $n > m$ we have

$$\begin{aligned}
\|\psi_n - \psi_m\| &\leq \|\psi_n - \psi_{n-1}\| + \|\psi_{n-1} - \psi_{n-2}\| + \cdots + \|\psi_{m+1} - \psi_m\| \\
&\leq \left(\delta^{n-1} + \delta^{n-2} + \cdots + \delta^m \right) \|\psi_1 - \psi_0\| \\
&\vdots \\
&\leq \delta^m \Big(\delta^{n-m-1} + \delta^{n-m-2} + \cdots + \delta^2 + \delta + 1) \|\psi_1 - \psi_0\| \\
&= \delta^m \left(\frac{1 - \delta^{n-m}}{1 - \delta} \right) \|f_1\|
\end{aligned}$$

Since $0 < \delta < 1$ so, $1 - \delta^{n-m} < 1$, and $\|f_1\| < \infty$. It follows that

$$\|\psi_n - \psi_m\| \leq \left(\frac{\delta^m}{1-\delta}\right)\|f_1\| \tag{2.9.12}$$

which converges to zero as $m \longrightarrow +\infty$. This implies that there exits a ψ such that

$$\lim_{n \longrightarrow +\infty} \psi_n = \psi \tag{2.9.13}$$

Since, we have

$$f = \sum_{i=0}^{+\infty} f_j = \lim_{n \longrightarrow +\infty} \psi_n = \psi \tag{2.9.14}$$

which is the exact solution of (2.9.5). ∎

Theorem 2.9.3 (*Estimation of Error*)

Let f be the exact solution of (2.9.5) and ψ_m be the series solution. Then there holds

$$\|f - \psi_m\| \leq \left(\frac{\delta^m}{1-\delta}\right)\|f_1\| \tag{2.9.15}$$

Where $\|f_1\| = \sup\limits_{t\in[0,t_0]} \int\limits_0^{+\infty} e^{\lambda v}|f_1(t,v)|\, dv$

Proof : From the estimate (2.9.12), for $n \geq m$, $n, m \in \mathbb{N}$.

$$\|\psi_n - \psi_m\| \leq \left(\frac{\delta^m}{1-\delta}\right)\|f_1\| \tag{2.9.16}$$

Fixing m and letting $n \longrightarrow +\infty$, and using $\lim\limits_{n \longrightarrow +\infty} \psi_n = f$, we obtain the desired result of theorem.

∎

2.9.2 For Aggregation equation

Consider PBE for pure Aggregation

$$\frac{\partial f(t,v)}{\partial t} = \frac{1}{2}\int_0^v \omega(v-u,u)f(t,v-u)f(t,u)\, du - \int_0^{+\infty} \omega(v,u)f(t,v)f(t,u)\, du \tag{2.9.17}$$

Subjected to the initial condition

$$f(0,v) = f_0(v) \tag{2.9.18}$$

Rewrite (2.9.17) in operator form as follows:

$$\mathcal{L}f(t,v) = \frac{1}{2}\int_0^v \omega(v-u,u)\mathcal{N}_1(f)(t,v-u,u)\,du - \int_0^{+\infty} \omega(v,u)\mathcal{N}_2(f)(t,v,u)\,du \tag{2.9.19}$$

The nonlinear functions are denoted by

$$\mathcal{N}_1(f)(t,v-u,u) = f(t,v-u)f(t,u)$$
$$\mathcal{N}_2(f)(t,v,u) = f(t,v)f(t,u)$$

Operating with $\mathcal{L}^{-1}$ defined previously on both sides of (2.9.19) yields

$$f(t,v) = f_0(v) + \mathcal{L}^{-1}\left[\frac{1}{2}\int_0^v \omega(v-u,u)\mathcal{N}_1(f)(s,v-u,u)\,du - \int_0^{+\infty} \omega(v,u)\mathcal{N}_2(f)(s,v,u)\,du\right] \tag{2.9.20}$$

The ADM introduces the solution $f(t,v)$ and the nonlinear functions $\mathcal{N}_1(f)$ and $\mathcal{N}_2(f)$ as

$$f(t,v) = \sum_{m=0}^{+\infty} f_m(t,v) \tag{2.9.21}$$

$$\mathcal{N}_1(f) = \sum_{m=0}^{+\infty} \mathcal{A}_m \tag{2.9.22}$$

$$\mathcal{N}_2(f) = \sum_{m=0}^{+\infty} \mathcal{B}_m \tag{2.9.23}$$

where $\mathcal{A}_m$ and $\mathcal{B}_m$ are the Adomian polynomials and defined by

$$\mathcal{A}_m(f_0,f_1,...,f_m) = \frac{1}{m!}\left[\!\left[\frac{d^m}{d\xi^m}\mathcal{N}_1\left(\sum_{i=0}^{+\infty}\xi^i f_i\right)\right]\!\right]_{\xi=0}, m \geq 0 \tag{2.9.24}$$

$$\mathcal{B}_m(f_0,f_1,...,f_m) = \frac{1}{m!}\left[\!\left[\frac{d^m}{d\xi^m}\mathcal{N}_2\left(\sum_{i=0}^{+\infty}\xi^i f_i\right)\right]\!\right]_{\xi=0}, m \geq 0 \tag{2.9.25}$$

Thus, the Adomian recursion scheme is

$$\begin{cases} f_0(t,v) &= f_0(v) \\ f_i(t,v) &= \mathcal{L}^{-1}\left(\frac{1}{2}\int\limits_0^v \omega(v-u,u)\mathcal{A}_{i-1}\,du - \int\limits_0^{+\infty} \omega(v,u)\mathcal{B}_{i-1}\,du\right), \quad i \geq 1 \end{cases} \tag{2.9.26}$$

To discuss the convergence of the previous recursive scheme (2.9.26), Let $\mathbb{X} = \Big(\mathcal{C}\big([0,T] : L^1[0,+\infty)\big), \|.\|\Big)$ be a Banach space with the norm defined as

$$\|f\| = \sup_{t\in[0,t_0]} \int_0^{+\infty} |f(t,v)|\,dv < \infty \tag{2.9.27}$$

Rewriting (2.9.20) in the form of an operator equation

$$f = \mathcal{N}(f) \tag{2.9.28}$$

where $\mathcal{N} : \mathbb{X} \longrightarrow \mathbb{X}$ is a nonlinear operator given by

$$\mathcal{N}(f)(t,v) = f_0(v) + \mathcal{L}^{-1}\left[\frac{1}{2}\int_0^v \omega(v-u,u)\mathcal{N}_1(f)(t,v-u,u)\,du - \int_0^{+\infty} \omega(v,u)\mathcal{N}_2(f)(t,v,u)\,du\right] \tag{2.9.29}$$

With the same approach, rewrite equation (2.9.29) in an equivalent form to show that $\mathcal{N}$ is contractive.

$$\frac{\partial}{\partial t}\Big(f(t,v)e^{\mathcal{F}(t,v,f)}\Big) = e^{\mathcal{F}(t,v,f)}\left(\int_0^v \omega(v-u,u)f(t,v-u)f(t,u)\,du\right) \tag{2.9.30}$$

where $\mathcal{F}(t,v,f) = \int_0^t \int_0^{+\infty} \omega(v,u)f(s,u)\,du\,ds$. Thus, we have

$$\tilde{\mathcal{N}}(f)(t,v) = f_0(v)e^{-\mathcal{F}(t,v,f)} + \frac{1}{2}\int_0^t e^{\mathcal{F}(s,v,f)-\mathcal{F}(t,v,f)}\left(\int_0^v \omega(v-u,u)f(s,v-u)f(s,u)\,du\right)ds \tag{2.9.31}$$

Since $\mathcal{N}$ and $\tilde{\mathcal{N}}$ are equivalent, it is enough to show $\tilde{\mathcal{N}}$ is contractive.

Theorem 2.9.4

Let the nonlinear operator $\tilde{\mathcal{N}}$ defined by (2.9.28) be contractive, that is,

$$\big\|\tilde{\mathcal{N}}(f) - \tilde{\mathcal{N}}(f^*)\big\| \le \delta\|f - f^*\|, \qquad \forall f, f^* \in \mathbb{X}$$

With

1. $\omega(v,u) = 1, \qquad \forall v, u \in (0, +\infty)$.
2. $\delta := t_0 e^{2t_0 L}\Big(\|f_0\| + 2t_0L^2 + 2t_0L\Big) < 1$, where $L = \|f_0\|(T+1)$.

Proof : Let $f, f^* \in \mathbb{X}$, consider

$$\tilde{\mathcal{N}}(f) - \tilde{\mathcal{N}}(f^*) = f_0(v)\mathcal{K}(t,0,v) + \frac{1}{2}\int_0^t \mathcal{K}(t,s,v)\int_0^v f(s,v-u)f(s,u)\,du\,ds$$
$$-\frac{1}{2}\int_0^t e^{\mathcal{F}(s,v,f^*)-\mathcal{F}(t,v,f^*)}\Bigg(\int_0^v f^*(s,v-u)\Big(f(s,u)-f^*(s,u)\Big)\,du \tag{2.9.32}$$
$$+\int_0^v f(s,u)\Big(f(s,v-u)-f^*(s,v-u)\Big)\,du\Bigg)\,ds$$

Where $\mathcal{K}(t,s,v) = e^{\mathcal{F}(s,v,f)-\mathcal{F}(t,v,f)} - e^{\mathcal{F}(s,v,f^*)-\mathcal{F}(t,v,f^*)}$. It can be shown that,

$$|\mathcal{K}(t,s,v)| \le e^{-\int_s^t\int_0^{+\infty} f^*(\tau,u)\,du\,d\tau}(t-s)\|f-f^*\| \tag{2.9.33}$$
$$\le (t-s)e^{(t-s)B}\|f-f^*\| \tag{2.9.34}$$
$$\le L_1\|f-f^*\| \tag{2.9.35}$$

where $L_1 = te^{tB}$ and $B = \max(\|f\|, \|f^*\|)$. In order to show that the operator $\tilde{\mathcal{N}}$ is contractive, let us define the set $D = \{f \in \mathbb{X} : \|f\| \le 2L\}$. It can be shown that the operator $\tilde{\mathcal{N}}$ maps D into itself.

For $f, f^* \in D$ we have $B \le 2L$. Taking norm on both sides of (2.9.32), we get

$$\big\|\tilde{\mathcal{N}}(f) - \tilde{\mathcal{N}}(f^*)\big\| \le L_1\|f_0\|\|f-f^*\| + \|f-f^*\|L_1\int_0^t\left(\frac{1}{2}\|f_0\|^2\right)ds$$
$$+\int_0^t L_1\left(\frac{1}{2}(\|f\|+\|f^*\|)\|f-f^*\|\right)ds$$
$$\le L_1\left(\|f_0\| + \frac{1}{2}t\|f\|^2 + \frac{1}{2}t(\|f\|+\|f^*\|)\right)\|f-f^*\|$$
$$\le \delta\|f-f^*\|$$

if $\delta = t_0e^{2t_0L}\Big(\|f_0\| + 2t_0L^2 + 2t_0L\Big) < 1$ under suitably chosen t_0 the operator $\tilde{\mathcal{N}}$ is a contraction map. ■

Theorem 2.9.5 (*Convergence theorem*)

1. Assume that all the conditions of Theorem 2.9.4 hold.
2. Let $f_0, f_1, f_2, \ldots$ be the components of the solution f by the recursive scheme (2.9.26)
3. Let $\psi_n = \sum\limits_{i=0}^{n} f_i$ be the n-terms series solution defined by $f = \sum\limits_{i=0}^{+\infty} f_i$

Then, the series solution ψ_n converges whenever $\delta := t_0 e^{2t_0 L}\Big(\|f_0\| + 2t_0 L^2 + 2t_0 L\Big) < 1$ and $\|f_1\| < \infty$

Proof : Using (2.9.26), we have

$$\psi_n(t,v) = \sum_{i=0}^{n} f_i(t,v) \tag{2.9.36}$$

$$= f_0(v) + \sum_{i=1}^{n} \mathcal{L}^{-1}\Bigg(\frac{1}{2}\int_0^v \omega(v-u,u)\mathcal{A}_{i-1}\,du - \int_0^{+\infty} \omega(v,u)\mathcal{B}_{i-1}\,du\Bigg) \tag{2.9.37}$$

$$= f_0(v) + \mathcal{L}^{-1}\Bigg(\frac{1}{2}\int_0^v \omega(v-u,u)\Big(\sum_{i=0}^{n-1}\mathcal{A}_i\Big)\,du - \int_0^{+\infty} \omega(v,u)\Big(\sum_{i=0}^{n-1}\mathcal{B}_i\Big)\,du\Bigg) \tag{2.9.38}$$

As given in ([68] pp. 945)

$$\sum_{i=0}^{n} \mathcal{A}_i \le \mathcal{N}_1(\psi_n) \tag{2.9.39}$$

$$\sum_{i=0}^{n} \mathcal{B}_i \le \mathcal{N}_2(\psi_n) \tag{2.9.40}$$

and use it in (2.9.38), we get

$$\psi_n \le f_0(v) + \mathcal{L}^{-1}\Bigg(\frac{1}{2}\int_0^v \omega(v-u,u)\mathcal{N}_1(\psi_{n-1})\,du - \int_0^{+\infty} \omega(v,u)\mathcal{N}_2(\psi_{n-1})\,du\Bigg) \tag{2.9.41}$$

This corresponds to the following operator form

$$\psi_n \le \mathcal{N}(\psi_{n-1}) \tag{2.9.42}$$

Following the steps of Theorem 2.9.4, we obtain

$$\|\psi_{n+1} - \psi_n\| \le \delta\|\psi_n - \psi_{n-1}\| \tag{2.9.43}$$

Thus, we have

$$\|\psi_{n+1}-\psi_n\| \le \delta\|\psi_n-\psi_{n-1}\| \le \delta^2\|\psi_{n-1}-\psi_{n-2}\| \le \ldots \le \delta^n\|\psi_1-\psi_0\|$$

Using the triangle inequality for all $n, m \in \mathbb{N}$ with $n > m$, we have

$$\begin{aligned}\|\psi_n-\psi_m\| &\le \|\psi_n-\psi_{n-1}\| + \|\psi_{n-1}-\psi_{n-2}\| + \cdots + \|\psi_{m+1}-\psi_m\| \\ &\le \left(\delta^{n-1}+\delta^{n-2}+\cdots+\delta^m\right)\|\psi_1-\psi_0\| \\ &= \delta^m\left(\frac{1-\delta^{n-m}}{1-\delta}\right)\|f_1\|\end{aligned}$$

Since $0 < \delta < 1$ so, $1-\delta^{n-m} < 1$, and $\|f_1\| < \infty$. It follows that

$$\|\psi_n-\psi_m\| \le \left(\frac{\delta^m}{1-\delta}\right)\|f_1\| \tag{2.9.44}$$

which converges to zero as $m \longrightarrow +\infty$. This implies that there exits a ψ such that

$$\lim_{n\longrightarrow+\infty}\psi_n = \psi \tag{2.9.45}$$

Since, we have

$$f = \sum_{i=0}^{+\infty} f_j = \lim_{n\longrightarrow+\infty}\psi_n = \psi \tag{2.9.46}$$

which is the exact solution of (2.9.28). ■

Theorem 2.9.6 (*Estimation of Error*)

Let u be the exact solution of (2.9.28) and ψ_m be the series solution. Then there holds

$$\|f-\psi_m\| \le \left(\frac{\delta^m}{1-\delta}\right)\|f_1\| \tag{2.9.47}$$

Where $\|f_1\| = \sup\limits_{t\in[0,t_0]} \int\limits_0^{+\infty} |f_1(t,v)|\, dv < \infty$

Proof : The proof is similar to that of Theorem 2.9.3. ■

3

Mathematical Background of Adomian Decomposition Method

The Adomian Decomposition Method (ADM), including its presentation and convergence, is thoroughly reviewed in this chapter.

Contents

3.1 Introduction

The Adomian's decomposition method or simply the ADM was initiated by the American mathematician **George Adomian** in the early eighties, this powerful methodology was presented for a practical solution of both linear or nonlinear and deterministic or even stochastic operator equations. It is worth mentioning that he introduced his method empirically and without theoretical foundations [11,12]. Professor **Y. Cherruault** was the first to establish rigorous foundations for this method, to justify its convergence, and to generalize it [30, 31].
Many phenomena in diverse fields such as engineering, physics, chemistry, biology ...etc, can be very successfully described by models using mathematical tools.

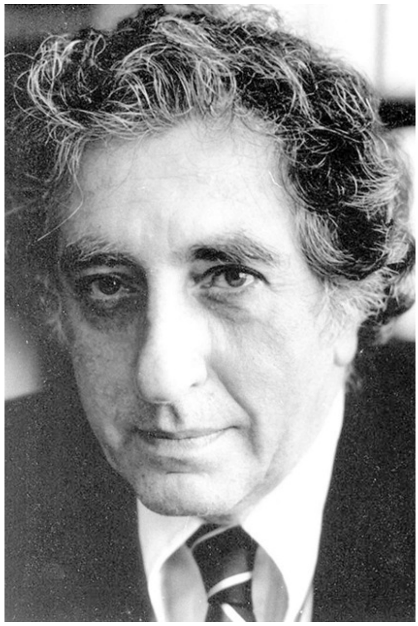

Figure 3.1: G. ADOMIAN (1922-1996) an American mathematician

The majority of these models illustrate nonlinear problems and nonlinear phenomena play a crucial role in the aforementioned fields. Consequently, explicit solutions of nonlinear equations are of fundamental importance to maintain the actual physical character of the problem and to understand deeply the described process.

Adomian [10,12–14] and others have successfully applied the ADM to algebraic, ordinary, partial, delay, and non-integer order or fractional differential equations for a wide range of nonlinearities, including polynomial, exponential, trigonometric, hyperbolic, composite, negative power, radical, and even decimal power nonlinearities.

The ADM solves nonlinear differential equations for any analytic nonlinearities. It permits one to solve nonlinear differential equations without having to appeal to the decidedly questionable practices of perturbation or linearization.

The ADM can be applied directly for all types of differential and integral equations, linear or nonlinear, homogeneous or inhomogeneous, with constant coefficients or with variable coefficients. It does not require discretization of the variables. Hence, the solution is not affected by computation round-off errors and the necessity of large computer memory. It is capable of greatly reducing the size of computation work while still maintaining the high accuracy of the numerical solution. Another advantage of this method is the avoidance of simplifications and restrictions, which change the nonlinear problem to a mathematically tractable one, whose solution is not consistent with the physical solution.

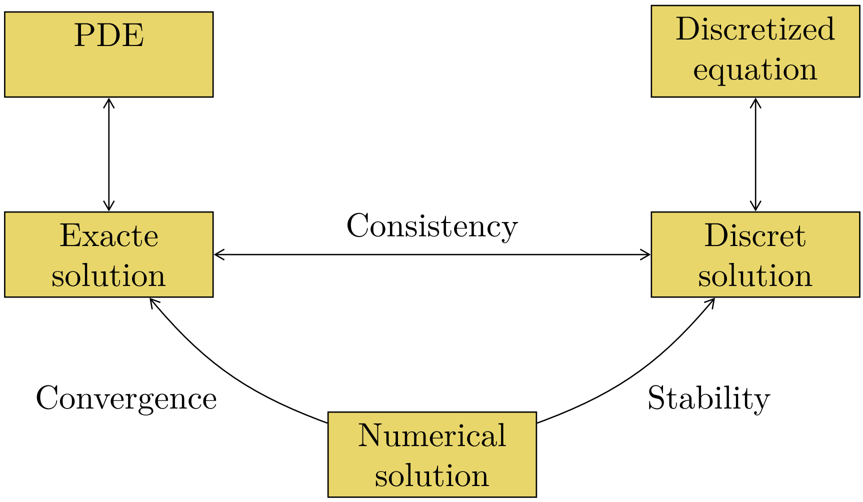

Figure 3.2: Exact, numerical, and discrete solution.

3.2 Presentation of the Method

Let us introduce this methodology by considering the nonlinear differential equation

$$\mathcal{G}f = g \tag{3.2.1}$$

The operator $\mathcal{G}$ is generally composed of linear and nonlinear parts. Therefore, 3.2.1 rewritten as

$$\mathcal{L}f + \mathcal{R}f + \mathcal{N}(f) = g \tag{3.2.2}$$

Here,

- $\mathcal{G}$: is a general nonlinear operator.
- $\mathcal{L}$: is a linear operator to be inverted, which usually is just the highest-order differential

operator.

- $\mathcal{R}$: is a linear remainder operator .
- $\mathcal{N}$: is the nonlinear operator, which is assumed to be analytic.
- g : is the given function .
- f : is the unknown function satisfying (3.2.2).

The ADM decomposes both the solution and the nonlinear term into infinite series [9]

$$f = \sum_{m=0}^{+\infty} f_m \tag{3.2.3}$$

$$\mathcal{N}(f) = \sum_{m=0}^{+\infty} \mathcal{A}_m(f_0, f_1, ..., f_m) \tag{3.2.4}$$

Where the $\mathcal{A}_m$ are known as Adomian polynomials. In the first approach given by Adomian [11] the $\mathcal{A}_m$ are obtained using the analytic parametrization with ξ of f and $\mathcal{N}(f)$

$$f = \sum_{m=0}^{+\infty} f_m \xi^m \tag{3.2.5}$$

$$\mathcal{N}(f) = \mathcal{N}(\sum_{m=0}^{+\infty} f_m \xi^m) = \sum_{i=0}^{+\infty} \mathcal{A}_i(f_0, f_1, ..., f_i)\xi^i \tag{3.2.6}$$

$\mathcal{A}_m$'s are formally defined by the classical formula[1] [16]

$$\mathcal{A}_m(f_0, f_1, ..., f_m) = \frac{1}{m!} \left[\!\left[\frac{d^m}{d\xi^m} \mathcal{N}\left(\sum_{i=0}^{+\infty} \xi^i f_i \right) \right]\!\right]_{\xi=0}, m \geq 0 \tag{3.2.7}$$

The analytic parameter ξ In (3.2.7) was a convenient parameter useful for the development of the definitions for $\mathcal{A}_m$, and is convenient for grouping terms. For instance, if $\mathcal{N}(f) = e^f$

$$\begin{aligned} \mathcal{A}_0 &= \mathcal{N}(f_0 + \xi f_1 + \xi^2 f_2 + \dots)\Big|_{\xi=0} \\ &= \mathcal{N}(f_0) \\ &= e^{f_0} \end{aligned}$$

[1] the relationship 3.2.7 are only formal and are not at all practical for computation

$$\begin{aligned}
\mathcal{A}_1 &= \frac{d}{d\xi}\mathcal{N}(f_0 + \xi f_1 + \xi^2 f_2 + \dots)\Big|_{\xi=0} \\
&= \frac{d}{d\xi}\Big(e^{(f_0+\xi f_1+\xi^2 f_2+\dots)}\Big)\Big|_{\xi=0} \\
&= \Big((f_1 + \xi f_2 + \dots)e^{(f_0+\xi f_1+\xi^2 f_2+\dots)}\Big)\Big|_{\xi=0} \\
&= f_1 e^{f_0} \\
&\vdots
\end{aligned}$$

Transform Eq. (3.2.2) to the integral Adomian equation (the canonical form) by applying the inverse linear operator $\mathcal{L}^{-1}$ on both sides

$$f = \gamma - \mathcal{L}^{-1}\big(\mathcal{R}f + \mathcal{N}(f)\big) \tag{3.2.8}$$

Where $\gamma = \Phi + \mathcal{L}^{-1}(g)$, and Φ satisfies the homogeneous equation $\mathcal{L}\Phi = 0$, and it is identified in terms of initial and/or boundary conditions.

The substitution of the Adomian decomposition series for the solution f and the series of Adomian polynomials tailored to the nonlinearity $\mathcal{N}(f)$ Eq. (3.2.3) and (3.2.4) into Eq. (3.2.8), leads to

$$\sum_{m=0}^{+\infty} f_m = \gamma - \mathcal{L}^{-1}\Big(\mathcal{R}(\sum_{m=0}^{+\infty} f_m) + \sum_{m=0}^{+\infty} \mathcal{A}_m(f_0, f_1, \dots, f_m)\Big) \tag{3.2.9}$$

Or equivalently,

$$\begin{aligned}
f_0 + f_1 + f_2 + \dots + f_m + \dots = \gamma - \mathcal{L}^{-1}\Big(\mathcal{R}(f_0) + \mathcal{A}_0(f_0)\Big) - \mathcal{L}^{-1}\Big(\mathcal{R}(f_1) \\
+\mathcal{A}_1(f_0, f_1)\Big) - \dots - \mathcal{L}^{-1}\Big(\mathcal{R}(f_{m-1}) + \mathcal{A}_{m-1}(f_0, f_1, \dots, f_{m-1})\Big) \\
+\dots
\end{aligned} \tag{3.2.10}$$

The series solution components f_m are then identified using the classical Adomian recursion scheme

$$\begin{cases}
f_0 &= \gamma \\
f_m &= -\mathcal{L}^{-1}\Big(\mathcal{R}(f_{m-1}) + \mathcal{A}_{m-1}(f_0, f_1, \dots, f_{m-1})\Big), m \geq 1
\end{cases} \tag{3.2.11}$$

Although the schema is not unique (3.2.11), it is the only scheme that permits an explicit definition of f_m. Practically, It is almost always impossible to calculate the sum of the series $\sum_{i=0}^{+\infty} f_i$. Therefore, through the relations from (3.2.11), the m-term approximation [2] of the series solution is defined as

$$\varphi_m = \sum_{i=0}^{m-1} f_i \tag{3.2.12}$$

In summary, after determining the $\{\mathcal{A}_m\}_{m\geq 0}$, a summation gives the approximate solution of the equation. However, the question that can already be asked is how to determine the $\{\mathcal{A}_m\}_{m\geq 0}$ and under what conditions the method converges.

Now, two important observations can be stated.

Remark :

- *$\mathcal{A}_0$ depends only on f_0, $\mathcal{A}_1$ depends only on f_0 and f_1, $\mathcal{A}_2$ depends only on f_0, f_1, f_2 and so on.*
- *The substitution of the first four Adomian polynomials into Eq. (3.2.4), see the list of the first 10 Adomian polynomials in Appendix A. gives*

$$\begin{aligned}
\mathcal{N}(f) &= \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \dots \\
&= \mathcal{N}(f_0) + \Big(f_1 + f_2 + f_3 + \dots\Big)\mathcal{N}'(f_0) \\
&+ \frac{1}{2!}\Big(f_1^2 + 2f_1f_2 + 2f_1f_3 + f_2^2 + \dots\Big)\mathcal{N}''(f_0) \\
&+ \frac{1}{3!}\Big(f_1^3 + 3f_1^2f_2 + 3f_1^2f_3 + 6f_1f_2f_3 + \dots\Big)\mathcal{N}'''(f_0) \\
&= \mathcal{N}(f_0) + \Big(f - f_0\Big)\mathcal{N}'(f_0) + \frac{1}{2!}\Big(f - f_0\Big)^2\mathcal{N}''(f_0) \\
&+ \frac{1}{3!}\Big(f - f_0\Big)^3\mathcal{N}'''(f_0) + \dots \\
&= \sum_{m=0}^{+\infty} \frac{\mathcal{N}^{(m)}(f_0)}{m!}\Big(f - f_0\Big)^m
\end{aligned}$$

The above expansion confirms that the series of Adomian polynomials $\sum \mathcal{A}_m$ is a Taylor series expansion about a function f_0 and not about a point, as is usually used.

[2] For concrete problems, the m-term approximation can be used for numerical approximations.

3.3 Adomian's polynomials

ADomian's polynomials, a concept credited to Adomian [15], which are essential in solving nonlinear equations, were termed as such by **Rach** [67] in clear recognition of Adomian's groundbreaking contribution to mathematics. In the literature, Several schemes have been introduced by researchers to calculate the Adomian polynomials. Adomian introduced a scheme for calculating the Adomian polynomials that were formally and widely regarded as straightforward and practical.

In the sequel, the general Adomian algorithm for calculating the Adomian polynomials was presented in addition to a summary of the necessary steps to calculate the first few Adomian polynomials.

3.3.1 Implicit formula of Adomian's Polynomials

Let $\mathcal{F}$ be a Banach space.

Theorem 3.3.1 (*Adomian's polynomials*)

Let $\mathcal{M}$ be an analytical function and $\sum f_m$ a convergent series in $\mathcal{F}$. The Adomian's Polynomials are defined by

$$\mathcal{A}_m(f_0, f_1, ..., f_m) = \frac{1}{m!}\left[\!\left[\frac{d^m}{d\xi^m}\mathcal{M}\left(\sum_{i=0}^{+\infty}\xi^i f_i\right)\right]\!\right]_{\xi=0}, m \geq 0 \tag{3.3.1}$$

Remark :

- *It is clear from (3.3.1) that $\mathcal{A}_m$ are polynomials of $f_1, f_2, \ldots, f_m$.*
- *The dependence of $\mathcal{A}_m$ on the independent variables and f_0 may be non-polynomial.*
- *The $\mathcal{A}_m$ are not unique there are other version of Adomian polynomials.(the accelerated Adomian polynomials $\bar{\mathcal{A}}_m$, the modified Adomian polynomials $\tilde{\mathcal{A}}_m$).*
- *The $\mathcal{A}_m$ can be calculated for a wide class of nonlinearities such as polynomial, negative power, composite, and even decimal power nonlinearities, among other classes of strong nonlinearities.*

Example : *Here are some examples of the computation of Adomian polynomials for certain* *nonlinear functions.*

1. *Consider the nonlinear function $\mathcal{N}(f) = f^2$, the first few Adomian polynomials are*

$$
\begin{aligned}
\mathcal{A}_0(f_0) &= f_0^2 \\
\mathcal{A}_1(f_0, f_1) &= 2f_1 f_0 \\
\mathcal{A}_2(f_0, f_1, f_2) &= f_1^2 + 2f_0 f_2 \\
\mathcal{A}_3(f_0, f_1, f_2, f_3) &= 2f_1 f_2 + 2f_0 f_3 \\
\mathcal{A}_4(f_0, f_1, f_2, f_3, f_4) &= f_2^2 + 2f_1 f_3 + 2f_0 f_4 \\
&\vdots
\end{aligned}
$$

2. *For the nonlinear function $\mathcal{N}(f) = e^f$*

$$
\begin{aligned}
\mathcal{A}_0(f_0) &= e^{f_0} \\
\mathcal{A}_1(f_0, f_1) &= f_1 e^{f_0} \\
\mathcal{A}_2(f_0, f_1, f_2) &= \left(f_2 + \frac{1}{2} f_1^2\right) e^{f_0} \\
\mathcal{A}_3(f_0, f_1, f_2, f_3) &= \left(f_3 + f_1 f_2 + \frac{1}{6} f_1^3\right) e^{f_0} \\
\mathcal{A}_4(f_0, f_1, f_2, f_3, f_4) &= \left(f_4 + \frac{1}{2} f_2^2 + f_1 f_3 + \frac{1}{2} f_1^2 f_2 + \frac{1}{24} f_1^4\right) e^{f_0} \\
&\vdots
\end{aligned}
$$

3. *For the nonlinear function $\mathcal{N}(f) = cos(f)$*

$$
\begin{aligned}
\mathcal{A}_0(f_0) &= cos(f_0) \\
\mathcal{A}_1(f_0, f_1) &= -f_1 sin(f_0) \\
\mathcal{A}_2(f_0, f_1, f_2) &= -f_2 sin(f_0) - \frac{1}{2} f_1^2 cos(f_0) \\
\mathcal{A}_3(f_0, f_1, f_2, f_3) &= -f_3 sin(f_0) - f_1 f_2 cos(f_0) - \frac{1}{3!} f_1^3 sin(f_0) \\
\mathcal{A}_4(f_0, f_1, f_2, f_3, f_4) &= -f_4 sin(f_0) - \left(\frac{1}{2} f_2^2 + f_1 f_2\right) cos(f_0) + \frac{1}{2} f_1^2 f_2 sin(f_0) + \frac{1}{4!} f_1^4 cos(f_0) \\
&\vdots
\end{aligned}
$$

3.3.2 Explicit formulae of Adomian polynomials

In 1994, **Abbaoui** and **Cherruault** [1, 2] proved that the Adomian polynomials $\mathcal{A}_m$ are defined by an explicit formula

Definition 3.3.2

The Adomian's polynomials are given by

$$\begin{cases} \mathcal{A}_0 = & \mathcal{N}(f_0) \\ \mathcal{A}_m = & \sum_{i=1}^{m} \mathcal{N}^{(i)}(f_0)\mathcal{P}_m^i(f_1, ..., f_m) \quad , \quad \forall m \geq 1 \end{cases} \tag{3.3.2}$$

With

$$\mathcal{P}_m^i(f_1, ..., f_m) = \sum_{n_1+n_2+\cdots+n_i=m} \frac{\left(\prod_{l=1}^{i} f_{n_l}\right)}{\left(\prod_{j=1}^{p}(k_j)!\right)} \tag{3.3.3}$$

Where $\mathcal{P}_m^i$ are homogeneous polynomials of m variables whose degree is i, p is the number of distinct n_j and k_j is their frequency.

In [1], a simpler formula for calculating Adomian's polynomials in function of the first term only (which is always known). Let us recall this formula:
For every sequence $\mathcal{U}_m(\xi) = \sum_{i=0}^{m} \xi^i f_i$, define $\mathcal{N}(\mathcal{U}_m(\xi))$ by [28]:

$$\mathcal{N}(\mathcal{U}_m(\xi)) = \sum_{i=0}^{m} \xi^i \mathcal{A}_i \tag{3.3.4}$$

Theorem 3.3.3

Suppose that $\mathcal{N}(f)$ is differentiable up to the m^{th} order, $\mathcal{A}_m$ are given by

$$\begin{cases} \mathcal{A}_0 & = \mathcal{N}(f_0) \\ \mathcal{A}_m & = \sum_{|mk|=m} \mathcal{N}^{(|k|)}(f_0)\frac{f_1^{k_1}}{k_1!}\frac{f_2^{k_2}}{k_2!}\cdots\frac{f_m^{k_m}}{k_m!}, \quad m \geq 1 \end{cases} \tag{3.3.5}$$

Where $|k| = k_1 + \cdots + k_m$, and $|mk| = k_1 + 2k_2 + \cdots + mk_m$.

Proof : Applying the classical formula [44] given the m^{th} derivative of composed function

$\mathcal{A}(\xi) = (\mathcal{N} \circ \mathcal{U}_m)(\xi)$, we obtain

$$\begin{aligned}
\mathcal{A}_m(f_0, f_1, \ldots, f_m) &= \left[\!\!\left[\frac{d^m(\mathcal{N} \circ \mathcal{U}_m)}{d\xi^m}(\xi) \right]\!\!\right] \\
&= \frac{1}{m!} \sum_{|mk|=m} \mathcal{N}^{(|k|)}(f_0) \frac{m! f_1^{k_1} (2! f_2)^{k_2} \ldots (m! f_m)^{k_m}}{(1!)^{k_1} (2!)^{k_2} \ldots (m!)^{k_m} k_1! \ldots k_m!} \\
&= \sum_{|mk|=m} \mathcal{N}^{(|k|)}(f_0) \frac{f_1^{k_1}}{k_1!} \frac{f_2^{k_2}}{k_2!} \cdots \frac{f_m^{k_m}}{k_m!}
\end{aligned}$$

■

Corollary 3.3.4

$$\begin{cases} \mathcal{A}_0 &= \mathcal{N}(f_0) \\ \mathcal{A}_m &= \sum\limits_{\alpha_1+\alpha_2+\cdots+\alpha_m=m} \mathcal{N}^{(\alpha_1)}(f_0) \frac{f_1^{(\alpha_1-\alpha_2)}}{(\alpha_1-\alpha_2)!} \frac{f_2^{(\alpha_2-\alpha_3)}}{(\alpha_2-\alpha_3)!} \cdots \frac{f_{m-1}^{(\alpha_{m-1}-\alpha_m)}}{(\alpha_{m-1}-\alpha_m)!} \frac{f_m^{(\alpha_m)}}{\alpha_m!}, \quad m \geq 1 \end{cases} \tag{3.3.6}$$

Where $(\alpha_i)_{i=1,2,\ldots,m}$ is a decreasing sequence.

Proof : It is sufficient to choose

$$\begin{aligned}
k_1 &= \alpha_1 - \alpha_2 \\
k_2 &= \alpha_2 - \alpha_3 \\
k_3 &= \alpha_3 - \alpha_4 \\
&\vdots \\
k_{m-1} &= \alpha_{m-1} - \alpha_m \\
k_m &= \alpha_m
\end{aligned}$$

Which leads to

$$k_1 + 2k_2 + 3k_3 + \cdots + mk_m = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_m = m$$

And

$$k_1 + k_2 + k_3 + \cdots + k_m = \alpha_1$$

■

Remark :

- *The sum in 3.3.6 has to be done on all the solutions of the equation*

$$\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_m = m, \quad \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots \geq \alpha_m \tag{3.3.7}$$

- *The formula (3.3.6) allows us to calculate quickly $\mathcal{A}_m$. The number of all possible nonnegative solutions of equation (3.3.7) $\mathcal{P}(m)$ is easily obtained using Number theory [27].*

Theorem 3.3.5

$$\mathcal{P}(m) < e^{\pi\sqrt{\frac{2m}{3}}}, \text{for} \qquad m \in \mathbb{N}^* \tag{3.3.8}$$

Proof : See [3]. ■

The following theorem gives some properties of the $\mathcal{A}_m$.

Theorem 3.3.6

1.
$$\frac{\partial}{\partial f_0}\mathcal{A}_{m-k} = \frac{\partial}{\partial f_k}\mathcal{A}_m, \quad \forall m, k, \quad m \geq k \tag{3.3.9}$$

2.
$$\mathcal{A}_{m+1} = \frac{1}{m+1}\sum_{k=0}^{m}(k+1)f_{k+1}\frac{\partial}{\partial f_k}\mathcal{A}_m \tag{3.3.10}$$

Proof : See [3]. ■

Definition 3.3.7

$$\mathcal{A}_m = \sum_{\alpha_1+\cdots+\alpha_m=m} c_{\alpha_1,\ldots,\alpha_m}(\mathcal{N}(f_0))^{(m+1-\alpha_1)}(\mathcal{N}^{(1)}(f_0))^{(\alpha_1-\alpha_2)}\ldots (\mathcal{N}^{(m-1)}(f_0))^{(\alpha_{m-1}-\alpha_m)}(\mathcal{N}^{(m)}(f_0))^{(\alpha_m)} \tag{3.3.11}$$

Where

$$c_{\alpha_1,\ldots,\alpha_m} = \frac{m!}{(\alpha_1-\alpha_2)!\ldots(\alpha_{m-1}-\alpha_m)!(\alpha_m)!(1!)^{(\alpha_1-\alpha_2)}\ldots((m-1)!)^{(\alpha_{m-1}-\alpha_m)}(m+1-\alpha_1)!} \tag{3.3.12}$$

Proof : We first remark that

$$\mathcal{A}_m = \frac{1}{(m+1)!}\frac{d^m}{d\xi^m}[\mathcal{N}(f_0)^{m+1}]$$

which can be proved easily by induction. Then, using a classical formula [44] for calculating the n th derivative of a function, it leads to (3.3.11), with parameters $c_{\alpha_1,\ldots,\alpha_m}$. given by (3.3.12). ■

3.4 Convergence of the Method

Yves Cherruault and his research team at the **Medimat laboratory** proved the convergence of ADM in the context of abstract functional equations; he introduced a new formulation of the method and used an important theorem, which is **Fixed point theorem** involving sufficient conditions in his proof [28]. All these conditions relate to the nonlinear operator $\mathcal{N}$.

Among the substantial questions that arise about the Adomian method, the following ones

- Under which conditions do the series $\sum f_i$ and $\sum \mathcal{A}_i$ converge?
- is $\sum f_i$ a solution to the original equation?

Figure 3.3: Yves Cherruault (1937-2010) was a French mathematician

3.4.1 For Functional Equations

Consider the general functional equation.

$$f - \mathcal{N}(f) = g, \qquad f \in \mathbb{H} \tag{3.4.1}$$

Where

- $\mathbb{H}$ is a Hilbert space.

- The nonlinear operator $\mathcal{N}$ is defined from $\mathbb{H}$ into itself.

3.4.1.1 Convergence based on the fixed point theorem

For every sequence $\{\Phi_m\}_{m\geq 0}$ defined as the partial sum of the series $\sum f_i$

$$\Phi_m = \sum_{i=0}^{m} f_i \tag{3.4.2}$$

Also, $\mathcal{N}(f)$ is approximated by

$$\mathcal{N}_m(f) = \sum_{i=0}^{m} \mathcal{A}_i \tag{3.4.3}$$

The Adomian method is equivalent to find the Sequence $\{\mathcal{S}_m\}_{m\geq 0}$ defined by

$$\mathcal{S}_m = \sum_{i=1}^{m} f_i \tag{3.4.4}$$

Using the following iterative scheme:

$$\begin{cases} \mathcal{S}_0 & = 0 \\ \mathcal{S}_{m+1} & = \mathcal{N}_m(f_0 + \mathcal{S}_m), \qquad m \geq 0 \end{cases} \tag{3.4.5}$$

Theorem 3.4.1 (*Classical theorem of convergence*)

If the following conditions are satisfied:

✓ The nonlinear operator $\mathcal{N}$ be a contraction (i.e: $\delta < 1$)

✓ $\left\|\mathcal{N}_m - \mathcal{N}\right\| = \mu_m \longrightarrow 0 \qquad \text{as } n \to +\infty$

Then, the sequence $\{\mathcal{S}_m\}_{m\geq 0}$ defined by the previous iterative scheme (3.4.5) converges towards to the solution of **Fixed Point Equation**

$$\mathcal{S} = \mathcal{N}(f_0 + \mathcal{S}) \tag{3.4.6}$$

Proof : We have the following equalities and inequalities:

$$\begin{aligned}\left\|\mathcal{S}_{m+1}-\mathcal{S}\right\| &= \left\|\mathcal{N}_m(f_0+\mathcal{S}_m)-\mathcal{N}(f_0+\mathcal{S})\right\| \\ &= \left\|\mathcal{N}_m(f_0+\mathcal{S}_m)-\mathcal{N}(f_0+\mathcal{S}_m)+\mathcal{N}(f_0+\mathcal{S}_m)-\mathcal{N}(f_0+\mathcal{S})\right\| \\ &\leq \left\|\Big(\mathcal{N}_m-\mathcal{N}\Big)(f_0+\mathcal{S}_m)\right\| + \left\|\mathcal{N}\Big(\mathcal{S}_m-\mathcal{S}\Big)\right\| \\ &\leq \left\|\mathcal{N}_m-\mathcal{N}\right\|\left\|f_0+\mathcal{S}_m\right\| + \delta\left\|\mathcal{S}_m-\mathcal{S}\right\| \\ &\leq \mu_m\Big(\left\|f_0\right\|+\left\|\mathcal{S}_m\right\|\Big)+\delta\left\|\mathcal{S}_m-\mathcal{S}\right\|\end{aligned}$$

Let us make the hypothesis

$$\left\|\mathcal{S}\right\| \leq \frac{N_0}{2}, \qquad \text{and} \qquad \left\|f_0\right\| \leq N_0$$

And the recurrent hypothesis

$$\left\|\mathcal{S}_m-\mathcal{S}\right\| \leq \frac{N_0}{2}, \qquad \text{which involves} \qquad \left\|\mathcal{S}_m\right\| \leq N_0 \tag{3.4.7}$$

This leads to the inequality

$$\left\|\mathcal{S}_{m+1}-\mathcal{S}\right\| \leq \frac{N_0}{2}(1+4\mu_m)$$

If we need to have $\left\|\mathcal{S}_{m+1}-\mathcal{S}\right\| \leq \frac{N_0}{2}\big(\delta+\mu\big)$ where $\delta+\mu$ is such that $(\delta+\mu)<1$. It suffices to choose $m \geq N_\mu$ such that $\left\|\mathcal{N}_m-\mathcal{N}\right\| = \mu_m \leq \frac{\mu}{4}$. The recurrent hypothesis is thus satisfied, and the theorem is proved. ■

As a remark, it is clear that the second condition in Theorem 3.4.1 implies the convergence of the series $\sum \mathcal{A}_i$.

3.4.1.2 Convergence based on properties of entire series

Another proof of convergence was given in [29] by using properties of the entire series substituted in another series.

Theorem 3.4.2

Consider an entire series

$$\sum_{m=0}^{\infty} a_m x^m \tag{3.4.8}$$

with a convergence radius R. Suppose that

$$x = \sum_{m=0}^{\infty} b_m \xi^m \tag{3.4.9}$$

If we replace x in (3.4.8) by the expression in (3.4.9), we have an entire series

$$x = \sum_{m=0}^{\infty} c_m \xi^m \tag{3.4.10}$$

where the c_m are given by

$$\begin{cases} c_0 &= a_0 + a_1 b_0 + a_2 b_0^2 + \cdots + a_m b_0^m + \dots \\ c_1 &= a_1 b_1 + 2a_2 b_1 b_0 + \cdots + m a_m b_0^{m-1} b_1 + \dots \\ c_2 &= a_1 b_2 + a_2(b_1^2 + 2b_0 b_2) + \dots \\ & \vdots \end{cases} \tag{3.4.11}$$

If we have

$$\begin{cases} |b_m| & \leq \frac{M}{1+\varepsilon}, \qquad m \geq 0, \qquad \varepsilon > 0 \\ M & < R \\ \varepsilon & \geq \frac{M}{R-M} \end{cases} \tag{3.4.12}$$

Then, the series in (3.4.10) has a radius of convergence $(R \geq 1)$.

Proof : It is sufficient to prove that

$$\sum_{m=0}^{\infty} |b_m||\xi|^m < R, \qquad \text{for} \qquad |\xi| < 1 \tag{3.4.13}$$

From (3.4.9) we have

$$\sum_{m=0}^{\infty} |b_m||\xi|^m \leq \sum_{m=0}^{\infty} M \left(\frac{|\xi|}{1+\varepsilon}\right)^m \tag{3.4.14}$$

Suppose we let $|\xi| < 1 + \varepsilon$, then from (3.4.14) it follows that

$$\sum_{m=0}^{\infty} |b_m||\xi|^m \leq \frac{M}{1 - \frac{|\xi|}{1+\varepsilon}} \tag{3.4.15}$$

$$\frac{1}{1-\frac{|\xi|}{1+\varepsilon}} < R \longleftrightarrow |\xi| < (1+\varepsilon)(1-\frac{M}{R}) \tag{3.4.16}$$

From (3.4.15) we have

$$(1+\varepsilon)(1-\frac{M}{R}) \geq \left(1+\frac{M}{R-M}\right)(1-\frac{M}{R}) = 1 \tag{3.4.17}$$

so that the result is proved. ■

Lemma 3.4.3

If

$$\sum_{|i|=n} i^0 = \sum_{|i|=n} 1 \tag{3.4.18}$$

Then

$$\sum_{|i|=n} 1 = \frac{(m+n)!}{n!m!} \tag{3.4.19}$$

where $i = (i_1, \ldots, i_m) \in \mathbb{N}^m$.

Lemma 3.4.4

Suppose that $\mathcal{N}$ is an analytic function in $]x_0 R; x_0 + R[$, and furthermore

$$\|\mathcal{N}^n(x_0)\| \leq n! M \alpha^n \tag{3.4.20}$$

Then the Adomian polynomials satisfy the following expression:

$$\|\mathcal{A}_n\| \leq \frac{(2n)!}{(n+1)!n!} M^{n+1} \alpha^n, \qquad (n \geq 0) \tag{3.4.21}$$

The above lemmas lead to the following theorem.

Theorem 3.4.5

Suppose that $\mathcal{N}$ satisfies the following condition

$$\|\mathcal{N}^n(x_0)\| \leq n! M \alpha^n \tag{3.4.22}$$

Then the sufficient conditions for the convergence of the method are

1. $4M\alpha \leq 1$ if R is infinite.
2. $5M\alpha \leq 1$ if R is finite.

Proof :

Case 1: It is sufficient to prove that

$$\sum_{n=0}^{\infty} \|\mathcal{A}_n\| < \infty \tag{3.4.23}$$

$$\sum_{n=0}^{\infty} \|\mathcal{A}_n\| \leq \sum_{n=0}^{\infty} \frac{(2n)!}{(n+1)!n!} M^{n+1} \alpha^n \tag{3.4.24}$$

Using the Stirling [49] formula we obtain

$$\frac{(2n)!}{(n+1)!n!} M^{n+1} \alpha^n \sim \frac{(4M\alpha)^n M}{\sqrt{\pi}(n+1)^{\frac{3}{2}}}, \qquad n \to +\infty \tag{3.4.25}$$

Case 2:

$$\|\mathcal{A}_n\| \leq \frac{(2n)!}{(n+1)!n!} M^{n+1} \alpha^n = \frac{(2n)!M}{4^n (n+1)!n!} (4M\alpha)^n \tag{3.4.26}$$

If

$$X_n = \frac{(2n)!}{4^n (n+1)!n!} \tag{3.4.27}$$

Then we have

$$\frac{X_{n+1}}{X_n} = \frac{2n+1}{2n+4} < 1 \Longrightarrow X_n < X_0 = 1 \tag{3.4.28}$$

Consequently,

$$\|\mathcal{A}_n\| \leq \frac{(2n)!}{(n+1)!n!} M^{n+1} \alpha^n = M(4M\alpha)^n = \frac{M}{(1+\varepsilon)^n} \tag{3.4.29}$$

where

$$\varepsilon = \frac{1}{4M\alpha - 1}, \qquad (4M\alpha < 1) \tag{3.4.30}$$

It follows that

$$\varepsilon \geq \frac{M}{R-M} \Longleftrightarrow 5M\alpha < 1 \tag{3.4.31}$$

where R is a convergence radius.

■

3.4.2 For Initial Value Problems

In 2009, Abdelrazec and Pelinovsky [6] prove the convergence of ADM for an abstract initial value problem for differential equations with analytic vector fields using Cauchy-Kowalevskaya theorem.

Consider the abstract initial value problem:

$$\begin{cases} \frac{\partial f}{\partial t} &= \mathcal{L}f + \mathcal{N}(f), \qquad t > 0 \\ f(0) &= g \end{cases} \tag{3.4.32}$$

Where

- $\mathcal{L} : X \longrightarrow Y$: is a linear operator from Banach space X to a Banach space Y $(X \subseteq Y)$.
- $\mathcal{N}(f) : X \longrightarrow X$: is a nonlinear function on the Banach space X.
- $g \in X$: is an initial data.

Assumption 3.4.6

1. Let $\mathcal{L} : X \longrightarrow Y$ form a continuous semigroup $E(t) = e^{t\mathcal{L}} : X \longrightarrow X$ for $t \in \mathbb{R}_+$, and there is a constant $C > 0$ such that

$$\|E(t)f\|_X \leq C\|f\|_X, \qquad \forall f \in X, \qquad \forall t \in \mathbb{R}_+ \tag{3.4.33}$$

2. Let $\mathcal{N}(f) : X \longrightarrow X$ be an analytic near $f = g$
3. X be a Banach algebra with the property

$$\|fg\|_X \leq \|f\|_X\|g\|_X, \qquad \forall f, g \in X \tag{3.4.34}$$

Remark :

1. *The IVP (3.4.32) can be reformulated Using Duhamel's principle as an integral equation*

$$f(t) = E(t)g + \int_0^t E(t-s)\mathcal{N}(f(s))\, ds \tag{3.4.35}$$

2. *If $\mathcal{N}(f)$ is analytic near g, it satisfies a local Lipschitz condition in the ball $B_\delta(g)$ of radius $\delta > 0$ centered at g. i.e., there is a constant K_δ such that*

$$\left\|\mathcal{N}(f) - \mathcal{N}(\tilde{f})\right\|_X \leq K_\delta \left\|f - \tilde{f}\right\|_X, \qquad \forall f, \tilde{f} \in B_\delta(g) \tag{3.4.36}$$

The local well-posedness of solutions of the IVP (3.4.32) with Lipschitz vector field $\mathcal{N}(f)$ can be proved for small time intervals Using Picard's method of successive iterations adopted for partial differential equations by Kato [51].

Theorem 3.4.7 (***Picard kato***)

1. If $\mathcal{L}$ and $\mathcal{N}(f)$ satisfy Assumption 3.4.6 and $g \in X$.

Then, there exists a $T > 0$ and a unique solution $f(t)$ of the IVP on $[0, T]$ such that

$$f(t) \in C([0,T], X) \cap C^1([0,T], Y) \tag{3.4.37}$$

Moreover, the solution $f(t)$ depends continuously on the initial data g

Proof : Basically, In this proof successive iterations are used
Starting with the free solution

$$f^{(0)} = E(t)g \tag{3.4.38}$$

The sequence of Picard's approximations defined from $\left\{f^{(m)}(t)\right\}_{m\in\mathbb{N}}$ is defined from $f^{(0)}(t)$ on a small interval $[0, T]$ according to the following recurrence relation

$$f^{(m+1)}(t) = f^{(0)}(t) + \int_0^t E(t-s)\mathcal{N}(f^{(m)}(s))\, ds, \qquad m \geq 0 \tag{3.4.39}$$

For any $\delta > 0$, there exists a $T > 0$ such that

$$\left\|f^{(0)} - g\right\|_X \leq \frac{1}{2}\delta, \qquad \forall t \in [0, T] \tag{3.4.40}$$

By the induction method, we obtain for any $m \geq 0$

$$\begin{aligned}
\sup_{t\in[0,T]} \left\|f^{(m+1)}(t) - g\right\|_X &\leq \sup_{t\in[0,T]} \left\|f^{(m+1)}(t) - f^{(0)}\right\|_X + \sup_{t\in[0,T]} \left\|f^{(0)}(t) - g\right\|_X \\
&\leq CT \sup_{t\in[0,T]} \left\|\mathcal{N}(f^{(m)}(t))\right\|_X + \frac{\delta}{2} \\
&\leq \frac{\delta}{2} + \frac{\delta}{2} = \delta
\end{aligned}$$

provided that T satisfies the bound

$$CT \sup_{t\in[0,T]} \left\|\mathcal{N}(f^{(m)}(t))\right\|_X \leq \frac{\delta}{2} \tag{3.4.41}$$

Therefore, the iteration operator (3.4.39) maps $C([0,T], B_\delta(g))$ to $C([0,T], B_\delta(g))$ for small $T > 0$. Furthermore, if $CTK_\delta < 1$ the map iteration is lipschitz and contraction. using the Banach fixed point theorem, there exists a unique solution $f(t)$ of the integral equation (3.4.35) in a complete metric space $C([0,T], B_\delta(g))$. If $f \in C([0,T], X)$, then $\mathcal{L}f + \mathcal{N}(f) \in C([0,T], Y)$ so that $f \in C^1([0,T], Y)$. ■

Theorem 3.4.8 (*Cauchy-Kovalevskaya Theorem*)

1. Let the assumptions 3.4.6 be satisfied.
2. Let $f(t)$ be the unique solution of the IVP (3.4.32) in $C([0,T],X)$, where T is the maximal existence time.

Then, there exists $\tau \in (0,T)$ such that $f:[0,\tau] \longrightarrow X$ is also a real analytic function.

Proof : By Cauchy estimates as $\mathcal{N}(f)$ analytic near g, there exist constants $a, b > 0$ such that[3]

$$\frac{1}{m!}\left\|\partial_f^m \mathcal{N}(g)\right\|_X \leq \frac{b}{a^m}, \qquad \forall m \geq 0 \tag{3.4.42}$$

The Taylor series for $\mathcal{N}(f)$ at g converges for any $\|f-g\|_X \leq a$. Furthemore

$$\begin{aligned}
\|\mathcal{N}(f)\|_X &\leq \sum_{k=0}^{\infty} \frac{1}{k!}\left\|\partial_f^m \mathcal{N}(g)\right\|_X \left\|(f-g)^k\right\|_X \\
&\leq b \sum_{k=0}^{\infty} \frac{\|(f-g)\|_X^k}{k!} \\
&= \frac{ab}{a-\rho} \\
&=: h(\rho)
\end{aligned}$$

Where $0 \leq \rho = \|(f-g)\|_X < a$. From the majorant function $h(\rho)$, it is clear that

$$\frac{1}{m!}\left\|\partial_f^m \mathcal{N}(g)\right\|_X \leq \frac{1}{m!}\partial_\rho^m h(0), \qquad m \geq 0 \tag{3.4.43}$$

Let us consider the majorant problem

$$\begin{cases} \frac{d\rho}{dt} &= h(\rho), \qquad t > 0 \\ \rho(0) &= 0 \end{cases} \tag{3.4.44}$$

It has an explicit solution $\rho(t) = a - \sqrt{a^2 - 2abt}$ which is an analytic function of t on $(-\infty, \frac{a}{2b})$. If $f(t)$ solves the integral equation

$$f(t) = g + \int_0^t \mathcal{N}(f(s))\, ds \tag{3.4.45}$$

[3] $\partial_f^m \mathcal{N}(f)$ denote operators in the sense of Fréchet derivative.

Then

$$\begin{aligned}
\|f(t)-g\|_X &\le \int_0^t \|\mathcal{N}(f(s))\|_X\, ds \\
&\le \int_0^t h(\rho(s))\, ds \\
&= \rho(t) \\
&= \sum_{k=1}^{\infty} \frac{t^k}{k!}\partial_t^k \rho(0)
\end{aligned}$$

The majorant Taylor series is absolutely convergent for any $|t| < \frac{a}{2b}$. To achieve to the main result which is that $f(t)$ is analytic function in t on $[0, \frac{a}{2b})$, it remains to prove that

$$\|\partial_t^m f(0)\|_X \le \partial_t^m \rho(0), \qquad m \ge 1 \tag{3.4.46}$$

To prove the bound above for $m = 1, 2, 3$, we compute

$$\begin{aligned}
\partial_t f(t) &= \mathcal{N}(f(t)) \\
\partial_t^2 f(t) &= \mathcal{N}'(f(t))\mathcal{N}(f(t)) \\
\partial_t^3 f(t) &= \mathcal{N}''(f(t))\mathcal{N}(f(t))\mathcal{N}(f(t)) + \mathcal{N}'(f(t))\mathcal{N}'(f(t))\mathcal{N}(f(t))
\end{aligned}$$

As a result

$$\begin{aligned}
\|\partial_t f(0)\|_X &\le \|\mathcal{N}(f(0))\|_X \le h(\rho(0)) = \partial_t \rho(0) \\
\left\|\partial_t^2 f(0)\right\|_X &\le \|\mathcal{N}'(f(0))\|_X \|\mathcal{N}(f(0))\|_X \le h'(\rho(0))h(\rho(0)) = \partial_t^2 \rho(0) \\
\left\|\partial_t^3 f(0)\right\|_X &\le \|\mathcal{N}''(f(0))\|_X \|\mathcal{N}(f(0))\|_X \|\mathcal{N}(f(0))\|_X + \|\mathcal{N}'(f(0))\|_X \|\mathcal{N}'(f(0))\|_X \|\mathcal{N}(f(0))\|_X \\
&\le h''(\rho(0))h(\rho(0))h(\rho(0)) + h'(\rho(0))h'(\rho(0))h(\rho(0)) = \partial_t^3 \rho(0)
\end{aligned}$$

Genereally,

$$f^{m+1}(t) = P_m(\mathcal{N}(f(t))), \qquad \forall m \ge 0$$

Where $P_m(\mathcal{N})$ is a polynomial of $\mathcal{N}$ and its Fréchet derivatives up to the m^{th} order with positive coefficients. As a result

$$\begin{aligned}\left\|\partial_t^{m+1} f(0)\right\|_X &= \|P_m(\mathcal{N}(f(0)))\|_X \\ &\le P_m(\|\mathcal{N}(f(0))\|_X) \\ &\le P_m(h(\rho(0))) \\ &= \partial_t^{m+1}\rho(0)\end{aligned}$$

If this is the case, then the Taylor series for $f(t)$ has the majorant series and hence it converges, by the Weierstrass M-test. ■

Remark :

1. *Existence and uniqueness of the solution $f(t)$ of the IVP (3.4.32) in $C^1([0,T),X)$ is proved in Theorem 3.4.7 for $Y = X$.*

According to ADM the solution f and the analytic function $\mathcal{N}(f)$ near $f = g$ are written in the series form.

$$f(t) = \sum_{m=0}^{\infty} f_m(t) \tag{3.4.47}$$

$$\mathcal{N}(f) = \sum_{m=0}^{\infty} \mathcal{A}_m(f_0, f_1, \ldots, f_m) \tag{3.4.48}$$

With the Adomian recursion scheme

$$\begin{cases} f_0(t) &= E(t)g \\ f_{m+1}(t) &= \int_0^t E(t-s)\mathcal{A}_m\Big(f_0(s), f_1(s), \ldots, f_m(s)\Big)\, ds, \qquad m \ge 0 \end{cases} \tag{3.4.49}$$

Theorem 3.4.9

- Let Assumption 3.4.6 be satisfied
- Let $f(t)$ be a unique solution of the integral equation (3.4.35) in $C([0,T),X)$, where T is the maximal existence time.
- Let $f_n(t)$ be defined by the recursion scheme (3.4.49).

Then, There exists a $\tau \in (0,T)$ such that the n^{th} partial sum $U_m(t) = \sum_{i=0}^{m} f_i(t)$ of the Adomian series (3.4.47) converges to the solution $f(t)$ in $C([0,\tau],X)$.

Proof : from the Theorem 3.4.7, for any given $\delta > 0$, there exist a $t_0 \in (0,T)$ such that

$$\sup_{t\in[0,t_0]} \|f_0(t) - g\|_X \le \frac{\delta}{2} \tag{3.4.50}$$

By choosing $\frac{\delta}{2} < a$, where a is the radius of analyticity of $\mathcal{N}(f)$ near g. The Cauchy estimates (3.4.42), (3.4.43) are generalized as

$$\frac{1}{m!}\left\|\partial_f^m \mathcal{N}(f_0)\right\|_X \leq \sum_{j\geq m} \frac{j(j-1)\dots(j-m+1)}{j!m!}\left\|\partial_f^j \mathcal{N}(g)\right\|_X \|f_0 - g\|_X^{j-m} \tag{3.4.51}$$

$$\leq b\sum_{j\geq m} \frac{j(j-1)\dots(j-m+1)}{m!}\frac{\|f_0 - g\|_X^{j-m}}{a^j} \tag{3.4.52}$$

$$= \frac{1}{m!}\partial_\rho^m h(\rho), \qquad m \geq 0 \tag{3.4.53}$$

Where $\rho = \|f_0 - g\|_X < a$ and $h(\rho) = \frac{ab}{a-\rho}$. Given $h(\rho)$, and let $\rho(t)$ satisfy the majorant problem 3.4.44) for $t \in [0, \frac{a}{2b})$. Using of the new Cauchy estimates (3.4.53) and the semigroup property (3.4.33) gives for any $t \in [0, \frac{a}{2b})$

$$\begin{aligned}
\|f_1(t)\|_X &\leq \int_0^t \|E(t-s)\mathcal{A}_0(f_0(s))\|_X\, ds \\
&\leq C\int_0^t \|\mathcal{N}(f_0(s))\|_X\, ds \\
&\leq C\int_0^t h(\rho(s))\, ds \\
&\leq (Ct)h(\rho(t)) = Ct\rho'(t)
\end{aligned}$$

And

$$\begin{aligned}
\|f_2(t)\|_X &\leq \int_0^t \|E(t-s)\mathcal{A}_1(f_0(s), f_1(s))\|_X\, ds \\
&\leq C\int_0^t \|\mathcal{N}'(f_0(s))f_1(s)\|_X\, ds \\
&\leq C^2\int_0^t h'(\rho(s))h(\rho(s))s\, ds \\
&\leq \frac{t^2C^2}{2}h'(\rho(t))h(\rho(t)) = \frac{t^2C^2}{2}\rho''(t)
\end{aligned}$$

By induction, Assuming that

$$\|f_i(t)\|_X \leq \frac{t^i C^i}{i!}\partial_t^i \rho(t), \qquad t \in \Big[0, \frac{a}{2b}\Big), \qquad \forall i \in \{1, 2, \dots, m\} \tag{3.4.54}$$

and demonstrate that the same relation remains true at $i = m + 1$

$$\|f_{m+1}(t)\|_X \leq \frac{t^{m+1} C^{m+1}}{(m+1)!}\partial_t^{m+1} \rho(t), \qquad t \in \Big[0, \frac{a}{2b}\Big) \tag{3.4.55}$$

As $\rho(t)$ analytic in t for all $t \in \Big[0, \frac{a}{2b}\Big)$, for any small $\xi > 0$, there exists a C^∞-function $\tilde{\rho}^\xi(t)$ on $\Big[0, \frac{a}{2b}\Big)$

$$\rho\Big((1 + \xi C)t\Big) = \sum_{i=0}^{m} \frac{\xi^i C^i t^i}{i!}\partial_t^i \rho(t) + \frac{\xi^{m+1} C^{m+1} t^{m+1}}{(m+1)!}\tilde{\rho}^\xi(t)$$

for any $\xi > 0$, let

$$U_m^\xi(t) = \sum_{i=0}^{m} \xi^i f_i(t)$$

then

$$\begin{aligned}
\big\|U_m^\xi(t)\big\|_X &\leq \sum_{i=0}^{m} \xi^i \|f_i(t)\|_X \\
&\leq \sum_{i=0}^{m} \frac{\xi^i C^i t^i}{i!}\partial_t^i \rho(t) \\
&= \rho\Big((1 + \xi C)t\Big) - \frac{\xi^{m+1} C^{m+1} t^{m+1}}{(m+1))!}\tilde{\rho}^\xi(t)
\end{aligned}$$

By definition of Adomian polynomials (3.2.7), we obtain

$$\mathcal{A}_m = \frac{1}{m!}\left[\!\!\left[\frac{d^m}{d\xi^m}\mathcal{N}\Big(\sum_{i=0}^{+\infty} \xi^i f_i\Big)\right]\!\!\right]_{\xi=0} = \frac{1}{m!}\left[\!\!\left[\frac{d^m}{d\xi^m}\mathcal{N}\Big(U_m^\xi\Big)\right]\!\!\right]_{\xi=0}$$

So that

$$\begin{aligned}
\|\mathcal{A}_m(t)\|_X &\leq \frac{1}{m!}\left[\!\!\left[\left\|\frac{d^m}{d\xi^m}\mathcal{N}\Big(U_m^\xi(t)\Big)\right\|_X\right]\!\!\right]_{\xi=0} \leq \frac{1}{m!}\left[\!\!\left[\frac{d^m}{d\xi^m}h\Big(\rho\Big((1 + \xi C)t\Big)\Big)\right]\!\!\right]_{\xi=0} \\
&\leq \frac{C^m t^m}{m!}\left[\!\!\left[\frac{d^m}{d\mu^m}h\Big(\rho(\mu)\Big)\right]\!\!\right]_{\mu=t} = \frac{C^m t^m}{m!}\left[\!\!\left[P_m\Big(h(\rho(\mu))\Big)\right]\!\!\right] \\
&= \frac{C^m t^m}{m!}\partial_t^{m+1} \rho(t)
\end{aligned}$$

Where $P_m(h)$[4] is a polynomial of h and its derivatives up to the m^{th} order with positive coefficients. Using the iterative formula (3.4.49)

$$\|f_{m+1}(t)\|_X \leq \int_0^t \|E(t-s)\mathcal{A}_m(s)\|_X \, ds \tag{3.4.56}$$

$$\leq \frac{C^{m+1}t^{m+1}}{(m+1)!}\partial_t^{m+1}\rho(t) \tag{3.4.57}$$

As a result, the series solution (3.4.47) is majorant in X by the power series

$$\rho\Big((1+C)t\Big) = \sum_{i=0}^{+\infty} \frac{C^i t^i}{i!}\partial_t^i \rho(t) \tag{3.4.58}$$

$$= a - \sqrt{a^2 - 2ab(1+C)t} \tag{3.4.59}$$

which converges for all $|t| < \frac{a}{2b(1+C)}$. Recall the constraint $t_0 \in (0,T)$ in bound (3.4.50). By the Weierstrass M-test, the series solution (3.4.47) converges to the unique solution $f(t)$ of (3.4.35) in $C([0,\tau],X)$ for any $\tau \in (0,\tau_0)$, where $\tau_0 = min\{t_0, \frac{a}{2b(1+C)}\}$. ∎

3.5 Order of convergence.

In [26], Babolian and Biazar discussed the order of convergence of the ADM.

Definition 3.5.1

Let $\{\mathcal{S}_m\}_{m\in\mathbb{N}}$ be a sequence converges to $\mathcal{S}$. If there exist two constants $C \geq 0$, $p \in \mathbb{N}$ such that

$$\lim_{m\longrightarrow+\infty}\left|\frac{\mathcal{S}_{m+1}-\mathcal{S}}{(\mathcal{S}_m-\mathcal{S})^p}\right| = C \tag{3.5.1}$$

Then the order of convergence of $\{\mathcal{S}_m\}_{m\in\mathbb{N}}$ is p.

Theorem 3.5.2

Suppose that $\mathcal{N} \in C^p([a,b])$, if $\mathcal{N}^{(i)}(f_0+\mathcal{S}) = 0, \quad \forall i \in \{1,2,\ldots,p-1\}$, and $\mathcal{N}^{(p)}(f_0+\mathcal{S}) \neq 0$, then the sequence $\{\mathcal{S}_m\}_{m\in\mathbb{N}}$ is of order p.

[4] the same as in the proof of Theorem 3.4.8.

Proof : For determining the order of convergence of $\{\mathcal{S}_m\}_{m\in\mathbb{N}}$ consider the Taylor expansion of $\mathcal{N}(f_0+\mathcal{S}_m)$

$$\mathcal{N}(f_0+\mathcal{S}_m)=\mathcal{N}(f_0+\mathcal{S})+\frac{\mathcal{N}^{(1)}(f_0+\mathcal{S})}{1!}(\mathcal{S}_m-\mathcal{S})+\frac{\mathcal{N}^{(2)}(f_0+\mathcal{S})}{2!}(\mathcal{S}_m-\mathcal{S})^2 +\dots+\frac{\mathcal{N}^{(i)}(f_0+\mathcal{S})}{i!}(\mathcal{S}_m-\mathcal{S})^i+\dots$$

Using (3.4.5), (3.4.6) we have

$$\mathcal{S}_{m+1}-\mathcal{S}=\frac{\mathcal{N}^{(1)}(f_0+\mathcal{S})}{1!}(\mathcal{S}_m-\mathcal{S})+\frac{\mathcal{N}^{(2)}(f_0+\mathcal{S})}{2!}(\mathcal{S}_m-\mathcal{S})^2 +\dots+\frac{\mathcal{N}^{(i)}(f_0+\mathcal{S})}{i!}(\mathcal{S}_m-\mathcal{S})^i+\dots \tag{3.5.2}$$

By the hypothesis of the theorem, (3.5.2) becomes

$$\mathcal{S}_{m+1}-\mathcal{S}=\frac{\mathcal{N}^{(p)}(f_0+\mathcal{S})}{p!}(\mathcal{S}_m-\mathcal{S})^p+\frac{\mathcal{N}^{(p+1)}(f_0+\mathcal{S})}{(p+1)!}(\mathcal{S}_m-\mathcal{S})^{p+1}+\dots \tag{3.5.3}$$

Which leads to

$$\frac{\mathcal{S}_{m+1}-\mathcal{S}}{(\mathcal{S}_m-\mathcal{S})^p}=\frac{\mathcal{N}^{(p)}(f_0+\mathcal{S})}{p!}+\frac{\mathcal{N}^{(p+1)}(f_0+\mathcal{S})}{(p+1)!}(\mathcal{S}_m-\mathcal{S})+\dots \tag{3.5.4}$$

By taking the limit when $m\to+\infty$, and since $\lim\limits_{m\to+\infty}\mathcal{S}_m=\mathcal{S}$

$$\lim_{m\to+\infty}\left|\frac{\mathcal{S}_{m+1}-\mathcal{S}}{(\mathcal{S}_m-\mathcal{S})^p}\right|=\left|\frac{\mathcal{N}^{(p)}(f_0+\mathcal{S})}{p!}\right|=C \tag{3.5.5}$$

Then, from the definition 3.5.1 the order of convergence of $\{\mathcal{S}_m\}_{m\in\mathbb{N}}$ is p. ■

Part II

The Main Results

4

ILLUSTRATIVE TEST EXAMPLES

The results of this chapter have been the subject of a publication:

☞ ***I. Achour, A. Bellagoun***, *Adomian Decomposition Method For Solving Spatially Inhomogenuous Population Balance Equation. Advances in Mathematics: Scientific Journal 12 (**2023**), no.1, 115-136.*

Contents

4.1 Introduction

ADM has been applied to solve PBEs for pure breakage for both batch and continuous systems [45], and for coalescence but only for the batch system [46]. In [47], the Adomian and the variational iteration methods have been applied to particle breakage equation for both batch and continuous flow systems. Furthermore, the variational iteration and projection methods have been used to solve certain spatially distributed population balance equations [48].

The results of this chapter have been published in Advances in Mathematics: Scientific Journal [7]. Our objective is to derive analytical solutions of spatially inhomogeneous PBEs that incorporate breakage, and coalescence in batch and continuous systems. Spatially inhomogeneous PBEs describe the time-space evolution of the particle number distribution function $f(t, v, z)$ under the simultaneous effect of breakage and coalescence processes in a continuous flow system [69, 78]. The PBE reads

$$\underbrace{\frac{\partial\left(f(t,v,z)\right)}{\partial t}}_{\text{the accumulation term}} + \underbrace{\frac{\partial(U_d f(t,v,z))}{\partial z}}_{\text{convection in physical space}} - \underbrace{\frac{\partial^2(D_d f(t,v,z))}{\partial z^2}}_{\text{diffusion in physical space}} = \frac{1}{\theta}\left(f^{feed}(v) - f(t,v,z)\right) + \underbrace{\phi(f,t,v,z)}_{\text{the source term}} \tag{4.1.1}$$

Where $\rho(f, t, v, z)$ represents the contribution to $f(t, v, z)$ of the change in the number of particles. due to particle breakage and coalescence [80, 81]:

$$\rho(f,t,v,z) = -\Gamma(v)f(t,v,z) + \int_v^{+\infty} \beta(v,v')\Gamma(v')f(t,v',z)\,dv' - \int_0^{+\infty} \omega(v,v')f(t,v',z)f(t,v,z)\,dv' + \frac{1}{2}\int_0^v \omega(v-v',v')f(t,v-v',z)f(t,v',z)\,dv' \tag{4.1.2}$$

The following is the list of relevant combinations of processes for which the continuous PBE has been solved analytically.

Case study I

$$\frac{\partial f(t,v,z)}{\partial t} + U_d\frac{\partial f(t,v,z)}{\partial z} = -\Gamma(v)f(t,v,z) + \int_v^{+\infty} \beta(v,v')\Gamma(v')f(t,v',z)\,dv' \tag{4.1.3}$$

Case study II

$$\frac{\partial f(t,v,z)}{\partial t} = D_d \frac{\partial^2 (f(t,v,z))}{\partial z^2} - \Gamma(v) f(t,v,z) + \int_v^{+\infty} \beta(v,v')\Gamma(v') f(t,v',z)\, dv' \tag{4.1.4}$$

Case study III

$$\begin{aligned}\frac{\partial f(t,v,z)}{\partial t} + U_d \frac{\partial f(t,v,z)}{\partial z} = \frac{1}{\theta}\left(f^{feed}(v) - f(t,v,z)\right) - \Gamma(v) f(t,v,z) \\ + \int_v^{+\infty} \beta(v,v')\Gamma(v') f(t,v',z)\, dv'\end{aligned} \tag{4.1.5}$$

Case study IV

$$\begin{aligned}\frac{\partial f(t,v,z)}{\partial t} + U_d \frac{\partial (f(t,v,z))}{\partial z} = \frac{1}{2}\int_0^v \omega(v-v',v') f(t,v-v',z) f(t,v',z)\, dv' \\ - \int_0^{+\infty} \omega(v,v') f(t,v',z) f(t,v,z)\, dv'\end{aligned} \tag{4.1.6}$$

In all case studies, the population balance equations are solved by the ADM, and all symbolic calculations are done using the MATHEMATICA SOFTWARE.

4.2 Linear case: Pure breakage

In a breakage process, particles break into two or many fragments. The total number of particles in this process increases while the total mass remains constant. Therefore, breakage has a significant effect on the number of particles.

In this section, we present three case studies for pure breakage in batch and continuous systems to illustrate how to use this technique. In all these cases, we have a linear breakage frequency $\Gamma(v) = v$ and a uniform daughter droplet distribution $\beta(v,v') = \frac{2}{v'}$.

4.2.1 PBE in batch system

4.2.1.1 Case study I: PBE with convection term

Consider the PBE

$$\frac{\partial f(t,v,z)}{\partial t} + U_d \frac{\partial f(t,v,z)}{\partial z} = -\Gamma(v) f(t,v,z) + \int_v^{+\infty} \beta(v,v')\Gamma(v') f(t,v',z)\, dv' \tag{4.2.1}$$

Subjected to the boundary condition $f(t,v,0) = te^{-v}$. And assuming that $U_d = 1$. The equation (4.2.1) becomes

$$\frac{\partial f(t,v,z)}{\partial t} + \frac{\partial f(t,v,z)}{\partial z} = -v f(t,v,z) + \int_v^{+\infty} 2f(t,v',z)\, dv' \tag{4.2.2}$$

Transforming equation (4.2.2) to the canonical form and operating both sides by $L_z^{-1}(.) = \int_0^z (.)\, dz$, we obtain

$$f(t,v,z) - f(t,v,0) = \int\limits_0^z \left(-\frac{\partial f(t,v,z)}{\partial t} - v f(t,v,z) \int_v^{+\infty} 2f(t,v',z)\, dv' \right) dz$$

Recall that the solution by ADM is written as:

$$f(t,v,z) = \sum_{m=0}^{+\infty} f_m(t,v,z)$$

where the solution components $f_m(t,v,z)$ are obtained by the Adomian recursion scheme. The first few terms of the series solution are

$$f_0(t,v,z) = te^{-v}$$

$$\begin{aligned} f_1(t,v,z) &= \int_0^z \left(-\frac{\partial f_0(t,v,z)}{\partial t} - v f_0(t,v,z) + \int_v^{+\infty} 2f_0(t,v',z)\, dv' \right) dz \\ &= -\frac{(1+t(-2+v))z}{e^v} \end{aligned}$$

$$
\begin{aligned}
f_2(t,v,z) &= \int_0^z \left(-\frac{\partial f_1(t,v,z)}{\partial t} - v f_1(t,v,z) + \int_v^{+\infty} 2 f_1(t,v',z)\, dv' \right) dz \\
&= \frac{(2(-2+v)+t(2-4v+v^2))z^2}{2e^v}
\end{aligned}
$$

$$
\begin{aligned}
f_3(t,v,z) &= \int_0^z \left(-\frac{\partial f_2(t,v,z)}{\partial t} - v f_2(t,v,z) + \int_v^{+\infty} 2 f_2(t,v',z)\, dv' \right) dz \\
&= -\frac{((6+v(3(-4+v)+t(6+(-6+v)v)))z^3)}{6e^v} \\
&\vdots
\end{aligned}
$$

Generally, $f_m(t,v,z)$ is the solution of

$$
f_m(t,v,z) = \int_0^z \left(-\frac{\partial f_{m-1}(t,v,z)}{\partial t} - v f_{m-1}(t,v,z) + \int_v^{+\infty} 2 f_{m-1}(t,v',z)\, dv' \right) dz \tag{4.2.3}
$$

Then we calculate the general term as :

$$
f_m(t,v,z) = (-vz)^m \left(\frac{vt\Big(v(v-2m)+(m-1)m\Big) + mv\Big(v-2(m-1)\Big)}{v^3 e^v m!} + \frac{(m-2)(m-1)m}{v^3 e^v m!} \right)
$$

So

$$
f(t,v,z) = \sum_{m=0}^{+\infty} (-vz)^m \left(\frac{vt\Big(v(v-2m)+(m-1)m\Big) + mv\Big(v-2(m-1)\Big)}{v^3 e^v m!} + \frac{(m-2)(m-1)m}{v^3 e^v m!} \right)
$$

Which converges to

$$
f(t,v,z) = \frac{(z+1)^2(t-z)}{e^{v(z+1)}}
$$

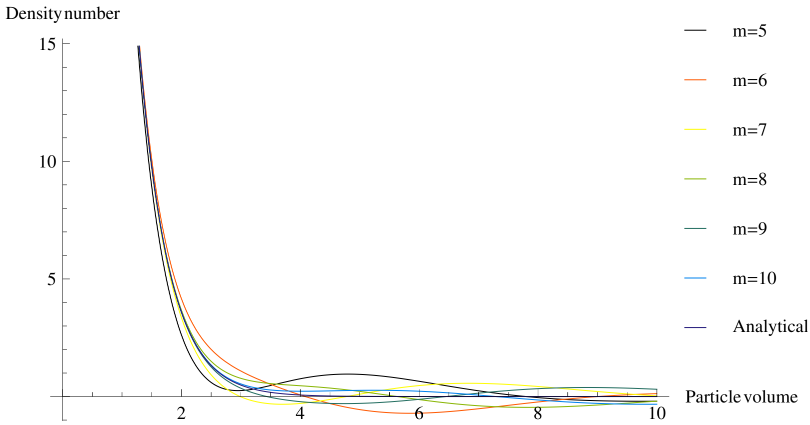

Figure 4.1: The effect of the truncation solution on the density number distribution for droplet breakage in a batch system with the boundary condition to be equal to $n(v,t,0) = te^{-v}$ at $z = 1$ and $t = 50$.

4.2.1.2 Case study II: PBE with diffusion term

Consider the PBE

$$\frac{\partial f(t,v,z)}{\partial t} = D_d \frac{\partial^2 (f(t,v,z))}{\partial z^2} - \Gamma(v) f(t,v,z) + \int_v^{+\infty} \beta(v,v')\Gamma(v') f(t,v',z)\,dv' \tag{4.2.4}$$

Subjected to the boundary conditions $f(t,v,0) = te^{-v}$ and $\frac{\partial f(t,v,0)}{\partial z} = e^{-v}$, and assuming that $D_d = 1$. Then, equation (4.2.4) becomes

$$\frac{\partial^2 (f(t,v,z))}{\partial z^2} = \frac{\partial f(t,v,z)}{\partial t} + v f(t,v,z) - \int_v^{+\infty} 2f(t,v',z)\,dv' \tag{4.2.5}$$

In this case $L_{zz}(.) = \frac{\partial^2}{\partial z^2}(.)$ and its inverse is $L_{zz}^{-1}(.) = \int_0^z \int_0^z (.)\,dz\,dz$. Applying the inverse operator to (4.2.5) gives

$$f(t,v,z) = f(t,v,0) + \frac{\partial f(t,v,0)}{\partial z} z + \int_0^z \int_0^z \left(\frac{\partial f(t,v,z)}{\partial t} + v f(t,v,z) - \int_v^{+\infty} 2f(t,v',z)\,dv' \right) dz\,dz$$

The solution by the ADM is determined by the following Adomian recursion scheme:

$$\begin{cases} f_0(t,v,z) &= te^{-v}+e^{-v}z \\ f_{m+1}(t,v,z) &= \int_0^z\int_0^z \left(\frac{\partial f_m(t,v,z)}{\partial t}+vf_m(t,v,z)-\int_v^{+\infty}2f_m(t,v',z)\,dv'\right)dz\,dz, \qquad m\geq 0 \end{cases}$$

From which, we calculate the solution components

$$f_1(t,v,z)=z^2\left(\frac{3+3t(-2+v)+(-2+v)z}{6e^v}\right)$$

$$f_2(t,v,z)=z^4\,\frac{5t((v-4)v+2)((v-4)v+2)z+10(v-2)}{120e^v}$$

$$f_3(t,v,z)=z^6\,\frac{(7v(t((v-6)v+6)+3(v-4))+v((v-6)v+6)z+42)}{5040e^v}$$

Then, the general term is

$$f_m(t,v,z)=\frac{e^{-v}v^{m-3}z^{2m}}{4^m(1)_m\left(\frac{3}{2}\right)_m}\Bigg(\Big(v(v-2m)+(m-1)m\Big)\Big((2m+1)tv+vz\Big) + m(2m+1)\Big(v(v-2(m-1))+(m-2)(m-1)\Big)\Bigg)$$

So

$$f(t,v,z)=\sum_{m=0}^{+\infty}\frac{e^{-v}v^{m-3}z^{2m}}{4^m(1)_m\left(\frac{3}{2}\right)_m}\Bigg(\Big(v(v-2m)+(m-1)m\Big)\Big((2m+1)tv+vz\Big) + m(2m+1)\Big(v(v-2(m-1))+(m-2)(m-1)\Big)\Bigg)$$

$$f(t,v,z)=\frac{z^2\,{}_3F_4\left(2,2,2;1,1,1,\frac{5}{2};\frac{vz^2}{4}\right)}{3e^vv^2}+\frac{\cosh\left(\sqrt{v}z\right)}{8e^vv^3}\Bigg(2tv^2\left(4v+z^2\right) - vz\Big(4v(z+2)+5z+6\Big)+1\Bigg)+\frac{\sinh\left(\sqrt{v}z\right)}{8e^vv^{7/2}z}\Bigg(2vz\Big(v\Big(2v(-2tz + z+2)+z(-t+z+2)+4\Big)+z+3\Big)-1\Bigg)$$

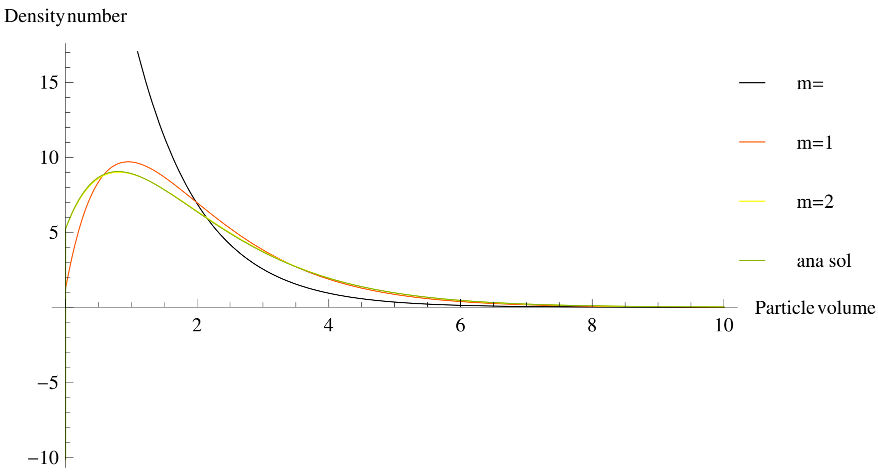

Figure 4.2: The effect of the truncation solution on the density number distribution for droplet breakage in batch system with the boundary condition to be equal to $f(t,v,0) = te^{-v}$, $\frac{\partial f(t,v,0)}{\partial z} = e^{-v}$ at $z = 1$ and $t = 50$.

4.2.2 PBE in continuous systems

4.2.2.1 Case study III: PBE with convection term

In this case, the boundary condition is $f(t,v,0) = te^{-v}$, the feed distribution is exponential and assuming that $U_d = 1$. We consider the following PBE:

$$\frac{\partial f(t,v,z)}{\partial t} + U_d \frac{\partial f(t,v,z)}{\partial z} = \frac{1}{\theta}\left(f^{feed}(v) - f(t,v,z)\right) - \Gamma(v) f(t,v,z) + \int_{v}^{+\infty} \beta(v,v')\Gamma(v') f(t,v',z)\, dv'$$

Rewrite the above equation to the following form

$$\frac{\partial f(t,v,z)}{\partial z} = -\frac{\partial f(t,v,z)}{\partial t} + \frac{1}{\theta} f^{feed}(v) - \left(v + \frac{1}{\theta}\right) f(t,v,z) + \int_{v}^{+\infty} 2 f(t,v',z)\, dv' \qquad (4.2.6)$$

Integrating the equation (4.2.6) with respect to z

$$f(t,v,z) = f(t,v,0) + \frac{1}{\theta} f^{feed}(v) z + \int_0^z \left(\frac{-\partial f(t,v,z)}{\partial t} - \left(v + \frac{1}{\theta} \right) f(t,v,z) + \int_v^{+\infty} 2 f(t,v',z)\, dv' \right) dz$$

Then the Adomian recursion scheme is

$$\begin{cases} f_0(t,v,z) & = te^{-v} + \frac{ze^{-v}}{\theta} \\ f_{m+1}(t,v,z) & = \int\limits_0^z \left(\frac{-\partial f_m(t,v,z)}{\partial t} - (v + \frac{1}{\theta}) f_m(t,v,z) + \int\limits_v^{+\infty} 2 f_m(t,v',z)\, dv' \right) dz, \qquad m \geq 0 \end{cases}$$

The second term is

$$f_1(t,v,z) = z \left(\frac{2 + (-2 + \frac{1}{\theta} + v)(2t + \frac{z}{\theta})}{e^v} \right)$$

The general term is

$$f_m(t,v,z) = \frac{(-1)^m (\frac{1}{\theta} + v)^{m-3} z^m}{(1)_{1+m} e^v} \Bigg((1+m) \Big(m(m-1)(m-2) + \frac{1}{\theta^3} t + \frac{1}{\theta^2} \Big(m + t(-2m + 3v) \Big)$$
$$+ v \Big(m(-2(m-1) + v) + t \big(m(m-1) + (-2m + v) v \big) \Big) + \frac{1}{\theta} \Big(2m((1-m) + v) + t \Big(m(m-1)$$
$$+ v(-4m + 3v) \Big) \Big) + \frac{1}{\theta} \left(\frac{1}{\theta} + v \right) \left(m(m-1) + (\frac{1}{\theta})^2 + \frac{2}{\theta}(-m + v) + (-2m + v) v \right) z \Bigg)$$

Then

$$f(t,v,z) = \sum_{m=0}^{+\infty} \frac{(-1)^m (\frac{1}{\theta} + v)^{m-3} z^m}{(1)_{1+m} e^v} \Bigg((1+m) \Big(m(m-1)(m-2) + \frac{1}{\theta^3} t + \frac{1}{\theta^2} \Big(m + t(-2m + 3v) \Big)$$
$$+ v \Big(m(-2(m-1) + v) + t \big(m(m-1) + (-2m + v) v \big) \Big) + \frac{1}{\theta} \Big(2m((1-m) + v) + t \Big(m(m-1)$$
$$+ v(-4m + 3v) \Big) \Big) + \frac{1}{\theta} \left(\frac{1}{\theta} + v \right) \left(m(m-1) + (\frac{1}{\theta})^2 + \frac{2}{\theta}(-m + v) + (-2m + v) v \right) z \Bigg)$$

So

$$f(t,v,z) = \frac{1}{(\frac{1}{\theta} + v)^3 e^v e^{z(\frac{1}{\theta} + v)}} \Bigg(\frac{1}{\theta^3} \Big(e^{z(\frac{1}{\theta} + v)} + (z+1)^2 (t - z - 1) \Big) + \frac{1}{\theta^2} \Big(2(v+1) e^{z(\frac{1}{\theta} + v)}$$
$$- (z+1) \Big(2 - v(z+1)(3t - 3z - 2) \Big) \Big) + \frac{1}{\theta} \Big(\Big(v(v+2) + 2 \Big) e^{z(\frac{1}{\theta} + v)}$$
$$- v(z+1) \Big(2 - v(z+1)(3t - 3z - 1) \Big) - 2 \Big) + v^3 (z+1)^2 (t - z) \Bigg)$$

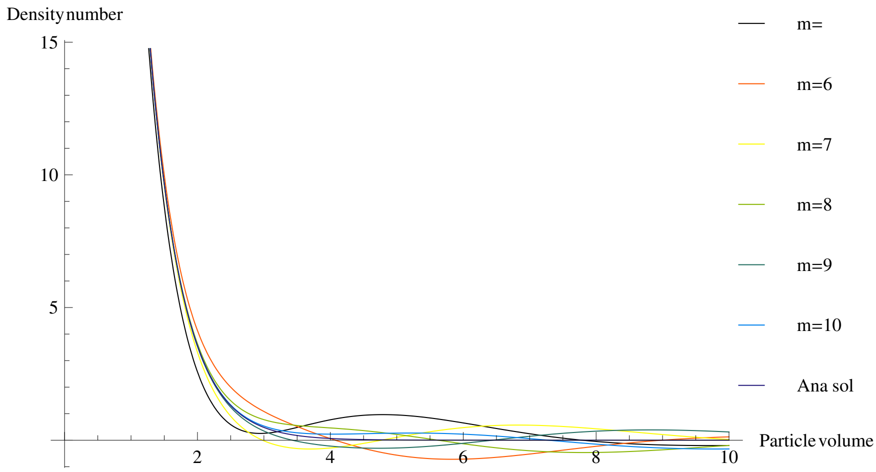

Figure 4.3: The effect of the truncation solution on the density number distribution for droplet breakage in continuous system with the boundary condition to be equal to $f(t, v, 0) = te^{-v}$ with exponential feed distribution at $z = 1$ and $t = 50$.

4.3 Nonlinear case: Pure Coalescence

In PBE the mechanism of coalescence poses the greatest numerical difficulty occurred by the non-linearity of this phenomena. We will consider the solution of equation (4.1.1) in the presence of only two terms, which are the convection and coalescence terms in a batch flow system in order to show the great accuracy of this technique.

4.3.1 PBE with convection term

In this section, we present only one case study for pure coalescence in batch systems.

$$\frac{\partial f(t,v,z)}{\partial t} + U_d \frac{\partial (f(t,v,z))}{\partial z} = \frac{1}{2} \int_0^v \omega(v-v',v') f(t,v-v',z) f(t,v',z)\, dv' - \int_0^{+\infty} \omega(v,v') f(t,v',z) f(t,v,z)\, dv' \tag{4.3.1}$$

Here we take $\omega(v,v')=1$, $U_d=1$ and the boundary condition $f(t,v,0)=te^{-v}$.
By considering $L_t=\frac{\partial}{\partial t}$ and $L_z=\frac{\partial}{\partial z}$, and operating both sides of equation (4.3.1) by L_z^{-1} (defined as $L_z^{-1}(.)=\int_0^z(.)\,dz$), then we obtain the canonical form

$$\begin{aligned}f(t,v,z)=f(t,v,0)+\int_0^z\Big(-\frac{\partial f(t,v,z)}{\partial t}&+\frac{1}{2}\int_0^v f(t,v-v',z)f(t,v',z)\,dv'\\&-\int_0^{+\infty}f(t,v',z)f(t,v,z)\,dv'\Big)\,dz\end{aligned}\tag{4.3.2}$$

And the nonlinear terms have the Adomian polynomial representation:

$$\begin{aligned}\frac{1}{2}\int_0^v f(t,v-v',z)f(t,v',z)\,dv'&=\frac{1}{2}\int_0^v\sum_{m=0}^{+\infty}A_m(v-v',v',t,x)\,dv'\\-\int_0^{+\infty}f(t,v',z)f(t,v,z)\,dv'&=-\int_0^{+\infty}\sum_{m=0}^{+\infty}B_m(v,v',t,x)\,dv'\end{aligned}\tag{4.3.3}$$

The polynomials A_m and B_m are obtained by the definitional formula (3.3.1). The solution by the ADM is calculated by the following Adomian recursion scheme:

$$\begin{cases}f_0(t,v,z) &= te^{-v}\\ f_{m+1}(t,v,z) &= \int\limits_0^z\left(\frac{\partial f_m(t,v,z)}{\partial t}+\frac{1}{2}\int\limits_0^v A_m(t,v-v',v',z)\,dv'-\int\limits_0^{+\infty}B_m(t,v,v',z)\,dv'\right)dz, \qquad m\geq 0\end{cases}$$

The first few terms are

$$f_1(t,v,z)=\left(\frac{-2-2t^2+t^2v}{e^v}\right)z$$

$$f_2(t,v,z)=t\left(\frac{-8(-2+v)+t^2(6-6v+v^2)}{8e^v}\right)z^2$$

$$\vdots$$

By rearrangement of terms, we can obtain the general term (See the Appendix C for more details)

$$f_m^*(t,v,z)=\frac{4v^m z^m(t-z)^{m+1}}{m!e^v(z(t-z)+2)^{m+2}}\tag{4.3.4}$$

Then

$$f(t,v,z)=\sum_{m=0}^{+\infty}\frac{4v^m z^m(t-z)^{m+1}}{m!e^v(z(t-z)+2)^{m+2}}\tag{4.3.5}$$

So the exact solution is

$$f(t, v, z) = \frac{4(t-z)e^{\frac{vz(t-z)}{tz-z^2+2}}}{e^v(z(t-z)+2)^2} \tag{4.3.6}$$

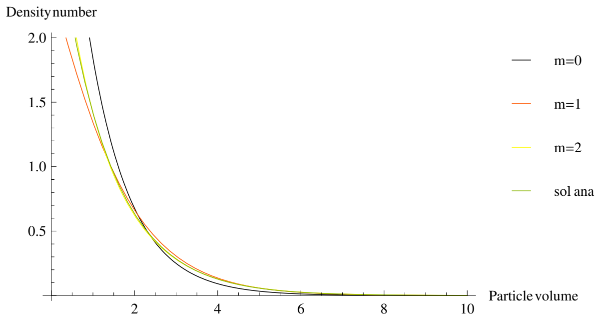

Figure 4.4: The effect of the truncation solution on the density number for droplet Coalescence with the boundary condition to be equal to $f(t, v, 0) = te^{-v}$ at $z = 0.1$ and $t = 5$.

For nonlinear models, more efficient algorithms and programs in MATHEMATICA for fast generation of Adomian polynomials to high orders have been provided by Duan in [38–40].

The Figure 4.2 and 4.4 shows that from the second iteration the m term approximant solutions were nearly identical to the analytical solutions. A rapid convergence was observed in these figures which shows the efficiency of the method. The Figure 4.1 and 4.3 represent the effect of the truncation on the density number for droplet breakage in batch and continuous flow particulate processes, respectively.

Conclusion

In this work, we are interested in the implementation of the ADM for the resolution of the population balance equation for continuous and discontinuous systems. The Adomian decomposition method has been successfully used to solve particle population balance equations in continuous and discontinuous flow systems with hypothetical functional forms of breakage/coalescence frequencies and daughter particle distributions. The solutions obtained by the ADM were infinite power series with appropriate boundary conditions. The method gave good approximations to the exact solutions with their series, which converge quickly for all the cases studied in this work. It is concluded that the ADM is robust and efficient and has a remarkable capacity to solve the population balance equation from an analytical or numerical point of view.

Some problems are still open until now. For instance, the practical convergence of the Adomian decomposition series may be ensured even if the hypotheses of the known method are not satisfied. which means that there still exist opportunities for further theoretical studies of convergence for more general situations. In addition, it is not always easy to take into account the boundary conditions for complex domains.

Bibliography

[1] Abbaoui, K and Cherruault, Y., *Convergence of Adomian's method applied to differential equations*, Computers & Mathematics with Applications, **28** (5): 103-109, 1994.

[2] Abbaoui, K and Cherruault, Y., *Convergence of Adomian's method applied to nonlinear equations*, Mathematical and Computer Modelling, **20**(9): 69-73, 1994.

[3] Abbaoui, K and Cherruault, Y., *New ideas for proving convergence of decomposition methods*, Computers & Mathematics with Applications, **29**(7): 103-108, 1995.

[4] Abbasbandy, S., *Numerical solutions of the integral equations: Homotopy perturbation method and Adomian's decomposition method*, Applied Mathematics and Computation, **173**(1): 493-500, 2006.

[5] Dudley, R. M. and Norvaiša R., *Concrete Functional Calculus*, Springer, 2011.

[6] Abdelrazec, A and Pelinovsky, D., *Convergence of the Adomian decomposition method for initial-value problems*, Numerical Methods for Partial Differential Equations, **27**(4): 749-766, 2011.

[7] Achour, I and Bellagoun, A., *ADOMIAN DECOMPOSITION METHOD FOR SOLVING SPATIALLY INHOMOGENUOUS POPULATION BALANCE EQUATION*, Advances in Mathematics: Scientific Journal, **12**: 115-136, 2023.

[8] Robert, A. Adams and John J. F. Fournier, *Sobolev Spaces*, Elsevier, 2003.

[9] Adomian, G., *Nonlinear stochastic dynamical systems in physical problems*, Journal of Mathematical Analysis and Applications, **111**(1): 105-113, 1985.

[10] Adomian, G., *Solving frontier problems of physics: the decomposition method*, **60**, Springer Science & Business Media, 2013.

[11] Adomian, G., *Nonlinear stochastic systems theory and applications to physics*, **46**, 1988.

[12] Adomian, G., *Convergent series solution of nonlinear equations*, Journal of Computational and Applied Mathematics, **11**(2): 225-230, 1984.

[13] Adomian, G., *Analytical solution of Navier-Stokes flow of a viscous compressible fluid*, Foundations of Physics Letters, **8**: 389-400, 1995.

[14] Adomian, G., *A review of the decomposition method in applied mathematics*, Journal of mathematical analysis and applications, **135**(2): 501-544, 1988.

[15] Adomian, G., *Nonlinear stochastic differential equations*, Journal of Mathematical Analysis and Applications, **55**(2): 441-452, 1976.

[16] Adomian, G., *Solving frontier problems of physics: the decomposition method*, Dordrecht, 1994.

[17] Aizenman, M and Bak, T A., *Convergence to equilibrium in a system of reacting polymers*, Communications in Mathematical Physics, **65**(3): 203-230, 1979

[18] Amann, H., *Coagulation-fragmentation processes*, Archive for rational mechanics and analysis, **151**:339-366, 2000.

[19] Amann, H., *Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications*, Mathematische nachrichten, **186**(1): 5-56, 1997.

[20] Amann, H and others., *Linear and quasilinear parabolic problems*, **1**, Springer, 1995.

[21] Amann, H., *Linear and Quasilinear Parabolic Problems: Volume II: Function Spaces and Linear Differential Operators*, Birkhauser, Basel, Boston, Berlin, 1997.

[22] Appell, J and Zabrejko, P., *Nonlinear superposition operators*, Cambridge university press, 1990.

[23] Arora, HL and Abdelwahid, FI., *Solution of non-integer order differential equations via the Adomian decomposition method*, Appl. Math. Lett, **6**(1): 21-23, 1993.

[24] Attarakih, M and Bart, H J and Faqir, N M., *Numerical solution of the spatially distributed population balance equation describing the hydrodynamics of interacting liquid-liquid dispersions*, Chemical Engineering Science, **59**(12): 2567-2592, 2004.

[25] Attarakih, M *Integral formulation of the population balance equation: Application to particulate systems with particle growth*, Computers & Chemical Engineering, **48**: 1-13, 2013.

[26] Babolian, E and Biazar, J., *On the order of convergence of Adomian method*, Applied Mathematics and Computation, **130**(2-3): 383-387, 2002.

[27] Chandrasekharan, K., *Arithmetical functions*,**167**, 2012.

[28] Cherruault, Y., *Convergence of Adomian's method*, Kybernetes, 1989.

[29] Cherruault, Y and Saccomandi, G and Some, B., *New results for convergence of Adomian's method applied to integral equations*, Mathematical and Computer Modelling, **16**(2): 85-93, 1992.

[30] Cherruault, Y., *Modèles et méthodes mathématiques pour les sciences du vivant*, Presses Universitaires de France-PUF, 1998.

[31] Cherruault, Y., *Optimisation: Méthodes locales et globales*, Presses Universitaires de France-PUF, 1999.

[32] Daftardar-Gejji, V and Jafari, H., *An iterative method for solving nonlinear functional equations*, Journal of mathematical analysis and applications, **316**(2): 753-763, 2006.

[33] Dehghan, M and Shakeri, F., *The use of the decomposition procedure of Adomian for solving a delay differential equation arising in electrodynamics*, Physica Scripta, **78**(6): 065-004, 2008.

[34] Dehghan, M and Salehi, R., *The use of variational iteration method and Adomian decomposition method to solve the Eikonal equation and its application in the reconstruction problem*, International Journal for Numerical Methods in Biomedical Engineering, **27**(4): 524-540, 2011.

[35] Dehghan, M and Shakourifar, M and Hamidi, A., *The solution of linear and nonlinear systems of Volterra functional equations using Adomian-Pade technique*, Chaos, Solitons & Fractals, **39**(5): 2509-2521, 2009.

[36] Dehghan, M and Hamidi, A and Shakourifar, M., *The solution of coupled Burgers' equations using Adomian-Pade technique*, Applied Mathematics and Computation, **189**(2): 1034-1047, 2007.

[37] Drake, R L., *A general mathematical survey of the coagulation equation (Editors G M Hidy and JR Brock). Chapter in Topics in Current Aerosol Research (Part 2), Volume 3 of International Reviews in Aerosol Physics and Chemistry*, 1972.

[38] Duan, J-S., *Recurrence triangle for Adomian polynomials*, Applied Mathematics and Computation, **216**(4): 1235-1241, 2010.

[39] Duan, J-S., *An efficient algorithm for the multivariable Adomian polynomials*, Applied Mathematics and Computation, **217**(6): 2456-2467, 2010.

[40] Duan, J-S., *Convenient analytic recurrence algorithms for the Adomian polynomials*, Applied Mathematics and Computation, **217**(13): 6337-3648, 2011.

[41] Dubovskiĭ, P B and Stewart, I W., *Existence, uniqueness and mass conservation for the coagulation-fragmentation equation*, Mathematical methods in the applied sciences, **19**(7): 571-591, 1996.

[42] El-Kalla, I L., *Error estimates for series solutions to a class of nonlinear integral equations of mixed type*, Journal of Applied Mathematics and Computing, **38**: 341-351, 2012.

[43] Friedlander, S K., *On the particle size spectrum of a condensing vapor*, The Physics of Fluids, **3** (5): 693-696, 1960.

[44] Schwartz, L., *Cours d'analyse I, Hermann*, Paris, France,1981.

[45] Hasseine, A and Bellagoun, A and Bart, H-J., *Analytical solution of the droplet breakup equation by the Adomian decomposition method*, Applied Mathematics and Computation, **218**(5): 2249-2258, 2011.

[46] Hasseine, A and Bart, H-J., *Adomian decomposition method solution of population balance equations for aggregation, nucleation, growth and breakup processes*, Applied Mathematical Modelling, **39**(7): 1975-1984, 2015.

[47] Hasseine, A and Barhoum, Z and Attarakih, M and Bart, H-J *Analytical solutions of the particle breakage equation by the Adomian decomposition and the variational iteration methods*, Advanced Powder Technology, **26**(1): 105-112, 2015.

[48] Hasseine, A and Athmani, K and Bart, H-J., *Variational iteration method and projection method solution of the spatially distributed population balance equation*, Arab Journal of Basic and Applied Sciences, **25**(3): 132-141, 2018.

[49] Himoun, N and Abbaoui, K and Cherruault, Y., *New results on Adomian method*, Kybernetes, **32**(4): 523-539, 2003.

[50] Hulburt, H M and Katz, S., *Some problems in particle technology: A statistical mechanical formulation*, Chemical engineering science, **19**(8): 555-574, 1964.

[51] Kato, T., *Quasi-linear equations of evolution, with applications to partial differential equations*. In *Spectral Theory and Differential Equations: Proceedings of the Symposium held at Dundee, Scotland, 1-19 July, 1974*, 25-70, Springer, 2006.

[52] Kopriwa, N and Buchbender, F and Ayesterán, J and Kalem, M and Pfennig, A., *A critical review of the application of drop-population balances for the design of solvent extraction columns: I. Concept of solving drop-population balances and modelling breakage and coalescence*, Solvent Extraction and Ion Exchange, **30**(7): 683-723, 2012.

[53] Kostoglou, M and Karabelas, A-J., *Evaluation of zero order methods for simulating particle coagulation*, Journal of colloid and interface science, **163**(2): 420-431, 1994.

[54] Krasnosel'skii, M A and Ladyzhenskii, L A., *Conditions for complete continuity of PS Uryson's operator acting in the space L^p*, Trudy Moskovskogo Matematicheskogo Obshchestva, **3**: 307-320, 1954.

[55] Kumar, S and Ramkrishna, D., *On the solution of population balance equations by discretization—I. A fixed pivot technique*, Chemical Engineering Science, **51**(8): 1311-1332, 1996.

[56] Kumar, S and Ramkrishna, D., *On the solution of population balance equations by discretization—II. A moving pivot technique*, Chemical Engineering Science, **51**(8): 1333-1342, 1996.

[57] Kumar, S and Ramkrishna, D., *On the solution of population balance equations by discretization—III. Nucleation, growth and aggregation of particles*, Chemical Engineering Science, **52**(24): 4659-4679, 1997.

[58] McLaughlin, D J and Lamb, W and McBride, A C., *An existence and uniqueness result for a coagulation and multiple-fragmentation equation*, SIAM Journal on Mathematical Analysis, **28**(5): 1173-1190, 1997.

[59] McLaughlin, D J and Lamb, W and McBride, A C., *Existence results for non-autonomous multiple-fragmentation models*, Mathematical methods in the applied sciences, **20**(15): 1313-1323, 1997.

[60] McLaughlin, D J and Lamb, W and McBride, A C., *A semigroup approach to fragmentation models*, SIAM Journal on Mathematical Analysis, **28**(5): 1158-1172, 1997.

[61] McLaughlin, D J and Lamb, W and McBride, A C *Existence and uniqueness results for the non-autonomous coagulation and multiple-fragmentation equation*, Mathematical methods in the applied sciences, **21**(11): 1067-1084, 1998.

[62] Melzak, Z A., *A scalar transport equation*, Transactions of the American Mathematical Society, **85** (2): 547-560, 1957.

[63] Melzak, Z A., *A scalar transport equation. II.*, Michigan Mathematical Journal, **4**(3): 193-206, 1957.

[64] Motz, S and Mitrović, A and Gilles, E D., *Comparison of numerical methods for the simulation of dispersed phase systems*, Chemical Engineering Science, **57**(20): 4329-4344, 2002.

[65] Müller, H., *Zur allgemeinen Theorie ser raschen Koagulation: Die Koagulation von Stäbchen-und Blättchenkolloiden; die Theorie beliebig polydisperser Systeme und der Strömungskoagulation*, Kolloidchemische Beihefte, **27**:223-250, 1928.

[66] Pathak, H. K., *An Introduction to Nonlinear Analysis and Fixed Point Theory*, Springer, 2018.

[67] Rach, R., *A convenient computational form for the Adomian polynomials*, Journal of mathematical analysis and applications, **102**(2): 415-419, 1984.

[68] Rach, R., *A new definition of the Adomian polynomials*, Kybernetes, 2008.

[69] Ramkrishna, D., *Population balances: Theory and applications to particulate systems in engineering*, Elsevier, 2000.

[70] Ramkrishna, D., *Statistical models of cell populations*, Advances in Biochemical Engineering, **11**: 1-47, 1997.

[71] Ramkrishna, D., *The status of population balances*, Reviews in chemical engineering, **3**(1): 49-95, 1985.

[72] Randolph, A D and Larson, M A., *Theory of Particulate Processes, Acad*, Press, New York, 1988.

[73] Santos, F-P and Senocak, I and Favero, J-L and Lage, P-LC., title=Solution of the population balance equation using parallel adaptive cubature on GPUs, Computers & chemical engineering, **55**: 61-70, 2013.

[74] Shawagfeh, N-T., *Analytical approximate solutions for nonlinear fractional differential equations*, Applied Mathematics and Computation, **131**(2-3): 517-529, 2002.

[75] Singh, R and Saha, J and Kumar, J *Adomian decomposition method for solving fragmentation and aggregation population balance equations*, Journal of Applied Mathematics and Computing, **48**: 265-292, 2015.

[76] Smoluchowski, M von., *Drei vortrage uber diffusion, brownsche bewegung und koagulation von kolloidteilchen*, Zeitschrift fur Physik, **17**: 557-585, 1916.

[77] Smoluchowski, M von., *Versuch einer mathematischen Theorie der Koagulationskinetik kolloider Lösungen*, Zeitschrift für physikalische Chemie, **92** (1): 129-168, 1918.

[78] Solsvik, J and Jakobsen, H-A., *The foundation of the population balance equation: a review*, Journal of Dispersion Science and Technology, **36**(4): 510-520, 2015.

[79] Su, J and Gu, Z and Xu, X-Y ., *Advances in numerical methods for the solution of population balance equations for disperse phase systems*, Science in China Series B: Chemistry, **52**(8): 1063-1079, 2009.

[80] Valentas, K-J and Amundson, N-R., *Breakage and coalescence in dispersed phase systems*, Industrial & Engineering Chemistry Fundamentals, **5**(4): 533-542, 1966.

[81] Valentas, K-J and Bilous, O and Amundson, N-R., *Analysis of breakage in dispersed phase systems*, Industrial & engineering chemistry fundamentals, **5**(2): 271-279, 1966.

[82] Ziff, R M *New solutions to the fragmentation equation*, Journal of Physics A: Mathematical and General, **24**(12): 2821-2828, 1991.

Appendix

Contents:

A

The Foundation of the Adomian Decomposition Method

The goal of the ADM is to solve an equation $u = G(u)$, in a Banach space E, where G is an operator which can be nonlinear. The Banach space E is not necessarily a finite-dimensional space, it can be a functional space. The ADM is an original approach to this kind of problem.

A.1 The basic concepts of the Decomposition Theory

Definition A.1.1 (***Decomposition series of finite-order*** r)

A Decomposition series of finite-order r is a series $\sum \mathcal{D}_k$, Where each $\mathcal{D}_k$ is an $\mathcal{E}$-valued function of $r(k+1)$ variables $U_0^{(1)}, \ldots, U_k^{(1)}, \ldots, U_0^{(r)}, \ldots, U_k^{(r)}$.

Decomposition series are decomposition series of first-order.

Definition A.1.2 (***Weak convergence of the decomposition series of finite-order*** r)

A Decomposition series of finite-order r is weakly convergent. If for each collection of r convergent series $\sum u_i^{(1)}, \ldots, \sum u_i^{(r)}$ in $\mathcal{E}$, the series $\sum \mathcal{D}_k \left(u_0^{(1)}, \ldots, u_k^{(1)}, \ldots, u_0^{(r)}, \ldots, u_k^{(r)} \right)$ in $\mathcal{E}$ converge.

Definition A.1.3 (***Sum of convergent decomposition series of finite-order*** r)

The sum of a decomposition series of finite-order r is function of r variables mapping the set of convergent series (in $\mathcal{E}$) into $\mathcal{E}$

$$\mathcal{S}\left(\sum u_i^{(1)},\ldots,\sum u_i^{(r)}\right)=\sum_{k=0}^{+\infty}\mathcal{D}_k\left(u_0^{(1)},\ldots,u_k^{(1)},\ldots,u_0^{(r)},\ldots,u_k^{(r)}\right)$$

Definition A.1.4 (*Strong convergence of the decomposition series of finite-order* r)

A decomposition series of finite-order r is strongly convergent. If it is weakly convergent and if its sum depends only on the sum of the series in $\mathcal{E}$, i.e:

$$\sum_{k=0}^{+\infty} u_k^{(i)}=\sum_{k=0}^{+\infty} v_k^{(i)}$$

$$\Longrightarrow \mathcal{S}\left(\sum u_k^{(1)},\ldots,\sum u_k^{(r)}\right)=\mathcal{S}\left(\sum v_k^{(1)},\ldots,\sum v_k^{(r)}\right),\quad \forall i\in[1,r]$$

Definition A.1.5 (*Degenerated sum of the decomposition series of finite-order* r)

Using the previously defined sum S of a convergent decomposition series of finite-order r, a new operator S^*, mapping E^r into E can be created when the convergence is strong. S and S^* can be identified.

Let S be the sum of a strongly convergent decomposition series of finite-order r. Then for each collection $(u^{(1)},\ldots,u^{(r)})$ of elements of E, $S^*(u^{(1)},\ldots,u^{(r)})$ can be defined (because of the strong convergence) by $S(\sum u^{(1)},\ldots,\sum u^{(r)})$, where each $\sum u_n^{(i)}$ is any convergent series in E the sum of which is $u^{(i)}$. As a series of this kind, the one which is reduced to one term equal to $u^{(i)}$ can be chosen. So, it can be written

$$S^*(u^{(1)},\ldots,u^{(r)})=S(u^{(1)},\ldots,u^{(r)})$$

Definition A.1.6 (*Decomposition scheme*)

Let $\sum\mathcal{D}_k\left(U_0,\ldots,U_k\right)$ be a strongly convergent decomposition series. The decomposition scheme associated with $\sum\mathcal{D}_k$ is the recurrent scheme

$$\begin{cases} u_0 & = 0\\ u_{n+1} & = \mathcal{D}_k\left(u_0,\ldots,u_n\right)\end{cases}$$

Which constructs a series $\sum u_n$ in $\mathcal{E}$. Such a series can be constructed because each $\mathcal{D}_n$ is a function of $u_0, \dots, u_n$ but not of the following terms.

Definition A.1.7 (*Decomposition method*)

The decomposition method is the method consisting of constructing the solution of an equation with a decomposition scheme

A.2 The basic Decomposition Series

Definition A.2.1 (*Basic decomposition series*)

The basic decomposition series associated with the operator G is the series $\sum B_n$, where

$$\begin{cases} B_0 &= 0 \\ B_n &= G(\sum_{i=0}^{n} X_i) - G(\sum_{i=0}^{n-1} X_i) \end{cases}$$

Each B_n is mapping E^{n+1} into E.

Theorem A.2.2 (*Convergence of the basic decomposition series*)

The basic decomposition series $\sum B_n$ associated with a continuous operator G is a decomposition series (of first order) that strongly converges and the degenerated sum of which is G.

Proof : If $\sum u_n$ converges, then the series in E, $\sum B_n(u_0, \dots, u_n)$ converges and its sum $\sum B_n G(\sum_{n=0}^{\infty} u_n)$ only depends on the sum of $\sum u_n$. ■

If G is a nonlinear operator, the basic decomposition series is useless because the Adomian decomposition method needs much more calculus than the successive approximations method to solve the equation $u = G(u)$. However, if G is a linear operator, the Adomian decomposition scheme becomes simpler as shown below.

Definition A.2.3 (*Basic decomposition series associated with a linear operator*)

The basic decomposition series $\sum B_n$ associated with the linear operator L is

$$\begin{cases} B_0 &= L(X_0) \\ B_n &= L(X_n) \end{cases}$$

A.3 The Adomian Decomposition Series

Definition A.3.1 (*Adomian's polynomials*)

Let G be an analytical function and $\sum u_m$ a convergent series in a Banach space $\mathcal{F}$. The Adomian's Polynomials are defined by

$$\mathcal{A}_m(f_0, f_1, ..., f_m) = \frac{1}{m!}\left[\!\left[\frac{d^m}{d\xi^m}G\left(\sum_{i=0}^{+\infty}\xi^i f_i\right)\right]\!\right]_{\xi=0}, m \geq 0 \quad \text{(A.3.1)}$$

Define $U = \sum\limits_{n=0} u_n$ and $u^+ = \sum\limits_{n=0} u_n\xi^n$. This power series converges when $\xi = 1$. But it is known that the sum of a power series, whose converge radius is ρ, is analytical over $OD(O, \rho)$ (open disk whose center is O and whose radius is ρ), then u^+ is analytical over $OD(O, \rho)$, i.e. there are $\mathcal{A}_m$ so that $G \circ u^+(\xi) = \sum_{m=0}^{\infty} \mathcal{A}_m\xi^m$ and these $\mathcal{A}_m$ verify

$$\mathcal{A}_m(f_0, f_1, ..., f_m) = \frac{1}{m!}\left[\!\left[\frac{d^m}{d\xi^m}G\left(\sum_{i=0}^{+\infty}\xi^i f_i\right)\right]\!\right]_{\xi=0}, m \geq 0 \quad \text{(A.3.2)}$$

Note. We do not need to assume that the convergence radius is greater than 1. If $\rho = 1$, as u^+ converges and its sum is U, then Abel's theorem leads to $\lim_{\xi \longrightarrow 1} u^+(\xi) = U$ (ξ being a real number) and so $\lim_{\xi \longrightarrow 1} G \circ u^+(\xi) = G(U)$.

Theorem A.3.2 (*Convergence of the Adomian decomposition series*)

The Adomian decomposition series $\sum \mathcal{D}_m$ associated with the analytical function G defines a decomposition series (of the first order) that strongly converges and the degeneration sum of which is G.

Proof : To verify that each $\mathcal{A}_m$ depends only on $u_0, \ldots, u_m$, we express $\mathcal{A}_m$ as a function of the coefficients of the two series that are composed using a classical theorem of power series composition [29]. We note that the expression obtained is only used to prove this dependence. We have just proved that the $\sum \mathcal{A}_m$ is a decomposition series. If $\sum u_m$ is a convergent series, we have seen that $\sum \mathcal{A}_m(u_0, \ldots, u_m)$ converges and that its sum is $G \circ u^+(1) = G(U)$, that is to say, that the decomposition series $\sum \mathcal{A}_m$ weakly converges, and its sum is G. If $\sum u_m$ and $\sum v_m$ are two series having the same sum U, and if their Adomian's polynomials are $\mathcal{A}_m^u$ and $\mathcal{A}_m^v$ respectively, then we write

$$\begin{aligned}\sum_{m=0}^{\infty} \mathcal{A}_m^u &= G \circ u^+(\xi = 1) \\ &= G \circ v^+(\xi = 1) \\ &= \sum_{m=0}^{\infty} \mathcal{A}_m^v\end{aligned}$$

So, the sum of the Adomian decomposition series depends only on the sum of the considered series the convergence is strong. ■

A.4 Reference table of $\mathcal{A}_m$

A reference table for computing 10 first Adomian polynomials is provided below [10]

$$\begin{aligned}
\mathcal{A}_0 &= \mathcal{N}(f_0) \\
\mathcal{A}_1 &= f_1\mathcal{N}^{(1)}(f_0) \\
\mathcal{A}_2 &= f_2\mathcal{N}^{(1)}(f_0) + \frac{1}{2}f_1^2\mathcal{N}^{(2)}(f_0) \\
\mathcal{A}_3 &= f_3\mathcal{N}^{(1)}(f_0) + f_1f_2\mathcal{N}^{(2)}(f_0) + \frac{1}{3}f_1^3\mathcal{N}^{(3)}(f_0) \\
\mathcal{A}_4 &= f_4\mathcal{N}^{(1)}(f_0) + \left(f_1f_3 + \frac{1}{2}f_0^2\right)\mathcal{N}^{(2)}(f_0) + \frac{1}{2}\left(f_1^2f_2\right)\mathcal{N}^{(3)}(f_0) + \frac{1}{24}f_1^4\mathcal{N}^{(4)}(f_0) \\
\mathcal{A}_5 &= f_5\mathcal{N}^{(1)}(f_0) + \left(f_1f_4 + f_2f_3\right)\mathcal{N}^{(2)}(f_0) + \frac{1}{2}\left(f_1^2f_3 + f_1f_2^2\right)\mathcal{N}^{(3)}(f_0) \\
&\quad + \frac{1}{2}f_1^3f_2\mathcal{N}^{(4)}(f_0) + \frac{1}{120}f_1^5\mathcal{N}^{(5)}(f_0) \\
\mathcal{A}_6 &= f_6\mathcal{N}^{(1)}(f_0) + \left(f_1f_5 + f_2f_4 + \frac{1}{2}f_3^2\right)\mathcal{N}^{(2)}(f_0) + \left(\frac{1}{2}f_1^2f_2f_3 + \frac{1}{6}f_2^3\right)\mathcal{N}^{(3)}(f_0) \\
&\quad + \left(\frac{1}{6}f_1^3f_3 + \frac{1}{4}f_1^2f_2^2\right)\mathcal{N}^{(4)}(f_0) + \frac{1}{24}f_1^4f_2\mathcal{N}^{(5)}(f_0) + \frac{1}{120}f_1^6\mathcal{N}^{(6)}(f_0)
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_7 &= f_7\mathcal{N}^{(1)}(f_0) + \Big(f_1f_6 + f_2f_5 + f_3f_4\Big)\mathcal{N}^{(2)}(f_0) + \Big(\frac{1}{2}f_1^2f_5 + f_1f_2f_4 + \frac{1}{2}f_1f_3^2\Big)\mathcal{N}^{(3)}(f_0)\\
&+ \Big(\frac{1}{6}f_1^3f_4 + \frac{1}{2}f_1^2f_2f_3 + \frac{1}{6}f_1f_2^3\Big)\mathcal{N}^{(4)}(f_0) + \frac{1}{24}\Big(f_1^4f_3 + f_1^3f_2^2\Big)\mathcal{N}^{(5)}(f_0)\\
&+ \frac{1}{120}f_1^5f^2\mathcal{N}^{(6)}(f_0) + \frac{1}{504}f_1^7\mathcal{N}^{(7)}(f_0)\\
\mathcal{A}_8 &= f_8\mathcal{N}^{(1)}(f_0) + \Big(f_1f_7 + f_2f_6 + f_3f_5 + \frac{1}{2}f_4^2\Big)\mathcal{N}^{(2)}(f_0) + \Big(\frac{1}{2}f_1^2f_6 + f_1f_2f_5 + f_1f_3f_5\\
&+ \frac{1}{2}f_2f_3^2 + \frac{1}{2}f_2^2f_4\Big)\mathcal{N}^{(3)}(f_0) + \Big(\frac{1}{6}f_1^3f_5 + \frac{1}{2}f_1^2f_2f_4 + \frac{1}{4}f_1^2f_3^2 + \frac{1}{2}f_1f_2^2f_3 + \frac{1}{24}f_2^4\Big)\mathcal{N}^{(4)}(f_0)\\
&+ \Big(\frac{1}{24}f_1^4f_4 + \frac{1}{6}f_1^3f_2f_3 + \frac{1}{12}f_1^2f_2^3\Big)\mathcal{N}^{(5)}(f_0) + \Big(\frac{1}{120}f_1^5f^3 + \frac{1}{48}f_1^4f_2^2\Big)\mathcal{N}^{(6)}(f_0)\\
&+ \frac{1}{720}f_1^6f_2\mathcal{N}^{(7)}(f_0) + \frac{1}{40320}f_1^8\mathcal{N}^{(8)}(f_0)\\
\mathcal{A}_9 &= f_9\mathcal{N}^{(1)}(f_0) + \Big(f_1f_8 + f_2f_7 + f_3f_6 + f_4f_5\Big)\mathcal{N}^{(2)}(f_0) + \Big(\frac{1}{2}f_1^2f_7 + f_1f_2f_6 + f_1f_3f_5\\
&+ \frac{1}{2}f_1f_4^2 + \frac{1}{2}f_2^2f_5 + f_2f_3f_5 + \frac{1}{6}f_3^3\Big)\mathcal{N}^{(3)}(f_0) + \Big(\frac{1}{6}f_1^3f_6 + \frac{1}{2}f_1^2f_2f_5 + \frac{1}{2}f_1^2f_3f_4\\
&+ \frac{1}{2}f_1f_2^2f_4 + \frac{1}{2}f_1f_2f_3^2 + \frac{1}{6}f_2^3f_3\Big)\mathcal{N}^{(4)}(f_0) + \Big(\frac{1}{24}f_1^4f_5 + \frac{1}{6}f_1^3f_2f_4 + \frac{1}{12}f_1^3f_3^3 + \frac{1}{4}f_1^2f_2^2f_3\\
&+ \frac{1}{4}f_1f_2^4\Big)\mathcal{N}^{(5)}(f_0) + \Big(\frac{1}{120}f_1^5f^4 + \frac{1}{24}f_1^4f_2f_3 + \frac{1}{36}f_1^3f_2^3\Big)\mathcal{N}^{(6)}(f_0) + \Big(\frac{1}{720}f_1^6f_3\\
&+ \frac{1}{240}f_1^5f_2^2\Big)\mathcal{N}^{(7)}(f_0) + \frac{1}{5040}f_1^7f_2\mathcal{N}^{(8)}(f_0) + \frac{1}{362880}f_1^9\mathcal{N}^{(9)}(f_0)\\
&\vdots
\end{aligned}$$

B

Semilinear parabolic evolution equations: Existence and uniqueness

The following theorem proves the existence, uniqueness, and continuity for semilinear parabolic evolution equations.

Theorem B.0.1

Suppose that $E_1 \overset{d}{\hookrightarrow} E_0$, and $0 < \gamma \le \beta < \alpha < 1$, and $(\ \cdot\ ,\ \cdot\ )_\theta$ are admissible interpolation functors for $\theta \in \{\gamma, \beta, \alpha\}$. Put

$$E_\theta := (E_0, E_1)_\theta$$

and suppose that

$$\Big(t \mapsto \big(A(t), g(t,\ \cdot\ )\big)\Big) \in C^\rho\Big(I, \mathcal{H}(E_1, E_0) \times C_b^{1-}(E_\beta, E_\gamma)\Big)$$

for some $\rho \in (0,1)$. Then, given $f^0 \in E_\alpha$, the IVP

$$\dot f + A(t) f = g(t, f), \qquad t \in \dot I, \qquad f(0) = f^0 \tag{B.0.1}$$

has a unique solution

$$f(\ \cdot\ , f^0) := f(\ \cdot\ , f^0, A, g) \in C(I(f^0), E_\alpha) \cap C(\dot I(f^0), E_1) \cap C^1(I(f^0), E_0)$$

The maximal interval of existence, $I(f^0) := I(f^0, A, g)$, is open in I. If

$$\sup_{t \in I(f^0) \cap [0,T]} \left\| f(t, f^0) \right\|_\alpha < \infty \tag{B.0.2}$$

for each $T \in I$, then $f(\,\cdot\,, f^0)$ is a global solution, that is, $I(f^0) = I$.
For each $T \in \dot{I}(f^0)$ there exists a neighborhood $\mathcal{U}$ of (f^0, A, g) in

$$E_\alpha \times C^\rho(I, \mathcal{H}(E_1, E_0)) \times C^\rho(I, C_b^{1-}(E_\beta, E_\gamma))$$

such that $[0, T] \subset I(\tilde{f}^0, \tilde{A}, \tilde{g})$ for $(\tilde{f}^0, \tilde{A}, \tilde{g}) \in \mathcal{U}$ and such that

$$f(\,\cdot\,, \tilde{f}^0, \tilde{A}, \tilde{g}) \Longrightarrow f(\,\cdot\,, f^0, A, g)$$

in $C([0, T], E_\alpha)$, as $(\tilde{f}^0, \tilde{A}, \tilde{g}) \Longrightarrow (f^0, A, g)$ in $\mathcal{U}$

Proof : Put $\delta := \rho \wedge (\alpha - \beta)$. Fix $T \in \dot{I}$ and set

$$g_u(t) := g(t, u(t)), \qquad 0 \le t \le T$$

for $u \in C^\delta([0, T], E_\beta)$. Then $g_u \in C^\delta([0, T], E_\gamma)$ and ([20], Theorem II.1.2.1 and II.5.3.1) guarantee the existence of a unique solution

$$f(\,\cdot\,; u) \in C(I(f^0), E_\alpha) \cap C(\dot{I}(f^0), E_1) \cap C^1(I(f^0), E_0) \tag{B.0.3}$$

of the linear Cauchy problem

$$\dot{f} + A(t) f = g_f(t), \qquad 0 \le t \le T, \qquad f(0) = f^0 \tag{B.0.4}$$

If $w \in C^\delta([0, T], E_\beta)$ then ([20], Theorem II.5.2.1) implies

$$\|f(t; u) - f(t; w)\|_\beta \le c T^{1-\beta} \|f - u\|_{C([0,T], E_\beta)}, \qquad 0 \le t \le T$$

where c is independent of u and w if $u([0, T])$ and $w([0, T])$remain in a given bounded subset of E_β. Thus, by making T smaller, if necessary, the contraction mapping principle implies the existence of a fixed point $\bar{f} \in C([0, T], E_\beta)$ of $u \mapsto f(\,\cdot\,; u)$. Next deducing from ([20], Theorem II.5.3.1) that $\bar{f} \in C([0, T], E_\alpha) \cap \in C^\delta([0, T], E_\beta)$. Hence $\bar{f} = f(\,\cdot\,; \bar{f})$ and (B.0.3) imply that $\bar{f}$ is a solution of (B.0.1) on $[0, T]$. Now a standard continuation argument shows that $\bar{f}$ has an extension $f(\,\cdot\,, f^0)$ to a maximal solution of (B.0.1), and the corresponding maximal interval of existence is open in I.

The uniqueness assertion is obvious.

Suppose that (B.0.2) is satisfied for each $T \in I$ and $I(f^0) \neq I$. Then the extension argument can be applied to the initial value $f(t^*, f^0)$, where t^* is sufficiently close to the right end point of $I(f^0)$, to obtain an extension of $f(\,\cdot\,, f^0)$ over an interval which is strictly larger than $I(f^0)$. Since this contradicts the maximality of $I(f^0)$ it follows that (B.0.2) implies $I(f^0) = I$.

Lastly, it is not difficult to deduce the stated continuity assertion from ([20], Theorem II.5.2.1). ■

C

Mathematical functions

Here we summarize only all different cases symbols and functions defined as

- **Pochhammer Symbol:**
  Given $a \in R \setminus Z_-$ and $n \in N$ the Pochhammer symbol is defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}$$

  Where Γ is the Gamma function which is given by

$$\Gamma(a) = \int_0^{+\infty} e^{-t} t^{a-1}\, dt$$

- **The generalized Hypergeometric function:**
  The generalized hypergeometric function, denoted ${}_pF_q(\alpha_1, ..., \alpha_p; \gamma_1, ..., \gamma_q; z)$ and defined by

$${}_pF_q(\alpha_1, ..., \alpha_p; \gamma_1, ..., \gamma_q; z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} (\alpha_i)_k}{\prod_{j=1}^{q} (\gamma_j)_k} \frac{z^k}{k!}$$

In the last example, by rearrangement of the terms according to the powers $t^i v^j z^m$, the series solution could be written as

$$f(t, v, z) = \sum_{k=0}^{+\infty} f_k(t, v, z) = \sum_{k=0}^{+\infty} f_k^*(t, v, z) = \sum_{j=0}^{+\infty} \sum_{i=0}^{+\infty} \sum_{m=0}^{+\infty} \left(a_{i,j,m} t^i v^j z^m \right) \tag{C.0.1}$$

$$= \sum_{j=0}^{+\infty} v^j \left(\sum_{i=0}^{+\infty} t^i \left(\sum_{m=0}^{+\infty} (a_{i,j,m} z^m) \right) \right) \tag{C.0.2}$$

$$= \sum_{j=0}^{+\infty} v^j \left(\sum_{i=0}^{+\infty} t^i b_{i,j}(z) \right) \tag{C.0.3}$$

Examples of $a_{i,j,m}$

- for $i = 0$
  - for $j = 0$

$$a_{0,0,0} = 0$$
$$a_{0,0,1} = -1$$
$$a_{0,0,2} = 0$$
$$\vdots$$

  - for $j = 1$

$$a_{0,1,0} = 0$$
$$a_{0,1,1} = 0$$
$$a_{0,1,2} = 0$$
$$a_{0,1,3} = \frac{1}{2}$$

$$\vdots$$
$$\vdots$$

- for $i = 1$
  - for $j = 0$

$$a_{1,0,0} = 1$$
$$a_{1,0,1} = 0$$
$$a_{1,0,2} = 2$$
$$\vdots$$

  - for $j = 1$

$$a_{1,1,0} = 0$$

$$a_{1,1,1} = 0$$

$$a_{1,1,2} = -1$$

$$a_{1,1,3} = 0$$

$$a_{1,1,4} = \frac{-9}{4}$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

Examples of $b_{i,j}(z)$

- for $i = 0$

$$b_{0,0}(z) = \frac{-4z}{e^v(-2+z^2)^2}$$

$$b_{0,1}(z) = \frac{-4vz^3}{e^v(-2+z^2)^3}$$

$$b_{0,2}(z) = \frac{-2v^2z^5}{e^v(-2+z^2)^4}$$

$$\vdots$$

- for $i = 1$

$$b_{1,0}(z) = \frac{-4t(2+z^2)}{e^v(-2+z^2)^3}$$

$$b_{1,1}(z) = \frac{-4vtz^2(4+z^2)}{e^v(-2+z^2)^4}$$

$$b_{1,2}(z) = \frac{-2v^2tz^4(6+z^2)}{e^v(-2+z^2)^5}$$

$$\vdots$$

$$\vdots$$

Therefore

$$
\begin{align}
f(t,v,z) &= \sum_{j=0}^{+\infty} v^j \left(\sum_{i=0}^{+\infty} t^i b_{i,j}(z) \right) \tag{C.0.4}\\
&= \sum_{j=0}^{+\infty} v^j c_j(t,z) \tag{C.0.5}\\
&= \sum_{j=0}^{+\infty} v^j \frac{4z^j(t-z)^{j+1}}{j! e^v (z(t-z)+2)^{j+2}} \tag{C.0.6}\\
&= \frac{4(t-z)e^{\frac{vz(t-z)}{tz-z^2+2}}}{e^v (z(t-z)+2)^2} \tag{C.0.7}
\end{align}
$$

Adomian Decomposition Method For Population Balance Equations And The Study Of Convergence

Presented by

Achour Imane

Presented in: 26/05/2024 in view of the composed jury:

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Faculty of Exact Sciences and Sciences of Nature and Life

Phd Thesis To Achieve the Doctoral Degree In Mathematics

Option: Applied Mathematics

Abstract

The Adomian decomposition method has received much attention in recent years in applied mathematics in general and in the area of series solutions in particular. It is an effective technique for the analytical solution of a wide class of dynamical systems. The population balance equation (PBE) has been used to model a variety of particulate Process. However, only a few cases where analytical solutions for the breakage/coalescence process exist, most of these solutions are for the spatially homogeneous system. The main objective of this thesis is to derive analytical solutions of spatially inhomogeneous PBE For breakage/ coalescence processes using the Adomian decomposition method which uses a specific kind of polynomials named "Adomian's polynomials" to decompose the nonlinear part of such equation. The results obtained indicate that the ADM avoids numerical stability problems that often characterize general numerical techniques in this area.

Key words: Population Balance Model, Adomian Decomposition Method, Adomian Polynomials, Convergence, Integro-differential Equation.

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