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Title

Contribution to the Qualitative Study of Some Generalized Boussinesq Problems

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Dedication

I dedicate this thesis

To my beloved parents, whose endless love, support and sacrifices have made this achievement possible. Their encouragement has been my guiding light and this achievement is as much yours as it is mine.

To my dear sister, Amina, and my dear brother, Charaf,my brother in law, Radhouane, whose unwavering support and boundless love have been the bedrock of my academic journey. I am truly grateful to have you in my life.

To my dear niece Tesnim.

To my big family, my grandparents, my uncles, my aunts.

To my friends and colleagues.

Garti Ines

Abstract

Research on Boussinesq equations has grown considerably over the past three decades due to their adaptability in explaining nonlinear phenomena, particularly in the analysis of water wave dynamics. These equations are widely used in various fields, including the study of oceanic waves and coastal engineering. This thesis explores the existence and finite-time blow-up of solutions in the Cauchy problem associated with a generalized Boussinesq equation. The study of the generalized Boussinesq equation and its solutions, including existence, non-existence, and blow-up, is of great interest. The thesis investigates solutions in both bounded and unbounded domains, as well as in the presence of logarithmic nonlinearity. Local solutions are proven to exist and be unique in both cases. Under certain restrictions on the initial data of our problems, we establish the existence and uniqueness of global solutions, as well as the possibility of blowing up solutions in finite time.

Keywords

Boussinesq equation, Cauchy problem, Stable set and Unstable set, Existence of global solution, Local existence, Finite time blow-up.

Résumé

La recherche sur les équations de Boussinesq a connu un développement significatif au cours des trois dernières décennies en raison de leur capacité à élucider les phénomènes non linéaires, notamment dans l'analyse de la dynamique des vagues. Ces équations jouent un rôle prépondérant dans divers domaines, tels que l'étude des vagues océaniques et l'ingénierie côtière. L'objectif de cette thèse est d'explorer l'existence et l'explosion en temps fini des solutions du problème de Cauchy lié à une équation de Boussinesq généralisée. L'investigation de l'équation de Boussinesq généralisée et de ses solutions, englobant des aspects tels que l'existence, la non-existence et l'explosion, revêt un intérêt majeur. La thèse approfondit l'examen des solutions dans des domaines bornés et non bornés, ainsi que dans le contexte d'une non-linéarité logarithmique. Il est démontré que les solutions locaux existent de manière unique dans les deux cas. Sous certaines contraintes sur les données initiales de nos problèmes, nous établissons l'existence et l'unicité des solutions globales, tout en considérant la possibilité d'une explosion en temps fini des solutions.

mots clés

Equation de Boussinesq, Problème de Cauchy, Ensemble stable et Ensemble instable, Existence d'une solution globale, Existence locale, Explosion en temps fini.

الملخص

تطورت الأبحاث حول معادلات بوسينسك بشكل كبير على مدى العقود الثلاثة الماضية بسبب قدرتها على التكيف في شرح الظواهر غير الخطية، لا سيما في تحليل ديناميكيات موجات المياه. تستخدم هذه المعادلات على نطاق واسع في مجالات مختلفة، بما في ذلك دراسة الموجات المحيطية والهندسة الساحلية. تستكشف هذه الأطروحة وجود الحلول وتفجيرها في الوقت المحدد في مشكلة كوشي المرتبطة بمعادلة بوسينسك المعممة. إن دراسة معادلة بوسينسك المعممة ووجود حلولها وعدم وجودها وتفجيرها ذات أهمية كبيرة. تبحث الأطروحة في الحلول في كل من المجالات المحدودة وغير المحدودة، وكذلك في وجود اللاخطية اللوغاريتمية. ثبت أن الحلول المحلية موجودة وفريدة من نوعها في كلتا الحالتين المدروستين. وفي ظل بعض القيود المفروضة على البيانات الأولية لمشاكلنا، نثبت وجود الحلول الشاملة وتفردها، فضلا عن إمكانية تفجير الحلول في وقت محدود

الكلمات المفتاحية

معادلة بوسينسك، مشكلة كوشي، المجموعة المستقرة والمجموعة غير المستقرة، وجود حل شامل، الوجود المحلي، تفجير زمني محدود

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To my **mother**, **father**, **sister** and **brother**: Thank you for everything.

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Symbols and Abbreviations

Sets:

$\mathbb{R}^n$	The real Euclidean space of dimension $n \geq 1$.

Functions and functions spaces:

$C([0,T],X)$	The space of continuous functions on $[0,T]$ to values in X.
$C_0(\mathbb{R}^n)$	Space of all continuous functions decaying to zero at infinity.
$AC([0,T])$	The space of absolutely continuous functions on $[0,T]$.
$AC^{n+1}[0,T]$	$\{f : [0,T] \to R, \text{and } \partial_t^n f \in AC[0,T]\}$ and ∂_t^n is the usual n times derivative.
$C_c(I,X)$	The space of continuous functions with compact support from I to X.
$C_b(I,X)$	The space of continuous and bounded functions from I to X.
$F(resp.F^{-1})$	Fourier transform(resp.Fourier transform inverse).
$\Lambda^{-s}\varphi$	$\mathcal{F}^{-1}\left(\lvert\xi\rvert^{-s}\mathcal{F}\varphi\right)$
$L^p(\mathbb{R}^n)$	The space of measurable functions on $\mathbb{R}^n$ such that $\lvert u\rvert^p$ is integrable.
$L^p([0,T],X)$	The space of measurable functions u on $[0,T]$ to values in X such that $\lvert\lvert u\rvert\rvert_X^p$ is integrable $(1 \leq p < \infty)$.
$L^\infty(\mathbb{R}^n)$	The space of measurable functions u on $\mathbb{R}^n$ such that there exists k
$W^{m,p}(\mathbb{R}^n)$	The usual Sobolev space.
$H^m(\mathbb{R}^n)$	$W^{m,2}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n), D^\alpha f \in L^2(\mathbb{R}^n)$ for all $\alpha \in N^n$ such that $\lvert\alpha\rvert \leq m\}$.

Norms:

$||u||_p := \left(\int_{\mathbb{R}^N} |u|^p\right)^{1/p}$ for $u \in L^p\left(\mathbb{R}^N\right)$.

$||u||_{p,q,T} := \sup\limits_{0\leq t\leq T} \left(\int_0^t ||u||_p^q\right)^{1/q}$.

$||u||_\infty := \inf\{k > 0, |u(x)| < k$ almost every where$\}$, for $u \in L^\infty\left(\mathbb{R}^N\right)$.

$||u||_{W^{m,p}} := \sum\limits_{\alpha\leq m} ||D^\alpha u||_{L^P}$ for $u \in W^{m,p}\left(\mathbb{R}^N\right)$.

$||u||_{H^m} := \left(\sum\limits_{\alpha\leq m} (||D^\alpha u||_{L^P}\ )^2\right)^{\frac{1}{2}}$ for $u \in H^m\left(\mathbb{R}^N\right)$. such that

Mathematical operators:

$*$ The convolution product.

$|.|$ Absolute value.

Δ The classical Laplace operator: $\Delta u(x,t) = \sum\limits_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}(x,t)$.

$a \lesssim b$ $i.e\ a \leq Cb$.

Contents

Chapter 1

General Introduction

During the 19th century, the investigation of water waves increased in importance due to their significance in naval architecture and industrial applications. In 1834, while observing a canal boat, Scottish engineer John Scott Russell made a significant discovery. He identified a distinctive water wave phenomenon, which he labelled as the "Wave of Translation." Russell confirmed the existence of this "solitary wave" in shallow water through his experiments in a homemade water tank and revealed several key attributes:

1. The stable waves investigated were capable of traveling long distances.
2. The wave speed depended on its size and the depth of the water.
3. In contrast to regular waves, these waves did not merge but instead saw smaller waves being surpassed by larger ones.
4. If a wave exceeded the water's depth, it splits into two waves, one large and one small.

The aforementioned observations challenged prevalent wave theories advocated by Isaac Newton and Daniel Bernoulli. Although some mathematicians and physicists objected, experimental evidence kept supporting the existence of these waves.

The turning point occurred in 1871, when French mathematician and physicist Joseph Boussinesq formulated the Boussinesq equation, a partial differential equation confirming the existence of these solitary waves. During the late 19th century, various publications offered comparable assistance through mathematical equations. Korteweg and De Vries proposed the KdV equation, while the Boussinesq equation and KdV equation were also acknowledged for their capability to provide solutions for solitary waves. These discoveries successfully solved the puzzle left by Russell.

The Boussinesq equation has been extensively studied in mathematics and physics since the early 1960s. Its significance in various fields, including physics, biology, electronics, and mechanics, especially through the use of soliton theory, has created curiosity in researchers. Moreover, the application of this equation has been identified in coastal and ocean engineering, for instance, in the simulation of tidal waves and tides.

However, during the first studies of the Boussinesq equation, despite its numerous useful applications and favorable properties, both mathematicians and physicists encountered a problem: the original Boussinesq equation was found to be linearly unstable, especially with respect to disturbances of short wavelength. In simpler terms, small differences in initial conditions can cause significant changes in the solution over time. Recognizing this unfavorable behavior, efforts were made to improve the Boussinesq equation. This resulted in the formulation of two revised versions, which rectify the shortcomings of the original while remaining similar to it:

$$u_{tt} - u_{xx} - u_{xxxx} = (u^2)_{xx}, \tag{1.1}$$

This equation is called **"Bad"** Boussinesq equation in comparison with the **"Good"** Boussinesq equation defined as

$$u_{tt} - u_{xx} + u_{xxxx} = (u^2)_{xx}. \tag{1.2}$$

The Bad Boussinesq equation finds application in the depiction of shallow-water wave flows with small amplitudes in a two-dimensional context. Additionally, it has been a focal point in research related to the Fermi-Pasta-Ulam **(FPU)** problem, which deals with the dynamics of lattice systems governed by equations similar to those presented in equation (1.1).

In contrast, the Good Boussinesq equation is employed to elucidate the characteristics of irrotational flows in two dimensions within a uniform rectangular channel for an inviscid liquid.

Apart from the standard Boussinesq equation, there are various mathematical models for small-amplitude, planar, and long waves on the ocean surface. Numerous options for dependent variables, as well as the ability to modify terms of lower order using primary order interactions, may result in several types of equations.

All of these models, however, have one point in common: they are all perturbations of linear wave equations that account for small nonlinearity and dispersion effects.

In 1872, Boussinesq, in [5] derived the first generalized wave equation for the flow in the shallow inviscid layer

$$u_{tt} - u_{xx} + \delta u_{xxxx} = \kappa (u^2)_{xx}, \tag{1.3}$$

where δ and κ depend on the depth of the fluid and the characteristic speed of long waves. Since then, extensive study has been conducted to investigate the equation's characteristics, solution, and initial boundary value problems.

The cauchy problem of the generalized boussinesq equation

$$u_{tt} - u_{xx} + (u_{xx} + f(u))_{xx} = 0, \tag{1.4}$$

various scholars have examined this problem (1.4) and its generations thoroughly, and the results of their study have given various qualitative conclusions, including the problem's existence and nonexistence of solutions locally and globally and the finite temporal blow-up. For example, Bona and Sachs [3] studied the Cauchy problem of the previous equation. By using Kato's abstract theory of quasilinear evolution equation, they proved the existence of a local $H^{r+2} \times H^r$ solution for $f \in \mathbb{C}^\infty$ with $f(0) = 0$ and for any $(u_0, u_1) \in H^{r+2} \times H^r$ with $r > \frac{1}{2}$, for $f(u) = |u|^{m-1}u$, $1 < m < 5$, they proved the global existence of $H^{r+2} \times H^r$ under some assumptions on initial data. Linares [31] determinated the local well-possedness of the Cauchy problem (1.4) with $f(u) = |u|^M u$ for $H^1 \times L^2$ solutions where $M > 0$ and for $L^2 \times H^{-1}$ when $0 < M < 4$ respectively, he also demonstrated how, when the data size is small, similar local solutions may be applied globally. After that, Liu [34] investigated the global existence and finite time blow-up of the same problem (1.4). When $f(u) = \pm |u|^m$ or $f(u) = \pm |u|^{m-1} u$ for $m > 1$, Liu and Xu in [36] dealt with the cauchy problem (1.4), they assert global existence, vacuum solitude, and finite time solutions blow up.

For the class of multidimentional generalized Boussinesq equations, we mention some recent works. In [59] and [60] Wang and Chen investigated the existence and nonexistence of both local and global solutions, as well as the global existence of small-amplitude solutions for the Cauchy problem of the multidmensional generalized Boussinesq equation

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta f(u). \tag{1.5}$$

Also, Polat and Artas [41] studied the existence and blow up of solution of the Cauchy problem for the generalized multidimensional Boussinesq equation

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u - \beta \Delta u_t = \Delta f(u), \tag{1.6}$$

where f is nonlinear function. By using the contraction mapping concept, they determined that the cauchy problem was locally well posed, and they subsequently obtained the appropriate constraints to ensure that each local solution of the cauchy problem with negative and nonnegative initial energy blows up in finite time using the concavity method.

When $f(u) = \pm p\,|u|^{\alpha}$ or $f(u) = -p\,|u|^{\alpha-1}\,u, p > 0$ Jihong and al [23] studied the multidimensional generalized Boussinesq equation (1.6). They demonstrated using the potential well method, the existence and nonexistence of the global weak solution at subcritical and critical initial energy levels and without establishing the local existence theory.
Then Hatice and al [52] investigated the following problem

$$u_{tt} - \Delta u + \Delta^2 u + \Delta^2 u_t = \Delta f(u), \tag{1.7}$$

where $f(u) = \beta\,|u|^{M}\,, \beta > 0$, They provided criteria that ensure the existence of global weak solutions with supcritical initial energy by defining new functionals and applying the potential well approach. After that, in [53], they generalized the results from one-dimension $(n = 1)$ to multidimensional.
Many reserchers investigated the sixth order generalized Bussinesq equation

$$u_{tt} - \Delta u + \Delta^2 u - \Delta u_{tt} + \Delta^2 u_{tt} = \Delta f(u). \tag{1.8}$$

Xu and al first [43] studied the existence and uniqueness of the local solution using the contraction mapping method. Then they demonstrated the global existence and finite time blow-up of the solution at subcritical and critical initial energy levels. When $f(u) = \pm\gamma\,|u|^{\beta}$ or $f(u) = -\gamma\,|u|^{\beta-1}\,u$ at three different initial energy levels and using the concavity method in the fram work of the potential well they demonstrated the global existence and blow up of the solution. Also Wang [65] studied the last problem (1.8) in high dimensional space, and he put some assumptions for f using Duhmel's principle and the Fourier transformation. He found the solutions, then he foxed on the existence and individuality of both local and global solutions. Furthmore, the problem's finite time blow-up was discussed using the potential well method. After that, Zaiyon Zhang and al [70] investigated the initial value problem (IVP) in conjunction with the generalized damped Boussinesq equation, which includes twofold rotational inertia.

$$u_{tt} - \Delta u - 2a\Delta u_t - \gamma\Delta^3 u + \delta\Delta^2 u - b\Delta u_{tt} + \lambda\Delta^2 u_{tt} = \Delta f(u).$$

They established decay and pointwise estimates using decay estimations from solutions to the relevant linear equation using the Fourier transform. Using the time-weighted norms approach and the contraction mapping concept, they achieved the existence and asymptotic behavior of global solutions in the relevant Sobolev spaces with minimal starting data requirements.

In recent years, mathematicians have focused their attention on studying nonlinearity, moving away from extensively researching polynomial nonlinearity. Logarithmic nonlinearity has been observed in various fields of physics, including optics, inflationary cosmology, and geophysics, leading to extensive investigation of equations involving nonlinearity and the discovery of new phenomena. These discoveries encompass aspects such as the global existence of solutions, exponential growth patterns, and situations where a blow-up occurs over a period of time.

An intriguing area of research explored by scientists involves the study of solitary wave solutions within the logarithmic KDV equation framework. Consequently, this exploration yielded the derivation of a Boussinesq-type equation that takes into consideration nonlinearity. For example, the concavity method and the potential well method with the logarithmic Sobolev inequality were used to derive results on the existence of local and global solutions and the infinite time blow-up of solutions by many researchers (Conf Nhan and Xuan [55], Ding and Zhou [10], Lian and Xu [30], Zhang and al [20],...). The using of the logarithmic Sobolev inequality because of the hurdles of applying the potential well approach when logarithmic nonlinearity is present. First, Wazwaz [66] introduced the generalized logarithmic Boussinesq equation in the form

$$u_{tt} + u_{xx} + u_{xxxx} + (u \log |u|^{\gamma})_{xx} = 0, \tag{1.9}$$

and the generalized logarithmic improved Boussinesq equation in the form

$$u_{tt} + u_{xx} + u_{xxtt} + (u \log |u|^{\gamma})_{xx} = 0. \tag{1.10}$$

He also derived the Gaussian solitary wave solutions for the logarithmic regularized Boussinesq equation. The aim of his work was to demonstrate that both logarithmic models are characterized by their Gaussian solutions.

Also, Hu and al in [21] and [20] studied the initial value problem of two Boussinesq equations with logarithmic nonlinearity

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} + (u \log |u|^{\gamma})_{xx} = 0, \tag{1.11}$$

and

$$u_{tt} - u_{xx} + u_{xxxx} + (u_x \log |u_x|^{\gamma})_x = 0. \tag{1.12}$$

They investigated the existence of the weak solution both locally and globally, using the Galerkin method, logarithmic Sobolev, and logarithmic Gronwall inequalities, and by the potential well method, they discussed the infinite time blow-up of solutions without proving that the maximal existence time of solutions can be extended to infinity.

Recently, Piskin and Irik [40] looked into the sixth order Boussinesq equation with logarithmic nonlinearity

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} + u_{xxxxtt} + u_{xxxxxx} + (u_x \log |u_x|^{k})_x = 0. \tag{1.13}$$

They demonstrated the existence of global solutions by virtue of the Feado-Galerkin method and logarithmic Sobolev inequality, then proved the existence of the infinite time blow-up solutions with subcritical initial energy.

Furthermore, Ding and Zhou in [11] studied the following Boussinesq equation

$$u_{tt} - \beta\Delta u_{tt} - \Delta u + \Delta^2 u - \alpha\Delta u_t + \gamma\Delta^2 u_t + \Delta(u\log|u|) = 0 \quad , x \in \Omega, t > 0. \qquad (1.14)$$

They also used the Faedo-Galerkin approach to determine the local existence of solutions. Then they demonstrated the dynamical behaviors of solutions. They also proposed a new approach to demonstrate the existence of infitite time blow-up of solutions with arbitrary high initial energies.

1.1 Structure of Thesis

- Chapter 2 revisits certain definitions and fundamental outcomes to provide a solid foundation for the research reader.
- Chapter 3 focuses on demonstrating the existence and uniqueness of both local and global solutions, as well as exploring blow-up results for a multidimensional dissipative Boussinesq equation within the bounded domain Ω. The examination delves into two situations concerning the function f, as proposed by Wang and Su in [51] and Ding and Zhou in [11].
- In Chapter 4, we address the Cauchy problem associated with the one-dimensional fifth-order Boussinesq equation featuring logarithmic nonlinearity. We establish the existence and uniqueness of the local mild solution within the energy space through a series of estimations. Additionally, we present results pertaining to the existence and uniqueness of global solutions, as well as the occurrence of finite-time blow-up of the solution, subject to certain constraints on the initial data.

Chapter 2

Preliminary Concepts

The aim of this chapter is to provide essential tools for understanding the concepts used throughout this thesis. We will introduce definitions, theorems and then outline their directly applicable properties. It is important to note that all statements in the first chapter are made without proof. (see [6], [27], [33], [39], [38], [49], [54], [71]).

2.1 Functional spaces

Definition 2.1 *$C^m(\Omega)$ is the space of functions u which are m times derivable and whose derivation of order m is continuous.*

Remark 2.1 *$C_0^\infty(\Omega)$ is the space of infinitely differentiable functions with compact support. We denote it also by $D(\Omega)$.*

Definition 2.2 • *Let $x = (x_1, x_2, ..., x_n) \in \Omega \subset \mathbb{R}^n$, and let u be a function defined from Ω to $\mathbb{R}^n$, we denote by*

$$D_i u(x) = u_{x_i}(x) = \frac{\partial u(x)}{\partial x_i},$$

to the partial derivative of u with respect to x_i.

- *We define the gradient of u and its module by*

$$\nabla u = (\frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}, ..., \frac{\partial u(x)}{\partial x_n})^T,$$
$$|\nabla u|^2 = \sum_{i=1}^{n} \left| \frac{\partial u(x)}{\partial x_i} \right|^2 .$$

2.1.1 Hilbert space

Definition 2.3 ***(Inner product)*** *An inner product on a complex linear space X is a map $(.,.) : X \times X \to \mathbb{C}$ such that for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{C}$*

a) $(x, \lambda y + \mu z) = \lambda(x, y) + \mu(x, z)$ *(Linear in the second argument).*

b) $(y, x) = \overline{(x, y)}$ *(Hermitian).*

c) $(x, x) \geq 0$ *(nonnegative).*

d) $(x, x) = 0$ *if and only if* $x = 0$ *(positive definite).*

Remark 2.2 *We call a linear space with an inner product a pre-Hilbert space.*

Proposition 2.1 *A Hilbert space is a complete inner product space.*

2.1.2 Banach space

Definition 2.4 *Let X be a victor space on $\Bbbk(\mathbb{R}$ or $\mathbb{C})$. We say that an application of X in $\mathbb{R}^+$ denoted by $\|.\|_X$ is a X norm if and only if the following conditions hold:*

a) $\forall x \in X,\ \|x\|_X = 0 \Longleftrightarrow x = 0$ *(positive definite).*

b) $\forall x \in X, \forall \lambda \in \Bbbk\ \|\lambda x\|_X = |\lambda|\, \|x\|_X$ *(Homogeneity).*

c) $\forall x, y \in X \quad \|x + y\|_X \leq \|x\|_X + \|y\|_X$ *(Triangular inequality).*

Definition 2.5 *Let X be a vector space and $\|.\|$ a norme on X. The paire $(X, \|.\|_X)$ is called normed space.*

Definition 2.6 *A normed linear space is a metric space with respect to the metric derived form its norm where*

$$d(x, y) = \|x - y\| .$$

Definition 2.7 *A Banach space is a normed linear space that is a complete metric space with the respect to the metric derived form its norm.*

Definition 2.8 *A Banach space X is said to be reflexive if the natural mapping from X into its bidual X^{**} is an isomorphism.*

2.1.3 Distributions spaces

Definition 2.9 *We define the support of a function $f : \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{C}$) by*

$$supp\ f = adh\{x \in \mathbb{R}^n \,:\, f(x) \neq 0\},$$

that is, the adhesion of the set of x such that is identically null, ***i.e.*** *the smallest closed set outside which f is identically null.*

Definition 2.10 *We refer to $D(\mathbb{R}^n)$, or simply, D, all the indefinitely differentiable functions with support*

$$D = \{\varphi \in C^{\infty} \,:\, suppf \quad borné\ \}.$$

This set is identified as the fundamental space and the essential functional elements of that space.

Definition 2.11 *We define the distribution T as a continuous linear functional on D.*
i) (Linearity) An application T of D on $\mathbb{R}$ (or $\mathbb{C}$) matching a function $\varphi \in D$, a number $\langle T, \varphi \rangle$ such as: for all $\varphi_1, \varphi_2 \in D$ and $\alpha, \beta \in \mathbb{C}$ we have

$$\langle T, \alpha\varphi_1 + \beta\varphi_2 \rangle = \alpha \langle T, \varphi_1 \rangle + \beta \langle T, \varphi_2 \rangle .$$

Instead of functional linear, we also say linear form.
ii) (Continuity) If (φ_k) converges in D to φ, then $\langle T, \varphi_k \rangle$ converges in the usual sens to $\langle T, \varphi \rangle$.

In other words, a linear function on D defines a distribution if for any $(\varphi_k) \in D$ that converges in D towards zero,and $\langle T, \varphi_k \rangle$ converges in the usual sens towards zero.

Proposition 2.2 *A linear functional on D is a distribution if and only if, for any compact K and for any function $\varphi \in D$ with supp$\varphi \subset K$, there is a constant $C > 0$ and an integer m such as:*

$$|\langle T, \varphi \rangle| \leq C \sum_{j=0}^{m} \sup_{x \in k} \left| \varphi_k^{(j)}(x) \right|. \tag{2.1}$$

In the case of several variables, the expression (2.1) is obviously replaced by this one

$$|\langle T, \varphi \rangle| \leq C \sum_{|\alpha| \leq m} \sup_{x \in k} |D^{\alpha} \varphi(x)|.$$

Definition 2.12 *Let $\Omega \subset \mathbb{R}^n$, the function $u : \mathbb{R} \to \mathbb{R}$ is locally integrable, if for all compact set $K \subset \Omega$*

$$\int_K |u(x)|\, dx < \infty.$$

We define $L^1_{loc}(\Omega)$ the space of locally integrable functions defined on Ω.

Definition 2.13 *Let $\Omega \subset \mathbb{R}, 1 \leq i \leq n$, a function $u \in L^1_{loc}(\Omega)$ has an i^{th} weak derivative in $L^1_{loc}(\Omega)$, if there exists $f_i \in L^1_{loc}(\Omega)$, $\left(\partial_i u = \frac{\partial u}{\partial x_i} = f_i \right)$ such that for all $\varphi \in C_0^{\infty}(\Omega)$, we have*

$$\int_{\Omega} u(x) \partial_i \varphi(x) dx = - \int_{\Omega} f_i(x) \varphi(x) dx,$$

this brings us to say that the i^{th} derevative within the meaning of distribbutions of u belongs to $L^1_{loc}(\Omega)$.

Definition 2.14 *We define $D'(\Omega)$ as the set of all distributions that are defined on the space $D(\Omega)$.*

Proposition 2.3 *The space $D(\Omega)$ is contained and dense in $L^p(\Omega), 1 \leq p < +\infty$ and every convergent sequence in D converges in L^p.*

Definition 2.15 *The space $S(\mathbb{R}^n)$ is defined by*

$$S(\mathbb{R}^n) = S = \left\{ u \in \mathbb{C}^\infty; \forall \alpha, \beta \in \mathbb{N}^n; x^\alpha D^\beta u \to 0 \ as \ \ |x| \to +\infty \right\},$$

S is the space of functions of class $\mathbb{C}^\infty$ with faster decay at infinity, although it is not a normed space. Its topology can be defined using the sequence of semi-norms:

$$u \to \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta u(x) \right| = d_{\alpha\beta}(u).$$

Definition 2.16 *A tempered distribution is a continuous linear functional on $S(\mathbb{R}^n)$, that is, a continuous linear map from $S(\mathbb{R}^n)$ to $\mathbb{C}$. The space of tempered distributions is denoted $S'(\mathbb{R}^n)$.*

2.1.4 Lebesgue space

Definition 2.17 *Let $\Omega \subset \mathbb{R}^n$ provided with the lebesgue mesure and $p \in \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$, we define the space $L^p(\Omega)$ by*

$$L^p(\Omega) = \begin{cases} \left\{ f : \Omega \to \mathbb{R}, f \quad measurable \ and \ \int_\Omega |f(x)|^p \, dx < +\infty \right\} & 1 \leq p < \infty \\ \{ f : \Omega \to \mathbb{R}, f \quad measurable, \ \exists c > 0, \ so \ that \ |f(x)| \leq c \} & p = \infty \end{cases},$$

equipped with norm

$$\|f\|_{L^p} := \begin{cases} (\int_\mathbb{R} |f(x)|^p \, dx)^{\frac{1}{p}} & , 1 \leq p < \infty \\ ess \sup_{x \in \mathbb{R}^n} |f| & , p = \infty \end{cases}.$$

If $\|f\|_{L^p} < \infty$ then $f \in L^p(\mathbb{R}^n)$.

Remark 2.3 *For any $1 \leq p \leq \infty$ the vector space L^p is a Banach space with the respect to the p-norm.*

2.1.5 Sobolev spaces

$\mathbf{H}^m(\Omega)$ space

Definition 2.18 *Let Ω be a nonempty open set in $\mathbb{R}^n, n \geq 1$, we define the sobolev space $H^1(\Omega)$ by*

$$H^1(\Omega) = \left\{ v : \Omega \subset \mathbb{R}^n \to \mathbb{R}, v \in L^2(\Omega), 1 \leq i \leq n \right\}.$$

Definition 2.19 •*The mapping $(.,.)_{H^1(\Omega)} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by:*

$$(u, v)_{H^1(\Omega)} = \int_\Omega u(x)v(x)dx + \int_\Omega \nabla u(x) \nabla v(x)dx,$$

defines the inner-product in $H^1(\Omega)$.

•*$H^1(\Omega)$ is a Hilbert space with a norm*

$$\forall v \in H^1(\Omega) : \|v\|^2_{H^1(\Omega)} = (v, v)_{H^1(\Omega)}.$$

With its norm

$$\forall v \in H^1(\Omega) : \|v\|^2_{H^1(\Omega)} = \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)},$$

where

$$\|\nabla v\|^2_{L^2(\Omega)} = \left\| \frac{\partial v}{\partial x_1} \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial v}{\partial x_2} \right\|^2_{L^2(\Omega)} + ... + \left\| \frac{\partial v}{\partial x_n} \right\|^2_{L^2(\Omega)}.$$

Definition 2.20 *Let Ω be nonempty open set in $\mathbb{R}^n$, $n \geq 1$, we define the sobolev space $H^1_0(\Omega)$ by*

$$H^1_0(\Omega) = \left\{ v \in H^1(\Omega), \textit{such that } v|_{\partial\Omega} = 0 \right\}.$$

Definition 2.21 *Let Ω be nonempty open set in $\mathbb{R}^n$, $n \geq 1$ and $m \in \mathbb{N}$, we say that $u \in H^m(\Omega)$ if $u \in L^2(\Omega)$ and if all its derevatives in the sense of the distributions up to the other m are still in $L^2(\Omega)$* ***i.e:***

$$H^m(\Omega) = \left\{ u \in L^2(\Omega), \forall \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}^n \textit{ where } |\alpha| \leq m, \textit{ we have } D^\alpha(u) \in L^2(\Omega) \right\}.$$

Theorem 2.1 ***a)*** *$H^m(\Omega)$ is equipped with the inner product $(.,.)_{H^m(\Omega)}$ is a Hilbert space.*
b) *If $p \geq q$, $H^p(\Omega) \hookrightarrow H^q(\Omega) \hookrightarrow H^0(\Omega)$ with continuous injection.*

Lemma 2.1 *Since $D(\Omega)$ is dense in $H_0^m(\Omega)$, we have*

$$D(\Omega) \hookrightarrow H_0^m(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-m}(\Omega) \hookrightarrow D'(\Omega),$$

where $H^{-m}(\Omega)$ the dual of $H_0^m(\Omega)$ in a low subspace Ω.

$\mathbf{W}^{k,p}(\Omega)$ spaces

Definition 2.22 *Let $k \in \mathbb{N} \cup \{0\}$ and p a real number such that $1 \leq p \leq \infty$, we define the $W^{k,p}(\Omega)$ space as following*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega), \textit{such that } \partial^\alpha u \in L^p(\Omega), \forall \alpha, |\alpha| \leq k\},$$

with the following norm

$$\|u\|_{W^{k,p}(\Omega)} = \|u\|_{L^p} + \sum_{0<|\alpha|\leq k} \|\partial^\alpha u\|_{L^p}.$$

Theorem 2.2 • *We suppose that Ω is a bounded open subset of $\mathbb{R}^n$ with C^1 boundary and $1 \leq p < n$, then*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega) \quad , 1 \leq q < p,$$

for each $q \in [p, p^*]$, where $p^* = \frac{np}{n-p}$.

- *If $p = n$, we have $W^{1,p}(\Omega) \subset L^q(\Omega)$ for each $q \in [p, \infty)$.*
- *If $p > n$, we have $W^{1,p}(\Omega) \subset L^\infty(\Omega) \cap \mathbb{C}^{0,\alpha}(\Omega)$ where $\alpha = \frac{p-n}{p}$.*

Theorem 2.3 *$W^{k,p}(\Omega)$ equipped with the norm*

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{0<|\alpha|\leq k} \|\partial^\alpha u\|_{L^p}, 1 \leq p \leq \infty, \textit{ for all } u \in W^{k,p}(\Omega),$$

is a Banach space.

2.1.6 Fourier transform

Definition 2.23 *(**Fourier transform of** L^1**-Functions for** $v \in L^1$) The Fourier transform of v is defined by the following formula:*

$$\hat{v}(y) = Fv(y) = \int_{\mathbb{R}^n} \exp(-2\pi ixy)v(x)dx \quad , \quad y \in \mathbb{R}^n,$$

and is abviously linear.

Remark 2.4 *If $u \in L^1$, $\hat{v}$ is a bounded continuous function on $\mathbb{R}^n$ with*

$$\|\hat{v}\|_{L^\infty} \leq \|v\|_{L^1} .$$

Remark 2.5 *We can also define $\bar{F}$ as*

$$\bar{F}v(y) = \int_{\mathbb{R}} \exp(2i\pi xy)v(x)dx \quad , y \in \mathbb{R}^n,$$

this transformation is known as the fourier inverse transformation, let us write symbolically

$$f(x) = \bar{F}[\hat{f}(v)] = F^{-1}[\hat{f}(v)].$$

Definition 2.24 *$\tau_b f$ is the translate of the function f of amplitude $b \in \mathbb{R}^n$, and defined as folowing:*

$$\tau_b f(x) = f(x - b) \quad , \quad x \in \mathbb{R}^n, b \in \mathbb{R}^n,$$

and we denote by $\check{f}$ the symmetrics of the function f defined by:

$$\check{f}(x) = f(-x), x \in \mathbb{R}^n.$$

Proposition 2.4 *For $u, v \in L^1$, we have*

(i) $\overline{\hat{u}} = \overline{F(u)} = \overline{F}(\overline{u}), \quad \overline{F}(\check{u}) = F(u) = \hat{u}.$

(ii) $\int_{\mathbb{R}^n} \hat{u}(x)v(x)dx = \int_{\mathbb{R}^n} u(y)\hat{v}(y)dy.$

(iii) $\begin{cases} \widehat{\tau_b u}(x) = \exp(-2i\pi bx)\hat{u}(x), b \in \mathbb{R}^n, x \in \mathbb{R}^n \\ F(\exp(-2i\pi bx).u) = \tau_{-b}\hat{u} \end{cases}.$

Remark 2.6 ***(Fourier Transform in*** L^2*) The space S being dense in L^2 and F and F^{-1} being continuous from S into it self when the space is furnished with the topology of L^2.*

Definition 2.25 *The fourier transform for tempered distributions by transposition: for each $T \in S'$, the Fourier transform of T, denote by $\hat{T}$ or $F(T)$ will be tempered distribution defined by*

$$\left\langle \hat{T}, \varphi \right\rangle = \langle T, \hat{\varphi} \rangle \text{ for all } \varphi \in S.$$

Definition 2.26 *Let $s \in \mathbb{R}$, $H^s(\mathbb{R}^n)$ is the space of tempered distributions u, such that*

$$(1 + |\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n),$$

we define $H^s(\mathbb{R}^n)$ with the scalar product

$$(u, v)_s = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \hat{u}(\xi)\overline{\hat{v}(\xi)} d\xi,$$

and the associated norm is

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s \left| \hat{u}(\xi) \right|^2 d\xi \right)^{1/2}.$$

Proposition 2.5 *For $s \in \mathbb{R}$, the space $H^s(\mathbb{R}^n)$ satisfies:*

i) *$H^s(\mathbb{R}^n)$ is a Hilbert space.*

ii) *If $s_1 \geq s_2$, then $H^{s_1}(\mathbb{R}^n) \subset H^{s_2}(\mathbb{R}^n)$.*

2.2 Some inequalities

Proposition 2.6 ***(Cauchy-Schwarz's inequality):*** *If $f, g \in L^2(\Omega)$ the Cauchy-Schwarz inequality is*

$$|f.g|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \cdot \|g\|_{L^2(\Omega)}.$$

Definition 2.27 ***(Hölder's inequality)****: Let E measurable space $p, q > 0$ where $\frac{1}{p}+\frac{1}{q} = 1$, $f \in L^p(E), g \in L^q(E)$, then, the product $f.g \in L^1(E)$ and the norm satisfy:*

$$\||f.g|\|_{L^1(E)} \leq \|f\|_{L^p(E)} \cdot \|g\|_{L^q(E)}.$$

In the other hand, for $0 < p, q < +\infty$ define by $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, if $f \in L^p(E), g \in L^q(E)$, then, the product $f.g \in L^1(E)$ and the norm satisfy:

$$\||f.g|\|_{L^r(E)} \leq \|f\|_{L^p(E)} \cdot \|g\|_{L^q(E)}.$$

Definition 2.28 ***(Young's Inequality)****: We assume that $f \in L^p(\Omega), g \in L^q(\Omega)$, where $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\||f.g|\|_{L^1(E)} \leq \frac{1}{p} \|f\|^p_{L^p(E)} \frac{1}{q} \|g\|^q_{L^q(E)}.$$

Theorem 2.4 ***(Poincare's inequality):*** *Suppose Ω is a bounded open subset of $\mathbb{R}^n$, $u \in W_0^{1,p}(\Omega)$ for some $1 \leq p < n$. Hence, we have the following estimate*

$$\|u\|_{L^q(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

for each $q \in [1, p^*]$, where $p^* = \frac{np}{n-p}$ and the constant C depends only on q, p, n and Ω

Lemma 2.2 *Let $1 \leq p \leq r \leq q$, $0 \leq \alpha \leq 1$ such that $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$. Then*

$$\|u\|_{L^r} \leq \|u\|^{\alpha}_{L^p} \|u\|^{1-\alpha}_{L^q}, \forall u \in L^p(\Omega).$$

Theorem 2.5 ***(Green Formula)*** *Let Ω be a nonempty open set in $\mathbb{R}^n, n \geq 1$. If u and v are in $H^1(\Omega)$, we have*

$$\int_\Omega u(x)\frac{\partial v}{\partial x_i}(x)dx = -\int_\Omega v(x)\frac{\partial u}{\partial x_i}(x)dx + \int_{\partial\Omega} u(x)v(x)\eta_i(x)dx,$$

where $(\eta_i)_{1\leq i\leq n}$ is a normal unit external a $\partial\Omega$.

Lemma 2.3 ***(Gronwell's inequality)*** *Let $M > 0, f \geq 0$ and $b_1, b_2 \geq 0$. Let $\varphi \geq 0$ almost every where such that $f.\varphi \in L^1(0, M)$ and*

$$\varphi(t) \leq b_1 + b_2 \int_0^M f(x)\varphi(x)dx, \qquad t \in (0, M),$$

then

$$\varphi(t) \leq b_1 \exp(b_2 \int_0^M f(x)dx), \qquad t \in (0, M).$$

Lemma 2.4 ***(Logarithmic Sobolev's inequality)*** *If $u \in H^1(\mathbb{R}^n)$, $a > 0$, then*

$$2\int_{\mathbb{R}^n} u(x)^2 \ln\left(\frac{|u(x)|}{\|u\|}\right) dx + n(1 + \ln a)\left\|u\right\|^2 \leq \frac{a^2}{\pi}\left\|u\right\|_{H^1}^2,$$

for $u \in H_0^1(\Omega)$, we define $u(x) = 0$ for $x \in \mathbb{R}^n\backslash\Omega$, then $u \in H^1(\mathbb{R}^n)$. So, for the general domain Ω, we have following logarithmic Sobolev inequality:

$$2\int_\Omega u(x)^2 \ln\left(\frac{|u(x)|}{\|u\|}\right) dx + n(1 + \ln a)\left\|u\right\|^2 \leq \frac{a^2}{\pi}\left\|u\right\|_{H^1}^2,$$

where $u \in H_0^1(\Omega)$ and $a > 0$.

2.3 Weak and Strong Convergence

Definition 2.29 *(**Weak convergence in E**) Let $x \in E$, and consider the sequence $\{x_n\} \subset E$. We say that $\{x_n\}$ weakly converges to x in E, denoted as $x_n \rightharpoonup x$ in E, if*

$$\langle f, x_n \rangle \to \langle f, x \rangle \quad \textit{for all } x \in E'.$$

Definition 2.30 *(**Weak convergence in E'**) Let $f \in E'$, and let $\{f_n\} \subset E'$. We say that $\{f_n\}$ weakly converges to f in E', and we write $f_n \rightharpoonup f$ in E', if*

$$\langle f_n, x_n \rangle \to \langle f, x \rangle \quad \textit{for all } x \in E''.$$

Definition 2.31 *(**Weak star convergence**) Suppose that $f \in E'$, and that $\{f_n\} \subset E'$. We assert that $\{f_n\}$ weakly star converges to f in E', denoted as $f_n \rightharpoonup^* f$ in E', if*

$$\langle f_n, x_n \rangle \to \langle f, x \rangle \quad \textit{for all } x \in E.$$

Definition 2.32 *(**Strong convergence**) Let $x \in E$ (resp. $f \in E'$) and let $\{x_n\} \subset E$ (resp. $\{f_n\} \subset E'$). We state that $\{x_n\}$ (resp. $\{f_n\}$) strong converges to x (resp. f), and we write $x_n \to x$ in E (resp. $f_n \to f$ in E'), if*

$$\lim_{n\to\infty} \|x_n - x\|_E = 0, \ \textit{(resp. } \lim_{n\to\infty} \|f_n - f\|_{E'} = 0\textit{)}.$$

Definition 2.33 *(**Strong convergence in** L^p **with** $1 \leq p < \infty$) Let Ω an open subset of $\mathbb{R}^n$. We say that the sequence $\{x_n\}$ of L^p weakly converges to $f \in L^p(\Omega)$, if*

$$\lim_{n\to\infty} \int_\Omega f_n(x) g(x) dx = \int_\Omega f(x) g(x) dx \textit{ for all } g \in L^q, \ \frac{1}{p} + \frac{1}{q} = 1.$$

Theorem 2.6 *(**Bolzano-Weierstrass**) If $dimE < \infty$ and if $\{x_n\} \subset E$ is bounded, then* *there exists $x \in E$ and a subsequence $\{x_{n_k}\}$ strongly converges to x.*

Theorem 2.7 ***(Weak compactness, Kakutani-Eberlin)*** *Assuming E is reflexive, let $\{x_n\}$ be a sequence in E. If $\{x_n\}$ is bounded, then there exists $x \in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$such that $\{x_{n_k}\}$ weakly star converges to x in E.*

Theorem 2.8 ***(Weak star Compactness, Banach-Alaoglu-Bourbaki)*** *Given that E is separable, let $\{f_n\}$ be a sequence in E'. If $\{x_n\}$ is bounded, then there exists $f \in E'$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$such that $\{f_{n_k}\}$ weakly star converges to f in E'.*

Theorem 2.9 ***(Weak compactness in $L^p(\Omega)$ with $1 < p < \infty$)*** *Let $\{f_n\} \subset L^p(\Omega)$, If $\{f_n\}$ is bounded, then there exists $f \in L^p(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \rightharpoonup^* f$ in $L^p(\Omega)$.*

Definition 2.34 *[38] E is a Banach space. If $(f_n)_n$ is a sequence of E', then $(f_n)_n$ converges to f in the sense of weak convergence if and only if $f_n(x)$ converges to $f(x)$ for all $x \in E$.*

Theorem 2.10 ***(The convergence theory)*** *[49] Assume (f_n) is a sequence of measurable functions with the property that $f_n(x) \to f(x)$ as $n \to \infty$. If $|f_n(x)| \leq g(x)$, where g is integrable, then*

$$\int |f_n - f| \to 0 \text{ as } n \to \infty,$$

and consequently

$$\int f_n \to \int f \text{ as } n \to \infty.$$

Theorem 2.11 ***(Density theorem)*** *[6] The space $C_0(\Omega)$ being dense in $L^1(\Omega)$, meaning that*

$$\forall f \in L^1(\Omega),\ \forall \epsilon > 0,\ \exists f_1 \in C_0(\Omega) \text{ such that } \|f - f_1\|_{L^1(\Omega)} < \epsilon.$$

Theorem 2.12 ***(Dunford-pettis)*** *[6] Let Ω denote a nonempty open set in $\mathbb{R}^n$, and let $X \subset L^1(\Omega)$ be a subset borne. X is compact so her topology $\sigma(L^1(\Omega), L^\infty(\Omega))$, then the following condition holds:*

$$\forall \epsilon > 0,\ \forall c > 0 \text{ such that } \int_A |f| < \epsilon, \forall f \in X, \forall A \subset \Omega \text{ where } |A| < c.$$

Lemma 2.5 ***(Aubin Lions lemma)*** *Consider three Banach spaces, denoted as X_0, X and X_1, with the inclusion relationships $X_0 \subseteq X \subseteq X_1$. Assume that X_0 is compactly embedded in X and that X is continuously embedded in X_1. Additionally, suppose that both X_0 and X_1 are reflexive spaces. For $1 < p,\ q < +\infty$, let $W = \{u \in L^p([0,t]; X_0)/u' \in L^q([0,t]; X_1)\}$. Then, it can be asserted that the embedding of W into $L^p([0,t]; X)$ is also compact.*

2.4 Abstract Cauchy problem

Lemma 2.6 *[54] Consider a fixed parameter $T > 0$. Assuming that u satisfies $u \in L^2(0,T;Z)$, $u_t \in L^2(0,T;S)$, and $u_{tt} + ßu \in L^2(0,T;S)$ (ß is a linear operator which is self-adjoint). After making adjustments on a set of measure zero, u attains continuity on $[0,T]$ in the space Z, and u_t becomes continuous on $[0,T]$ in the space S. Additionally, in terms of distributions over the interval $(0,T)$, the following relation holds:*

$$(u_{tt} + ßu, u_t)_S = \frac{1}{2}\frac{d}{dt}\left[\|u_t\|_S^2 + B(u,v)\right],$$

where $(.,.)_S$ denotes the inner product in the space S, and $\|.\|_S^2 = (.,.)_S$.

Lemma 2.7 *[54] Let $T > 0$ be fixed, Assume that $C \in \mathbb{R}, f \in L^2(0,T;S), u_0 \in Z$ and $u_1 \in S$, then the ordinary differential equation*

$$\begin{cases} u_{tt} + Cu_t + ßu = f, \quad t \in (0,T) \\ u(0) = u_0, u_t(0) = u_1 \end{cases},$$

admits a unique solution $u \in C([0,T];Z) \cap C^1([0,T];S)$.

Lemma 2.8 *[54] Let* V, H *and* V' *be three Hilbert spaces with* V' *being the dual space of* V *and each space included and dense in the following one:* $V \hookrightarrow H \cong H' \hookrightarrow V'$. *If an abstract function* u *belongs to* $L^2(0,T;V)$ *and its derivative* u_t *in the distribution sense belongs to* $L^2(0,T;V')$, *then* u *is almost everywhere equal to a function continuous from* $[0,T]$ *into* H.

Lemma 2.9 *Let* B *be a reflexive Banach space and* $0 < \tilde{T} < \infty$. *Suppose that* $1 < p < \infty$,$\phi \in L^p(0,\tilde{T};B)$, *and the sequence* $\{\phi_m\}_{m=1}^{\infty} \subset L^p(0,\tilde{T};B)$ *satisfies, as* $m \to \infty, \phi_m \rightharpoonup \phi$ *weakly in* $L^p(0,\tilde{T};B)$, $\phi_{mt} \rightharpoonup \phi_t$ *weakly in* $L^p(0,\tilde{T};B)$. *Then* $\phi_m(0) \rightharpoonup \phi(0)$ *weakly in* B.

Chapter 3

Initial Boundary Value Problem for the Dissipative Boussinesq Equation

This chapter presents the results of Boussinesq equation with damping, a topic that has attracted considerable research attention.

$$u_{tt} - \beta_1 \Delta u_{tt} - \Delta u + \Delta^2 u + \beta_2 \Delta^2 u_t - \beta_3 \Delta u_t + \Delta f(u) = 0. \tag{3.1}$$

3.1 Introduction

Numerous researchers have explored the equation (3.1) under various sets of initial conditions and the function f, as evidenced by studies [58], [15], [8], [26], [69], [51], [62], [37] and [11]. For instance, Varlamov in [58] investigated the initial boundary value problem (3.1) when $\beta_1 = \beta_2 = 0, \beta_3 > 0$, and $f(u) = Mu^2$, with M being a real number. He conducted a comprehensive analysis regarding the existence and long-term behavior of solutions. After setting the conditions to $\beta_1 > 0$, $\beta_2 = 0$, and $\beta_3 \geq 0$ and ensuring that f satisfies appropriate assumptions, Chen, Wang, and Wang in [8] successfully established the presence of global classical solutions for the initial boundary value problem (3.1) in a one-dimensional case. Furthermore, they substantiated the nonexistence of a global solution.

In the two-dimensional case, specifically when $\beta_1 = 1$, $\beta_2 = \beta_3 = 0$, and f exhibits growth at infinity, in [69], Zhang and Hu provided insights into both the existence and nonexistence of global solutions for the Cauchy problem (3.1). Su and Wang, in [51] and [62], undertook a thorough examination of the Cauchy problem (3.1) under distinct conditions. In this chapter, we will explore into their findings in [51] as well as those of Zhou and Ding, who also investigated the Cauchy problem with the function $f(u) = u \log |u|$ in [11].

3.2 Existence and Nonexistence of Global solution

In this part, we will talk about the result of the equation (3.1), with $\beta_1 = 0, \gamma \geq 0$ and $f(u) = s\,|u|^{p-1}\,u$ $(s \in \mathbb{R}^*, p > 1)$. So we have the following equation:

$$u_{tt} - \Delta u + \Delta^2 u - \alpha \Delta u_t + \gamma \Delta^2 u_t + \Delta f(u) = 0, \qquad (x,t) \in \Omega \times \mathbb{R}^+. \tag{3.2}$$

$$u|_{\partial\Omega} = 0, \qquad \Delta u|_{\partial\Omega} = 0. \tag{3.3}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x). \tag{3.4}$$

In examining the well-posedness of problems (3.2)-(3.4), we will apply restrictions to parameter p.

$$1 < p < \infty, \ \text{ if } n = 1,2, \qquad 1 < p \leq \begin{cases} \frac{n+2}{n-2}, & \text{for} \gamma > 0 \\ \frac{n}{n-2}, & \text{for} \gamma = 0 \end{cases}, \ \text{if } n \geq 3 \tag{3.5}$$

To proceed with presenting the results, we shall introduce the appropriate definition of a weak solution that satisfies a particular variational inequality. Applying the operator $(-\Delta)^{-1}$ to equation (3.2), we have

$$(-\Delta)^{-1} u_{tt} + u + (-\Delta)^{-1} \Delta^2 u + \alpha u_t + \gamma (-\Delta)^{-1} \Delta^2 u_t = f(u). \tag{3.6}$$

Since $(-\Delta)^{-1} \Delta^2 u = -\Delta u$, for any $u \in \{u \in H^4 \cap H_0^1 : \Delta u|_{\partial\Omega} = 0\}$ (according to lemma

1.7 of [24]), then equation (3.6) becomes

$$(-\Delta)^{-1}u_{tt} + u - \Delta u + \alpha u_t - \gamma \Delta u_t = f(u). \tag{3.7}$$

For this reason, the weak solution of equation (3.7) with the initial data (3.3) and boundary value condition $u|_{\partial\Omega} = 0$ is said to be the weak solution of the problem (3.2)-(3.4). This leads to the following definition:

Definition 3.1 *(**Weak Solution**) A function $u = u(x,t)$ is a weak solution to problem (3.2)-(3.4) on $\Omega \times [0, T_{\max})$, if and only if for any $T \in [0, T_{\max}), u \in C([0,T]; H_0^1)$ with $(-\Delta)^{-\frac{1}{2}} u_t \in C\left([0,T], L^2\right)$ and $u_t \in L^2(0,T; H_0^1)$ if $\gamma > 0$, $u_t \in L^2(0,T; L^2)$ if $\gamma = 0$ satisfying:*
i) For all test function $\varphi \in C([0,T]; H_0^1)$

$$\left\langle (-\Delta)^{-\frac{1}{2}} u_{tt}, (-\Delta)^{-\frac{1}{2}} \varphi \right\rangle_{H^{-2}, H^2 \cap H_0^1} + (u,\varphi) + (\nabla u, \nabla \varphi) + (u_t, \varphi)_* = \langle f(u), \varphi \rangle_{H^{-1}, H_0^1}. \tag{3.8}$$

ii) $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$.
Moreover, $u(x,t)$ is called a local solution if $T_{\max} < \infty$, and we say $u(x,t)$ is global if $T_{\max} = \infty$.

Theorem 3.1 *(**Local Well-Posedness**) [51] Let p satisfies (3.5) and $\beta \in \mathbb{R}$. If the initial data $u_0 \in H_0^1, (-\Delta)^{-\frac{1}{2}} u_1 \in L^2$, then there exists a maximal time*

$$T_{\max} = T(\left\| (-\Delta)^{-\frac{1}{2}} u_1 \right\|_{L^2}, \|u_0\|_{H_0^1}, \alpha, \gamma) > 0,$$

such that the initial boundary value problem (3.2)-(3.4) admits a unique local weak solution $u = u(x,t)$ such that

$$u \in C([0, T_{\max}); H_0^1), (-\Delta)^{-\frac{1}{2}} u_t \in C\left([0, T_{\max}), L^2\right), \tag{3.9}$$

and

$$\begin{cases} u_t \in L^2(0, T_{\max}; H_0^1), & \textit{for } \gamma > 0 \\ u_t \in L^2(0, T_{\max}; L^2), & \textit{for } \gamma = 0 \end{cases}. \tag{3.10}$$

Additionally, the law of conservation of the full energy of the weak solutions is u holds, i.e.,

$$E(t) + \int_0^t \|u_\tau\|_*^2 \, d\tau = E(0), \tag{3.11}$$

where

$$\begin{aligned} E(t) &= \frac{1}{2}(\left\|(-\Delta)^{-\frac{1}{2}} u_t\right\|^2 + \|u\|^2 + \|\nabla u\|^2) - \frac{\beta}{p+1} \|u\|_{L^{p+1}}^{p+1} \\ &= \frac{1}{2}(\left\|(-\Delta)^{-\frac{1}{2}} u_t\right\|^2 + \|u\|_{H_0^1}^2) - \frac{\beta}{p+1} \|u\|_{L^{p+1}}^{p+1} . \end{aligned}$$

Moreover, if

$$\sup_{t \in [0, T_{\max})} (\left\|(-\Delta)^{-\frac{1}{2}} u_t\right\| + \|u\|_{H_0^1}) < \infty, \tag{3.12}$$

then $T_{\max} = \infty$.

Proof. Let $\{b_j(x)\}$ for $j = 1, 2, ..., m$ be a base functions of $H^2 \cap H_0^1$, such that $\|b_j\| = 1$. By the elliptic operator theory, $\{b_j(x)\}$ forms base functions in $H_0^1 \cap L^p (1 < p < \infty)$, and $b_j \in C^\infty(\overline{\Omega})$. Let

$$u_m(x,t) = \sum_{j=1}^m a_{jm}(t) b_j(x). \tag{3.13}$$

By the Galerkin approximate solutions of the problem (3.2)-(3.4) satisfying:

$$((-\Delta)^{-1} u_{mtt} + u_m - \Delta u_m + \alpha u_{mt} - \gamma \Delta u_{mt}, b_j) = (f(u_m), b_j). \tag{3.14}$$

$$u_m(x,0) = u_{m0}(x) = \sum_{j=1}^m \rho_j b_j \to u_0 \quad \text{in } H_0^1, \text{ as } m \to \infty. \tag{3.15}$$

$$u_{mt}(x,0) = u_{m1}(x) = \sum_{j=1}^m \zeta_j b_j \to u_1 \quad \text{in } H_0^1, \text{ as } m \to \infty. \tag{3.16}$$

Substituting (3.13) into (3.14)-(3.16), it follows that

$$\lambda_j^{-1}a''_{jm} + a_{jm} + \lambda_j a_{jm} + \alpha a'_{jm} + \gamma\lambda_j a'_{jm} = (f(u_m), b_j), \tag{3.17}$$

$$a_{jm}(0) = \rho_j, \quad a'_{jm}(0) = \zeta_j \quad (j = 1, 2, ..., m). \tag{3.18}$$

In accordance with standard ordinary differential equations theory, the issue (3.17)-(3.18) admits a solution $a_{jm} \in C^2$ on a certain interval $[0, t_m]$ for each m.

Multiplying both sides of (3.14) with $a'_{jm}(t)$ and summing up for $j = 1, 2, ..., m$, we find

$$E_m(t) + \int_0^t \|u_{m\tau}\|_*^2 \, d\tau = E_m(0), \quad \forall t \in [0, t_m], \tag{3.19}$$

where

$$E_m(t) = \frac{1}{2}(\left\|(-\Delta)^{-\frac{1}{2}}u_{mt}\right\|^2 + \|u_m\|_{H_0^1}^2) - \frac{\beta}{p+1}\|u_m\|_{L^{p+1}}^{p+1}.$$

From (3.19), there holds

$$\begin{aligned}
&\frac{1}{2}(\left\|(-\Delta)^{-\frac{1}{2}}u_{mt}\right\|^2 + \|u_m\|_{H_0^1}^2) + \int_0^t \|u_{m\tau}\|_*^2 \, d\tau \\
= \;& \frac{1}{2}(\left\|(-\Delta)^{-\frac{1}{2}}u_{m1}\right\|^2 + \|u_{m0}\|_{H_0^1}^2) + \frac{\beta}{p+1}(\|u_m\|_{L^{p+1}}^{p+1} - \|u_{m0}\|_{L^{p+1}}^{p+1}) \\
= \;& \beta\int_0^t\int_\Omega |u_m|^{p-1} u_m u_{m\tau} dx d\tau + \frac{1}{2}(\left\|(-\Delta)^{-\frac{1}{2}}u_{m1}\right\|^2 + \|u_{m0}\|_{H_0^1}^2).
\end{aligned}$$

With the fact $H_0^1 \hookrightarrow L^{p+1}$, and using Hölder and Young inequalities, we find

$$\begin{aligned}
&\frac{1}{2}(\left\|(-\Delta)^{-\frac{1}{2}}u_{mt}\right\|^2 + \|u_m\|_{H_0^1}^2) + \int_0^t \|u_{m\tau}\|_*^2 \, d\tau \\
\leq \;& |\beta|\int_0^t \|u_m\|_{L^{p+1}}^p \|u_{m\tau}\|_{L^{p+1}} \, d\tau + \frac{1}{2}(\left\|(-\Delta)^{-\frac{1}{2}}u_{m1}\right\|^2 + \|u_{m0}\|_{H_0^1}^2) \\
\leq \;& \frac{|\beta|^2 C_*^{2(p+1)}\widetilde{c}}{2}\int_0^t \|u_m\|_{H_0^1}^{2p} \, d\tau + \frac{1}{2}\int_0^t \|u_{m\tau}\|_*^2 \, d\tau + \frac{1}{2}(\left\|(-\Delta)^{-\frac{1}{2}}u_{m1}\right\|^2 + \|u_{m0}\|_{H_0^1}^2),
\end{aligned} \tag{3.20}$$

then for sufficiently large m

$$\frac{1}{2}(\left\|(-\Delta)^{-\frac{1}{2}}u_{m1}\right\|^2+\|u_{m0}\|^2_{H^1_0})\leq\left\|(-\Delta)^{-\frac{1}{2}}u_1\right\|^2+\|u_0\|^2_{H^1_0},$$

therefore,

$$\begin{aligned}&\int_0^t\|u_{m\tau}\|_*^2\,d\tau+\left\|(-\Delta)^{-\frac{1}{2}}u_{mt}\right\|^2+\|u_m\|^2_{H^1_0}\\ \leq\ &|\beta|^2\,C_*^{2(p+1)}\widetilde{c}\int_0^t\|u_m\|^{2p}_{H^1_0}\,d\tau+2(\left\|(-\Delta)^{-\frac{1}{2}}u_1\right\|^2+\|u_0\|^2_{H^1_0}),\quad\forall t\in[0,t_m].\end{aligned}\tag{3.21}$$

By simple calculation, we can find

$$\left\|(-\Delta)^{-\frac{1}{2}}u_{mt}\right\|^2+\|u_m\|^2_{H^1_0}\leq[B_0^{1-p}-C_1(p-1)t]^{-\frac{1}{p-1}}\tag{3.22}$$

where

$$B_0=2(\left\|(-\Delta)^{-\frac{1}{2}}u_1\right\|^2+\|u_0\|^2_{H^1_0}),\quad C_1=|\beta|^2\,C_*^{2(p+1)}\widetilde{c}.$$

We remark that the right side of (3.22) will be blow-up, as $t\to\frac{B_0^{1-p}}{2C_1(p-1)}$, and for

$$T=\frac{B_0^{1-p}}{2C_1(p-1)},\tag{3.23}$$

we reach

$$\left\|(-\Delta)^{-\frac{1}{2}}u_{mt}\right\|^2+\|u_m\|^2_{H^1_0}\leq 2^{\frac{1}{p-1}}B_0,\quad\forall t\in[0,T],\tag{3.24}$$

we also find

$$\int_0^t\|u_{m\tau}\|_*^2\,d\tau\leq(2^{\frac{1}{p-1}}\frac{1}{p-1}+1)B_0,\quad\forall t\in[0,T].$$

Summarizing these estimates, we have uniform boundedness of $\left\|(-\Delta)^{-\frac{1}{2}}u_{mt}\right\|^2$, $\|u_m\|_{H^1_0}$ and $\int_0^t\|u_{m\tau}\|_*^2\,d\tau$ on $[0,T]$.

The prior estimate (3.24) implies the existence of a subsequence $\{u_m\}$, denoted again by $\{u_m\}$, and u and v such that $u\in L^\infty(0,T;H^1_0),v\in L^2(0,T;H^1_0)$ and $(-\Delta)^{-\frac{1}{2}}v\in$

$L^\infty(0,T;L^2)$, with $u_t = v$ such that

$$u_m \to u \quad \text{weakly-star in } L^\infty(0,T;H_0^1). \tag{3.25}$$

$$u_{mt} \to v \quad \text{weakly in } L^2(0,T;H_0^1). \tag{3.26}$$

$$(-\Delta)^{-\frac{1}{2}} u_{mt} \to (-\Delta)^{-\frac{1}{2}} v \quad \text{weakly-star in } L^\infty(0,T;L^2).$$

$$f(u_m) \to f(u) \quad \text{weakly-star in } L^\infty(0,T;H^{-1}). \tag{3.27}$$

It follows from equation (3.14) that $(-\Delta)^{-\frac{1}{2}} u_{mtt} \in L^2(0,T;H^{-2})$ and

$$(-\Delta)^{-\frac{1}{2}} u_{mtt} \to (-\Delta)^{-\frac{1}{2}} u_{tt} \quad \text{weakly in } L^2(0,T;H^{-2}).$$

Then, for $\psi \in C[0,T]$, $\varphi_j = \psi(t) b_j \in C([0,T];H_0^1)$, multiplying both sides of (3.14) by $\psi(t)$, and taking $m \to \infty$ for $t \in [0,T]$, we get

$$\left\langle (-\Delta)^{-\frac{1}{2}} u_{tt}, (-\Delta)^{-\frac{1}{2}} \varphi_j \right\rangle_{H^{-2},H^2\cap H_0^1} + (u,\varphi_j) + (\nabla u, \nabla \varphi_j) + (u_t,\varphi_j)_* = \left\langle f(u), \varphi_j \right\rangle_{H^{-1},H_0^1}.$$

The above equation holds for all j, and consequently, it holds for any linear combination of the φ_j's. Thus (3.8) is satisfied. Ultimately, in accordance with definition 3.1, establishing $u(x,0) = u_0$ and $u_t(x,0) = u_1$ demonstrates that u qualifies as a weak solution for the initial boundary problem posed by equations (3.2)-(3.4). Furthermore, we also deduce that $(-\Delta)^{-\frac{1}{2}} u_t \in C([0,T];L^2)$.

After verifying the existence of local solutions, we now focus on strengthening their uniqueness. Suppose u and v are two solutions to the problem (3.2)-(3.4) with identical initial data, u_0 and u_1. In this case, letting $w = u - v$, in the distributional sense, leads to the following relationship:

$$
\begin{aligned}
(-\Delta)^{-1}w_{tt} + w - \Delta w &= -\alpha w_t + \gamma \Delta w_t + (f(u) - f(v)), \\
w(x,0) &= 0, \quad w_t(x,0) = 0
\end{aligned}
$$

and

$$
\begin{gathered}
w \in C([0,T]; H_0^1); \quad w_t \in L^2(0,T; H_0^1); \quad (-\Delta)^{-\frac{1}{2}} w_t \in C([0,T]; L^2), \\
f(u) - f(v) \in L^\infty(0,T; L^{1+\frac{1}{p}}) \hookrightarrow L^\infty(0,T; H^{-1}). \qquad .
\end{gathered}
$$

We obtain

$$
\frac{1}{2}\left(\left\|(-\Delta)^{-\frac{1}{2}} w_t\right\|^2 + \|w\|_{H_0^1}^2\right) + \int_0^t \|w_\tau\|_*^2 \, d\tau = \int_0^t \langle f(u) - f(v), w_\tau \rangle_{L^{1+\frac{1}{p}}, L^{p+1}} \, d\tau. \quad (3.28)
$$

Thus, using Young's inequality, it comes

$$
\begin{aligned}
\left| \int_0^t \langle f(u) - f(v), w_\tau \rangle_{L^{1+\frac{1}{p}}, L^{p+1}} \, d\tau \right| &\leq \int_0^t \|f(u) - f(v)\|_{L^{1+\frac{1}{p}}} \|w_\tau\|_{L^{p+1}} \, d\tau \\
&\leq C \int_0^t (\|u\|_{L^{p+1}}^{p-1} + \|v\|_{L^{p+1}}^{p-1}) \|w\|_{L^{p+1}} \|w_\tau\|_{H_0^1} \, d\tau \\
&\leq C \int_0^t \|w\|_{H_0^1}^2 \, d\tau + \frac{1}{2} \int_0^t \|w_\tau\|_*^2 \, d\tau.
\end{aligned}
$$

The conclusion drawn from the Gronwall's inequality is that w is identically zero. Up to this point, we have successfully demonstrated the existence and uniqueness of a solution to the problem defined by equations (3.2)-(3.4) in the case where $\gamma > 0$.

In the instance where $\gamma = 0$ and $\alpha > 0$, the adaptation to the scenario with $\gamma > 0$ requires some appropriate modifications. To address the aspect of existence, the estimate in (3.20) is replaced by the following expression:

$$\begin{aligned}
&\frac{1}{2}\left(\left\|(-\Delta)^{-\frac{1}{2}}u_{mt}\right\|^2 + \|u_m\|^2_{H_0^1}\right) + \alpha\int_0^t \|u_{m\tau}\|_*^2\,d\tau \\
\leq\ & |\beta|\int_0^t \|u_m\|^p_{L^{2p}}\|u_{m\tau}\|_{L^2}\,d\tau + \frac{1}{2}\left(\left\|(-\Delta)^{-\frac{1}{2}}u_{m1}\right\|^2 + \|u_{m0}\|^2_{H_0^1}\right) \\
\leq\ & \frac{|\beta|^2 C_*^{2p}}{2\alpha}\int_0^t \|u_m\|^{2p}_{H_0^1}\,d\tau + \frac{\alpha}{2}\int_0^t \|u_{m\tau}\|_*^2\,d\tau + \frac{1}{2}\left(\left\|(-\Delta)^{-\frac{1}{2}}u_{m1}\right\|^2 + \|u_{m0}\|^2_{H_0^1}\right).
\end{aligned}$$

Then, for the continuity of solutions $u(x,t)$, since

$$(u, u_t) \in L^\infty(0,T;H_0^1) \times L^2(0,T;L^2),$$

thus

$$u \in L^\infty(0,T;H_0^1) \cap C_w(0,T;L^2) = C_w(0,T;H_0^1),$$

which gives us $u \in C(0,T;H_0^1)$.

Finally, concerning uniqueness, it is pertinent to recall that $H_0^1 \hookrightarrow L^{2p}$, $f(u) \in L^\infty(0,T;L^2)$, and $w_t \in L^2(0,T;L^2)$. Subsequently, (3.28) undergoes replacement with

$$\frac{1}{2}\left(\left\|(-\Delta)^{-\frac{1}{2}}w_t\right\|^2 + \|w\|^2_{H_0^1}\right) + \alpha\int_0^t \|w_\tau\|^2\,d\tau = \int_0^t \langle f(u) - f(v), w_\tau\rangle\,d\tau?$$

and with the help of the Hölder and Young inequalities, we get

$$\begin{aligned}
\left|\int_0^t \langle f(u) - f(v), w_\tau\rangle\,d\tau\right| &\leq |\beta|\int_0^t \left\|(|u|^{p-1} + |v|^{p-1})\,|w|\right\|\,\|w_\tau\|\,d\tau \\
&\leq |\beta|\int_0^t \left(\|u\|^{p-1}_{L^{2p}} + \|v\|^{p-1}_{L^{2p}}\right)\|w\|_{L^{2p}}\|w_\tau\|\,d\tau \\
&\leq C\int_0^t \|w\|^2_{H_0^1}\,d\tau + \frac{\alpha}{2}\int_0^t \|w_\tau\|^2\,d\tau.
\end{aligned}$$

To extend the interval of existence, it is crucial to observe (3.23). Recognizing that T is uniformly upper bounded, given fixed initial data and coefficients, a standard argument allows us to identify a T_{max} such that conditions (3.9)-(3.10) are satisfied. With this, the proof is concluded. ■

Remark 3.1 *Initially, it is observed that the energy equality stated in Theorem 3.1 can be formally derived from the variational equality (3.8) when applied with test function* $\varphi = u_t$*. Nevertheless, the restricted regularity of weak solutions obstruct the ability to rigorously substantiate this argument. The proof of (3.11) is established through regularization, with specific details omitted here for brevity; interested readers are directed to [[50], Theorem 4.1] or [[63], Lemma 3.1] for further information.*

Theorem 3.2 ***(Global Well-Posedness for*** $\beta < 0$***)*** *[51] Let* p *satisfies (3.5), and let* u *the unique local solution of (3.2)-(3.4) with initial data* $u_0 \in H_0^1$ *and* $(-\Delta)^{-\frac{1}{2}}u_1 \in L^2$*. If* $\beta < 0$*, then* u *is the unique global solution for the probem (3.2)-(3.4), and there exist a positive* $m > 0$ *independent of* t *such that*

$$E(t) \leq \exp(-mt)E(0), \forall t \in [0, \infty). \tag{3.29}$$

Proof. If $\beta < 0$, obviously, $E(t)$ is positive, we immediately get (3.12) from (3.11), then global solutions are obtained.

Due to (3.11), it follows that

$$\frac{d}{dt}E(t) + \|u_t\|_*^2 = 0.$$

Multiplying by $\exp(mt)$ and integrating over $[0, t]$, we find

$$\exp(mt)E(t) + \int_0^t \exp(m\tau) \|u_\tau\|_*^2 \, d\tau = m \int_0^t \exp(m\tau)E(\tau)d\tau + E(0).$$

Noting that $\beta < 0$, it follows that

$$m \int_0^t \exp(m\tau)E(\tau)d\tau \leq \frac{m}{2} \int_0^t \exp(m\tau)(\left\|(-\Delta)^{-\frac{1}{2}}u_\tau\right\|^2 + \|u\|_{H_0^1}^2)d\tau - \frac{m}{2} \int_0^t \exp(m\tau)\beta \|u\|_{L^{p+1}}^{p+1} \, d\tau.$$

By the equation (3.2), we get

$$\begin{aligned} m\int_0^t \exp(m\tau)E(\tau)d\tau \ &\leq \ \frac{m}{2}\int_0^t \exp(m\tau)(\left\|(-\Delta)^{-\frac{1}{2}}u_\tau\right\|^2 + \|u\|_{H_0^1}^2)d\tau \\ &\quad -\frac{m}{2}\int_0^t \exp(m\tau)\int_\Omega((-\Delta)^{-1}u_{\tau\tau} + u - \Delta u + \alpha u_\tau - \gamma\Delta u_\tau)udxd\tau \\ &\leq \ \frac{m}{2}\int_0^t \exp(m\tau)\left\|(-\Delta)^{-\frac{1}{2}}u_\tau\right\|^2 d\tau \qquad (3.30) \\ &\quad -\frac{m}{2}\int_0^t \exp(m\tau)\left\langle(-\Delta)^{-1}u_{\tau\tau}, u\right\rangle_{H^{-1},H_0^1} d\tau - \frac{m}{2}\int_0^t \exp(m\tau)(u_\tau, u)_* d\tau, \end{aligned}$$

using the integration by parts and the Poincare, Cauchy, and Granwall inequalities, we can get our result (3.29). Thus, the proof is completed. ■

When $\beta < 0$, the behavior of the function $E(t)$ shows positivity. The transition from equation (3.11) to equation (3.12) is straightforward under these circumstances. However, when $\beta > 0$, $E(t)$ becomes non-positive, leading to challenges in establishing global a priori estimates based on the conservation law of the total energy for weak solutions u. To address this issue, the potential wells theory is employed as a solution approach.

Therefore, we first introduce a continuous functional under the constraint (3.5). This function may be defined as long as $u \in H_0^1$,

$$J(u) = \frac{1}{2}\|u\|_{H_0^1}^2 - \frac{\beta}{p+1}\|u\|_{L^{p+1}}^{p+1}, \quad \forall u \in H_0^1.$$

Its alteration

$$I(u) = \|u\|_{H_0^1}^2 + \beta\|u\|_{L^{p+1}}^{p+1}.$$

And $I(u)$ is also called the Nehari functional. The stable sets and the instable sets are defined by

$$W := \left\{u \in H_0^1; I(u) > 0, J(u) < d\right\} \cup \{0\}. \qquad (3.31)$$

$$V := \left\{u \in H_0^1; I(u) < 0, J(u) < d\right\}. \qquad (3.32)$$

Where the potential well depth is defined as

$$d = \inf \left\{ \sup_{\lambda \geq 0} J(\lambda u) : u \in H_0^1, u \neq 0 \right\}.$$

Additionally, the depth of the potential well d is also defined by

$$d = \inf_{u \in \aleph} J(u) = \frac{p-1}{2(p+1)} \beta^{-\frac{2}{p-1}} C_*^{-\frac{2(p+1)}{p-1}} > 0.$$

Recall the Nehari manifold

$$\aleph = \left\{ u \in H_0^1; I(u) = 0, u \neq 0 \right\}. \tag{3.33}$$

The optimal Sobolev constant C_* of $H_0^1 \hookrightarrow L^q$, defined by

$$C_* = \sup_{u \in H_0^1, u \neq 0} \frac{\|u\|_{L^q}}{\|u\|_{H_0^1}}.$$

We also defined the sets

$$\begin{aligned} A &= \left\{ \phi \in H_0^1 : \phi \text{ is a stationary solution of (3.2)-(3.4)} \right\}. \\ A_l &= \{ \phi \in A : J(\phi) = l \} \, (l \in \mathbb{R}^+). \end{aligned}$$

Various outcomes have been obtained to characterize the global existence and non-existence of solutions at three distinct energy levels: subcritical initial energy where $E(0) < d$, critical initial energy where $E(0) = d$, and supercritical initial energy where $E(0) > d$.

Theorem 3.3 *[51] (**Global Existence for** $E(0) < d$) Let $\beta > 0$, p satisfies (3.5) and $u = u(x,t)$ be the unique local solution to (3.2)-(3.4) with $u_0 \in H_0^1$, $(-\Delta)^{-\frac{1}{2}} u_1 \in L^2$. Let $E(0) < d$ and assume either $I(u_0) > 0$ or $u_0 = 0$. Then u is a global solution for the problem (3.2)-(3.4) and $u \in W$ for all $t \in [0, \infty)$.*

Theorem 3.4 *[51] (**Global Existence for** $E(0) \le d$) Let $\beta > 0$, p satisfies (3.5) and $u = u(x,t)$ be the unique local solution to (3.2)-(3.4) with $u_0 \in H_0^1$, $(-\Delta)^{-\frac{1}{2}}u_1 \in L^2$. Let $E(0) \le d$ and $\|u\|_{H_0^1}^2 \le \frac{2(p+1)}{p-1}d$. Then u is a global solution for the problem (3.2)-(3.4) and $u \in \overline{W} = W \cup \partial W$, $\forall t \in [0,\infty)$. Moreover, there exists a positive constant ξ independent of t such that*

$$E(t) < C\exp(-\xi t), \quad \forall t \in [0,\infty). \tag{3.34}$$

Remark 3.2 *We note that based on the expression of $E(t)$ and the assumption $I(u_0) > 0$, the aforementioned inequality indicates that*

$$\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + \|u\|_{H_0^1}^2 \le C\exp(-\xi t), \quad \forall t \in [0,\infty).$$

Theorem 3.5 *[51] (**Blow-up for** $E(0) \le d$) Let $\beta > 0$, p satisfy (3.5) and $u = u(x,t)$ be the unique local solution to (3.2)-(3.4) with $u_0 \in H_0^1$, $(-\Delta)^{-\frac{1}{2}}u_1 \in L^2$. Let $E(0) \le d$, $I(u_0) < 0$, and $((-\Delta)^{-\frac{1}{2}}u_0, (-\Delta)^{-\frac{1}{2}}u_1) \ge 0$ when $E(0) = d$. Then the solution of the problem (3.2)-(3.4) blows up in finite time, i.e., $T_{\max} < \infty$,*

$$\lim_{t \to T_{\max}} \left(\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\| + \|u\|_{H_0^1}\right) = \infty.$$

Lemma 3.1 ([28],[29]) *(**Concavity method**) Suppose that $0 < T \le \infty$ and suppose a nonnegative function $F(t) \in C^2(0,T)$ satisfies*

$$F''(t)F(t) - (1+\gamma)(F'(t))^2 \ge 0,$$

for some constant $\gamma > 0$. If $F(0) > 0$, $F'(0) > 0$ then

$$T \le \frac{F(0)}{\gamma F'(0)} < \infty,$$

and $F(t) \to \infty$ as $t \to T$.

Proof. 3.5 Supose $u(t)$ is the unique local solution of the problem (3.2)-(3.4) with $E(0) \leq d$, $I(u_0) < 0$ and $((-\Delta)^{-\frac{1}{2}}u_0, (-\Delta)^{-\frac{1}{2}}u_1) \geq 0$ when $E(0) = d$, we shall prove by cntradiction that $T_{\max} < \infty$, we suppose that $T_{\max} = +\infty$, we set

$$\phi(t) = \left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + \int_0^t \|u_\tau\|_*^2 \, d\tau + (T-t)\|u_0\|_*^2 \ , \quad t \in [0,T].$$

Then, we derive ϕ twice, we get

$$\phi'(t) = 2\left((-\Delta)^{-\frac{1}{2}}u, (-\Delta)^{-\frac{1}{2}}u_t\right) + 2\int_0^t (u_\tau, u)_* d\tau \quad \forall t \in [0,T].$$

$$\begin{aligned}\phi''(t) &= \left\langle(-\Delta)^{-1}u_{tt}, u\right\rangle_{H^{-1},H_0^1} + 2(u_t,u)_* + 2\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 \\ &= -2I(u) + 2\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 .\end{aligned} \tag{3.35}$$

Simplifying (3.35) and by simple calculation with the definition of $I(u)$, it comes

$$\phi''(t) = (p-1)\|u\|_{H_0^1}^2 - 2(p+1)E(0) + (p+3)\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + 2(p+1)\int_0^t \|u_\tau\|_*^2 \, d\tau.$$

Then, we squar $\phi'(t)$, and by Schwarz inequality, we get

$$(\phi'(t))^2 \leq 4\phi(t)(\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + \int_0^t \|u_\tau\|_*^2 \, d\tau), \quad \forall t \in [0,T].$$

so, we have

$$\phi''(t)\phi(t) - (1 + \frac{p-1}{4})(\phi')^2(t) > 0,$$

from

$$\phi''(t) > 2(p+1)(d - E(0)),$$

and

$$\phi'(t) > \phi'(0) + 2(p+1)(d - E(0))t.$$

For sufficiently large T, noting that $\phi'(0) \geq 0$ when $E(0) = d$, we know that there is $t_0 > 0$ such that for any $t \in [t_0, T]$, $\phi'(t) \geq 0$. Consequently, $\phi(t)$ never vanishes on $[t_0, T]$.
So we can find $t_1 \leq T_0 = \frac{\phi(t_0)}{\frac{p-1}{4}\phi'(t_0)} + t_0$ such that

$$\lim_{t \to t_1} \phi(t) = +\infty.$$

Which contradicts the assumption $T_{\max} = \infty$. Thus, the proof is completed. ■

Theorem 3.6 *[51] (**Global Existence for** $E(0) > d$) Let $\beta > 0$, p satisfy (3.5) and $u = u(x,t)$ be the unique local solution to (3.2)-(3.4) with $u_0 \in H_0^1$, $(-\Delta)^{-\frac{1}{2}} u_1 \in L^2$. Assume that $E(0) > 0$, and the following conditions are true:*
i) $2((-\Delta)^{-\frac{1}{2}} u_0, (-\Delta)^{-\frac{1}{2}} u_1) + \frac{c_*(p-1)}{p+3} \widetilde{C}^{-1} \left\| (-\Delta)^{-\frac{1}{2}} u_0 \right\|^2 - \widetilde{C} \left\| u_0 \right\|_*^2 + \frac{2(p+1)}{(p-1)c_*} \widetilde{C} E(0) \leq 0.$
ii) $2((-\Delta)^{-\frac{1}{2}} u_0, (-\Delta)^{-\frac{1}{2}} u_1) + \left\| u_0 \right\|_*^2 < 0.$
iii) $K(u_0) = I(u_0) - \left\| (-\Delta)^{-\frac{1}{2}} u_1 \right\|^2 > 0.$
where $\widetilde{C} = \sqrt{\frac{2(p+1)}{p+3}} - 1$. *Then u is a global solution for the problem (3.2)-(3.4). Moreover, when $E(0) \geq d$ then there exists $l \in \mathbb{R}^+$ such that $A_l \neq \varnothing$,*

$$\lim_{t \to \infty} E(t) = l, \quad \lim_{t \to \infty} dist_{H_0^1}(u(t), A_l) = 0, \quad \lim_{t \to \infty} \left\| (-\Delta)^{-\frac{1}{2}} u_t \right\| = 0, \tag{3.36}$$

and there exist $\{t_j\} \subset \mathbb{R}^+$ with $t_j \to \infty$ and $\phi \in A_l$ such that

$$\lim_{j \to \infty} \left\| u(t_j) - \phi \right\|_{H_0^1} = 0. \tag{3.37}$$

Remark 3.3 *It's important to observe that the condition (iii) implies $I(u_0) > 0$. This observation does not conflict with the assertion in theorem 3.3. Specifically, when $0 < E(0) \leq d$, under the hypothesis (iii), we readily obtain global solutions using theorem 3.3. Therefore, hypotheses (i) and (ii) are necessary only when considering the case $E(0) > d$. Furthermore, in the case where $E(0) < d$, $\lim_{t \to \infty} E(t) = 0$, which is implied by (3.34).*

Theorem 3.7 *[51] (**Blow-up for** $E(0) > d$) Let p satisfy (3.5) and $u = u(x,t)$ be the* *unique local solution to (3.2)-(3.4) with* $u_0 \in H_0^1$, $(-\Delta)^{-\frac{1}{2}} u_1 \in L^2$. *Assume that* $E(0) > 0$, *and*

$$2((-\Delta)^{-\frac{1}{2}} u_0, (-\Delta)^{-\frac{1}{2}} u_1) + \|u_0\|_*^2 - \frac{2(p+1)}{k} E(0) > 0, \tag{3.38}$$

where

$$k = \frac{1}{2c_*}(\sqrt{A^2 + B} - A),$$

and

$$A = (p+3)\lambda_1(1+\lambda_1), \qquad B = 4(p-1)(p+3)\lambda_1(1+\lambda_1)c_*^2,$$

then the solution of the problem (3.2)-(3.4) blows up in finite time.

Remark 3.4 *[51]When* $0 < E(0) < d$, *the hypothesis (3.38) implies* $I(u_0) < 0$. *As a matter of fact, from (3.38), we have*

$$2((-\Delta)^{-\frac{1}{2}} u_0, (-\Delta)^{-\frac{1}{2}} u_1) + \|u_0\|_*^2 - \frac{2(p+3)}{k} E(0) + \frac{4}{k} E(0) > 0. \tag{3.39}$$

Using the fact

$$E(0) = \frac{1}{2} \left\|(-\Delta)^{-\frac{1}{2}} u_1\right\|^2 + \frac{p-1}{2(p+1)} \|u_0\|_{H_0^1}^2 + \frac{1}{p+1} I(u_0), \tag{3.40}$$

(3.39) can be rewritten as

$$2((-\Delta)^{-\frac{1}{2}} u_0, (-\Delta)^{-\frac{1}{2}} u_1) + \|u_0\|_*^2 - \frac{(p+3)}{k} \left\|(-\Delta)^{-\frac{1}{2}} u_1\right\|^2 - \frac{(p-1)(p+3)}{k(p+1)} \|u_0\|_{H_0^1}^2 + \frac{4}{k} E(0) > \frac{2(p+3)}{k(p+1)} I(u_0 \tag{3.41}$$

By means of Cauchy's inequality, we see that

$$2((-\Delta)^{-\frac{1}{2}} u_0, (-\Delta)^{-\frac{1}{2}} u_1) \le \frac{(p+3)}{k} \left\|(-\Delta)^{-\frac{1}{2}} u_1\right\|^2 + \frac{k}{p+3} \left\|(-\Delta)^{-\frac{1}{2}} u_0\right\|^2.$$

Poincare's inequality gives us

$$\left\|(-\Delta)^{-\frac{1}{2}}u_0\right\|^2 \leq \frac{1}{\lambda_1}\|u_0\|^2 \leq \frac{1}{\lambda_1(1+\lambda_1)}\left(\|u_0\|^2 + \|\nabla u_0\|^2\right).$$

Then, using the above two inequalities with $E(0) < d$, we get

$$\begin{aligned} 2((-\Delta)^{-\frac{1}{2}}u_0, (-\Delta)^{-\frac{1}{2}}u_1) + \|u_0\|_*^2 - \tfrac{(p+3)}{k}\left\|(-\Delta)^{-\frac{1}{2}}u_1\right\|^2 - \tfrac{(p-1)(p+3)}{k(p+1)}\|u_0\|_{H_0^1}^2 + \tfrac{4}{k}E(0) \\ \leq \tfrac{k}{p+3}\left\|(-\Delta)^{-\frac{1}{2}}u_0\right\|^2 + \|u_0\|_*^2 - \tfrac{(p-1)(p+3)}{k(p+1)}\|u_0\|_{H_0^1} + \tfrac{4}{k}d \\ \leq (\tfrac{k}{\lambda_1(1+\lambda_1)(p+1)} + \tfrac{1}{c_*} - \tfrac{(p-1)(p+3)}{k(p+1)})\|u_0\|_{H_0^1} + \tfrac{4}{k}d. \end{aligned} \tag{3.42}$$

It follows from (3.40) that

$$\|u_0\|_{H_0^1}^2 \leq \frac{2(p+1)}{p-1}d - \frac{2}{p-1}I(u_0). \tag{3.43}$$

Substituing (3.43) into (3.42), combining the result with (3.41), we can obtain

$$(\frac{k}{\lambda_1(1+\lambda_1)(p+1)} + \frac{1}{c_*})I(u_0) < 0,$$

which means $I(u_0) < 0$. This fact tells us that under the assumption (3.38), we get the result of theorem 3.5 when $0 < E(0) < d$.

3.3 Existence and Nonexistence of Global Solution for Boussinesq Equation With Logarithmic Nonlinearity ($f(u) = u \log |u|$)

In this part, we will talk about the results of the equation (3.1) where $\beta > 0$, $\gamma \geq 0$, $\alpha \begin{cases} \geq 0 & \text{if } \gamma = 0 \\ > -\gamma\lambda_1 & \text{if } \gamma > 0 \end{cases}$ and $f(u) = u \log |u|$, so we lead with the following equation:

$$\begin{cases} u_{tt} - \beta\Delta u_{tt} - \Delta u + \Delta^2 u + \gamma\Delta^2 u_t - \alpha\Delta u_t + \Delta(u \log |u|) = 0, & x \in \Omega, t > 0 \\ u = \Delta u = 0 \quad x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad x \in \Omega \end{cases} . \tag{3.44}$$

Definition 3.2 *[11](**Weak Solution**) Assume that $u_0 \in H_0^1(\Omega)$ and $u_1 \in X$. Let $0 < T \leq \infty$. A function*

$$u \in C([0,T]; H_0^1(\Omega)) \text{ with } u_t \begin{cases} \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; X), & \text{if } \gamma > 0 \\ \in C([0,T]; X), & \text{if } \gamma = 0 \end{cases}, \tag{3.45}$$
$$u_{tt} \begin{cases} \in L^2(0,T; H^{-1}(\Omega)), & \text{if } \gamma > 0 \\ \in L^\infty(0,T; H^{-1}(\Omega)), & \text{if } \gamma = 0 \end{cases},$$

is called a weak solution of problem (3.44) over $[0,T)$, if $u(x,0) = u_0(x)$ in $H_0^1(\Omega)$, $u_t(x,0) = u_1(x)$ in X and for all $t \in (0,T)$, the equality

$$\left\langle (-\Delta)^{-\frac{1}{2}} u_{tt}, (-\Delta)^{-\frac{1}{2}} \phi \right\rangle + \beta \langle u_{tt}, \phi\rangle + (u,\phi) + (\nabla u, \nabla\phi) + \gamma(\nabla u_t, \nabla\phi) + \alpha(u_t,\phi) = (u \log |u|, \phi) \tag{3.46}$$

holds for any $\phi \in H_0^1(\Omega)$, where $\langle ., . \rangle$ denotes the dual product between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Moreover, in the case where $T = \infty$, u is defined as a global weak solution to the problem referenced as (3.44). Conversely, if $T < \infty$, u is referred as a local weak solution to the problem denoted by (3.44).

Remark 3.5 *[11] Let's note that all the terms in (3.46) are well-defined. Specifically,* *according to (3.45), we ascertain that* $u(t) \in H_0^1(\Omega)$, $u_t \in H_0^1(\Omega)$ *when* $\gamma > 0$, $u_t(t) \in L^2(\Omega)$ *when* $\gamma = 0$, *and* $u_{tt} \in H^{-1}(\Omega)$ *for all* $t \in [0,T)$.

Consequently, the terms $\beta \langle u_{tt}, \phi \rangle, (u,\phi), (\nabla u, \nabla\phi), \gamma(\nabla u_t, \nabla \phi)$ *and* $\alpha(u_t,\phi)$ *in (3.46) are well-defined.*

For the term $\left\langle (-\Delta)^{-\frac{1}{2}} u_{tt}, (-\Delta)^{-\frac{1}{2}}\phi \right\rangle$, *it follows that*

$$\left\|(-\Delta)^{-\frac{1}{2}}\phi\right\|_{H_0^1}^2 = \left\|(-\Delta)^{-\frac{1}{2}}\phi\right\|_2^2 + \left\|(-\Delta)^{-\frac{1}{2}}\nabla\phi\right\|_2^2 \leq \frac{1}{\lambda_1}(\|\phi\|_2^2 + \|\nabla\phi\|_2^2) = \frac{1}{\lambda_1}\|\phi\|_{H_0^1}^2, \tag{3.47}$$

which, together with $\phi \in H_0^1(\Omega)$, *implies* $(-\Delta)^{-\frac{1}{2}}\phi \in H_0^1(\Omega)$. *Moreover, because* $(-\Delta)^{-\frac{1}{2}}$ *is a selfadjoint operator and* $u_{tt}(t) \in H^{-1}(\Omega)$ *for all.* $t \in [0,T)$, *then we get from (3.47) that*

$$\left\|(-\Delta)^{-\frac{1}{2}} u_{tt}\right\|_{H^{-1}} = \sup_{\psi \in H_0^1(\Omega), \|\psi\|_{H_0^1(\Omega)}=1} \left\langle u_{tt}(t), (-\Delta)^{-\frac{1}{2}}\psi \right\rangle \leq \frac{1}{\sqrt{\lambda_1}}\|u_{tt}(t)\|_{H^{-1}}, \tag{3.48}$$

which implies $(-\Delta)^{-\frac{1}{2}}u_{tt} \in H^{-1}(\Omega)$ *for all.* $t \in [0,T)$.

For the term $(u\log|u|, \phi)$, *it follows from* $\sup_{|\tau|\leq 1}(\tau\log|\tau|)^2 = \exp(-2)$, *for all* $t \in [0,T)$ *and any* $\epsilon > 0$,

$$\begin{aligned} \|u(t)\log|u(t)|\|_2^2 &\leq \int_{\{x\in\Omega:|u(x,t)|\leq 1\}} (u\log|u|)^2\,dx + \int_{\{x\in\Omega:|u(x,t)|\geq 1\}} (u\log|u|)^2\,dx \\ &\leq \exp(-2)|\Omega| + \frac{1}{\epsilon^2}\int_{\{x\in\Omega:|u(x,t)|\geq 1\}} |u|^{2+2\epsilon}\,dx \leq \exp(-2)|\Omega| + \frac{1}{\epsilon^2}\|u(t)\|_{2+2\epsilon}^{2+2\epsilon}. \end{aligned} \tag{3.49}$$

Obviously, for sufficiently small $\epsilon > 0$, *there holds* $H_0^1(\Omega) \hookrightarrow L^{2+2\epsilon}(\Omega)$, *which, together with (3.49) and* $u(t) \in H_0^1(\Omega)$ *for all* $t \in [0,T)$, *implies* $(u\log|u|, \phi)$ *is well-defined for all* $t \in [0,T)$.

We now introduce key sets and functionals, specifically the potential energy functional J and the Nehari functional I, defined as follows:

$$J(\phi) := \frac{1}{2}\left\|u\right\|_{H_0^1}^2 - \frac{1}{2}\int_\Omega u^2 \log|u|\, dx + \frac{1}{4}\left\|u\right\|_2^2. \tag{3.50}$$

$$I(\phi) := \left\|u\right\|_{H_0^1}^2 - \int_\Omega u^2 \log|u|\, dx. \tag{3.51}$$

It is evident that J and I are well-defined for $u \in H_0^1(\Omega)$. Furthermore, the relationship between J and I is established by the expressions (3.50) and (3.51), demonstrating that

$$J(\phi) = \frac{1}{4}\left\|u\right\|_2^2 + I(u). \tag{3.52}$$

The mountain-pass level d is defined by

$$d := \inf_{u \in \aleph} J(u). \tag{3.53}$$

Where $\aleph$ denotes the Nehari manifold, and we mentioned it in the first part (3.33) with W (3.31) and V (3.32).

Let u be a weak solution of problem (3.44), the Energy function is defined by

$$E(t) := \frac{1}{2}\left\|u_t\right\|_X^2 + \frac{1}{2}\left\|u\right\|_{H_0^1}^2 - \frac{1}{2}\int_\Omega u^2 \log|u|\, dx + \frac{1}{4}\left\|u\right\|_2^2. \tag{3.54}$$

We notice by (3.46) and the last remark that $E(t)$ is well defined.

Lemma 3.2 *Suppose $u \in H_0^1$ and $r := (2\pi)^{\frac{n}{4}} \exp(\frac{n}{2})$ the following statements hold:*

i) *If $0 < \left\|u\right\|_2 < r$ then $I(u) > 0$.*

ii) *If $I(u) < 0$ then $\left\|u\right\|_2 > r$.*

iii) *If $I(u) = 0$, $u \neq 0$, ie $u \in \aleph$, then $u \in \aleph$, then $\left\|u\right\|_2 \geq r$*

Proof. i) Using the logarithmic Sobolev inequality 2.4 for $a > 0$ we get

$$\begin{aligned} I(u) &= \|u\|_{H^1}^2 - \int_{-\infty}^{+\infty} u^2 \ln |u| \, dx \\ &= \|u\|_{H^1}^2 - \int_{-\infty}^{+\infty} u^2 (\ln \frac{|u|}{\|u\|} + \|u\|) dx \\ &\geq \|u\|_{H^1}^2 - \frac{a^2}{2\pi} \|u\|_{H^1}^2 + \frac{n(1+\ln a)}{2} \|u\|^2 - \|u\|^2 \ln \|u\| \\ &\geq (1 - \frac{a^2}{2\pi}) \|u\|_{H^1}^2 + \frac{n(1+\ln a)}{2} \|u\|^2 - \|u\|^2 \ln \|u\| . \end{aligned}$$

We put $a = \sqrt{2\pi}$.

$$\begin{aligned} I(u) &\geq \frac{n(2+\ln 2\pi)}{4} \|u\|_2^2 - \|u\|_2^2 \ln \|u\| \\ &= (\frac{n(2+\ln 2\pi)}{4} - \ln \|u\|) \|u\|^2 . \end{aligned}$$

If $0 < \|u\|_{H^1} < r$, then $\frac{n(2+\ln 2\pi)}{4} > \ln \|u\|$, which gives $I(u) > 0$

ii) We have $I(u) < 0$

$$(\frac{n(2+\ln 2\pi)}{4} - \ln \|u\|) \|u\|^2 < 0,$$

which means

$$\|u\| > (2\pi)^{\frac{n}{4}} \exp(\frac{n}{2}) = r.$$

iii) We have $I(u) = 0$ and $\|u\|_{H^1} \neq 0$, which means

$$\|u\| \geq (2\pi)^{\frac{n}{4}} \exp(\frac{n}{2}) = r.$$

■

Lemma 3.3 *The depth of potentiel d satisfies*

$$d \geq \frac{1}{4}(2\pi)^{\frac{n}{2}} \exp(n).$$

Proof. *From the definition of d and the last lemma, it means that $u \in \aleph$, we find*

$$0 = I(u) \geq (\frac{n(2 + \ln 2\pi)}{4} - \ln \|u\|) \|u\|^2,$$

so we obtain

$$\|u\| \geq (2\pi)^{\frac{n}{4}} \exp(\frac{n}{2}).$$

We know that

$$J(u) = \frac{1}{2} I(u) + \frac{1}{4} \|u\|^2,$$

we have $I(u) = 0$, so

$$\begin{aligned} J(u) &= \frac{1}{4} \|u\|^2 \\ &\geq \frac{1}{4} (2\pi)^{\frac{n}{2}} \exp(n). \end{aligned}$$

■

Lemma 3.4 *Let $u \in H_0^1(\Omega)$ and $\Omega \subset \mathbb{R}^n$ satisfy $I(u) < 0$, then it holds*

$$I(u) < 2(J(u) - d).$$

Theorem 3.8 *[11] (**The local well-posedness solution**) Suppose $u_0 \in H_0^1(\Omega)$ and $u_1 \in X$. Let $M_* > 0$ be a constant. If $\frac{1}{2} \|u_1\|_X^2 + \frac{1}{2} \|u_0\|_{H_0^1}^2 < M_*$, then there exists a constant $T_{M_*} > 0$ depending solely on M_* such that the problem (3.44) admits a weak solution u defined on $[0, T_{M_*}]$, and the existence interval $[0, T_{M_*}]$ can be extended maximally to $[0, T)$. In addition, there hold*

$$\int_0^t \|u_\tau\|_*^2 \, d\tau + E(t) = E(0), \quad 0 \leq t < T. \tag{3.55}$$

Theorem 3.9 *[11]**(The global existence and exponential energy decay estimates)***

Suppose $u_0 \in H_0^1(\Omega)$ *and* $u_1 \in X$. *Let* $u = u(t)$ *be a local weak solution to the problem (3.44).*

i) $E(0) < d$ *and* $I(u_0) > 0$, *then* u *is global and uniformly bounded for all* $t \in [0, \infty)$. *More precisely, we have*

$$\|u_t\|_X^2 + \frac{3 + n(1 + \log \sqrt{\pi})}{2} \|u\|_2^2 + \frac{1}{2} \|\nabla u\|_2^2 + 2 \int_0^t \|u_\tau\|_*^2 \, d\tau \leq 2d(1 + \log(4d), \quad (3.56)$$

for all $t \in [0, \infty)$, *and* $u \in W$.

ii) Let $\beta > 0$, $\gamma \geq 0$, *and* $\alpha > -\gamma\lambda_1$. *If* $I(u_0) > 0$ *and*

$$E(0) < \min \left\{ \frac{\delta^n}{4} \exp(n + 3), \frac{(2\pi)^{\frac{n}{2}}}{4} \exp(2 + n - \frac{M + 4|\alpha| b}{2 - M} \right\} < d, \quad (3.57)$$

for some $\delta \in (0, \sqrt{2\pi})$, $M \in (0, 2)$, $b \in (0, \infty)$, *then for any constant* a *satisfying*

$$0 < a < a_* := \min \left\{ \frac{1}{k}, \frac{4b\lambda_1(\alpha + \gamma\lambda_1)}{2b(M + 2)(\beta\lambda_1 + 1) + \lambda_1 |\alpha|} \right\},$$

there holds

$$E(t) \leq \frac{2a(u_0, u_1)_X + \alpha\gamma \|\nabla u_0\|_2^2 + 2E(0)}{2(1 - ak)} \exp(-\frac{Ma}{1 + ak} t), t \in [0, \infty),$$

where

$$k := \max \left\{ \frac{2(1 + \beta\lambda_1)}{\lambda_1[3 + n(1 + \log \delta) - \log(4E(0))]}, \frac{2\pi\gamma}{2\pi - \gamma^2}, 1 \right\}.$$

Corollary 3.1 *Assuming* $u_0 \in H_0^1(\Omega)$ *and* $u_1 \in X$, *consider* $u = u(t)$ *the local weak solution of the problem (3.44). If* $E(0) \leq d$ *and* $I(u_0) \geq 0$, *then* u *is both globally and uniformly bounded for all* $t \in [0, \infty)$. *In other words, condition (3.56) is satisfied, and* $u \in \overline{W}$ *for all* $t \in [0, \infty)$.

Theorem 3.10 *[11]* ***(Infinite time blow up with subcritical and critical initial energy)*** *Assume that* $u_0 \in H_0^1(\Omega)$ *and* $u_1 \in X$. *Consider* $u = u(t)$ *the local weak solution* *to the problem (3.44). If* $I(u_0) < 0, E(0) < d$ *or* $I(u_0) < 0,\ E(0) = d,\ (u_0, u_1)_X > 0$*, then* u *can be extended over time and blows up at infinite time in the sense of*

$$\begin{cases} \lim_{t\to\infty} \|u(t)\|_*^2 = \infty, & \text{if } \gamma > 0 \\ \lim_{t\to\infty} \|u(t)\|_X^2 = \infty, & \text{if } \gamma = 0 \end{cases}.$$

Theorem 3.11 *[11]* ***(Infinite time blow up with arbitrary high initial energy)*** *Assume that* $u_0 \in H_0^1(\Omega)$ *and* $u_1 \in X$*. consider* $u = u(t)$ *as a local weak solution to the problem (3.44).*

i) *If*

$$\begin{cases} d < E(0) < \frac{\sqrt{2(u_0,u_1)_X + \|u_0\|_*^2}}{2\sqrt{e\hbar}} \\ \qquad I(u_0) < \|u_1\|_X^2 \end{cases}, \tag{3.58}$$

then, u *can be extended over time and blows up at infinite time in the sense of*

$$\begin{cases} \lim_{t\to\infty} \|u(t)\|_*^2 = \infty, & \text{if } \gamma > 0 \\ \lim_{t\to\infty} \|u(t)\|_X^2 = \infty, & \text{if } \gamma = 0 \end{cases},$$

where $\hbar$ *satisfying*

$$\hbar = \max\left\{2, \frac{4(\beta\lambda_1 + 1)}{\lambda_1[3 + n(1 + \log\sqrt{\pi})]}, \frac{4(\gamma\lambda_1 + \alpha)}{\lambda_1}, 4\gamma, \frac{4\gamma^2}{\exp(1)}, \frac{(\rho+1)^2}{\exp(1)}\right\}, \tag{3.59}$$

and ρ *is some constant satisfying*

$$\rho = \max\left\{1, \frac{2|\alpha|\lambda_1 + 4(\beta\lambda_1 + 1)}{\lambda_1[3 + n(1 + \log\sqrt{\pi})]} - 1\right\}. \tag{3.60}$$

ii) *Furthermore, for any constant* $P > d$*, there exist infinitely many functions* $u_0^P \in H_0^1(\Omega)$ *and* $u_1^P \in X$*, which satisfy (3.58) and* $E(0; (u_0^P, u_1^P) = P$*, then,* u *blows up infinite time with initial data* $(u_0, u_1) = (u_0^P, u_1^P)$*.*

For the proofs of this section, we have intentionally omitted the repetition of previous ideas. Instead, we have made some simple changes to demonstrate the same concept. For additional information, refer to [11].

Chapter 4

On the Cauchy Problem for the Generalized Double Dispersion Equation With Logarithmic Nonlinearity

In this chapter, we study the Cauchy problem for the generalized double dispersion equations

$$u_{tt} - u_{xx} - u_{xxtt} + (u_{xx} + f(u))_{xx} - \alpha u_{xxt} + u_{xxxxt} = 0, \ x \in \mathbb{R}, \quad t > 0, \tag{4.1}$$

subject to initial conditions values

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x \in \mathbb{R}, \tag{4.2}$$

where $u = u(x,t)$ is the unknown real valued function of $x \in \mathbb{R}$ and $t > 0$, $f(u) = |u|^{p-2} u \ln |u|$, $(p > 2)$ represents the nonlinearity, $\alpha > 0$ is a constant, and $u_0(x)$ and $u_1(x)$ are given initial data.

4.1 Introduction

Several investigations have been carried out to examine the Cauchy problem associated with generalized Boussinesq equations. Efforts have been made to establish results related to local and global well-posedness, stability, instability, and the potential blow-up of solutions, as shown in various sources, including [8, 14, 25, 35, 61] [23, 67, 57, 62]. Logarithmic nonlinearity frequently arises in partial differential equations (PDEs), particularly in Boussinesq equations. References such as [1, 7, 11, 18, 21, 20, 30, 40, 66] delve into this topic. This type of nonlinearity is prevalent in inflation cosmology and various branches of physics, including geophysics, quantum mechanics, and nuclear physics.

The presence of the nonlinear logarithmic term $f(u) = |u|^{p-2} u \ln |u|$ in the studied problem poses challenges when applying Sobolev's logarithmic inequality. Unlike previous works where it is easily employed to address such problems (see for example, [1, 7, 11, 18, 21, 20, 30, 40, 66] and references therein), another difficulty arises from the absence of Poincaré's inequality in the interval $(-\infty, \infty)$. These difficulties give rise to significant questions regarding the local and global existence of solutions for the problem described in equations (4.1)-(4.2). Motivated by prior research, our primary objective is to initially investigate the existence and uniqueness of local mild solutions to the Cauchy problem (4.1)-(4.2) within the energy space $H^1 \times H^{-1}$ with the initial data $(u_0; u_1) \in H^1 \times H^{-1}$. We employ the contraction mapping principle, drawing on ideas from [62]. Using the potential well method inspired by [47] and the concavity method [28, 29], we investigate the global existence and non-existence of mild solutions with initial energy levels $E(0) < d$ and $E(0) = d$. Furthermore, the investigation into the sufficiency and necessary conditions for the blowup of solutions presents an intriguing aspect. Notably, to the best of our knowledge, there are no existing results on the global existence of a solution to the Cauchy problem for a one-dimensional Boussinesq-type equation of the fifth order.

4.2 Mild solution

Before stating our main results of this work, we first introduce an integral equation representation which serves for the definition of a mild solution of the problem (4.1)-(4.2). To do that, we shall introduce the solution operator of the following equation

$$\begin{aligned} u_{tt} - u_{xx} - Lu_{xxt} &= (I - \partial_{xx})^{-1} g_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \\ t = 0: \quad u(0) = u_0(x), & \qquad u_t(0) = u_1(x), \ x \in \mathbb{R} \end{aligned} \tag{4.3}$$

where $L := (I - \partial_{xx})^{-1} (\alpha I - \partial_{xx})$. Taking the Fourier transform with respect to x, 4.3 is reduced as the following ordinary differential equation with parameter

$$\hat{u}_{tt} + |\xi|^2 \hat{u} + \hat{L}(\xi) |\xi|^2 \hat{u}_t = \left(1 + |\xi|^2\right)^{-1} |\xi|^2 \hat{g} \tag{4.4}$$

with initial values given by

$$t = 0: \quad \hat{u} = \hat{u}_0(\xi), \qquad \hat{u}_t = \hat{u}_1(\xi). \tag{4.5}$$

where $\hat{L}(\xi) := \frac{\alpha + |\xi|^2}{1 + |\xi|^2}$. The corresponding characteristic equation is

$$\lambda^2 + \hat{L}(\xi) |\xi|^2 \lambda + |\xi|^2 = 0. \tag{4.6}$$

Let $\lambda = \lambda_{\pm}(\xi)$ be the corresponding roots, i.e

$$\lambda_{\pm}(\xi) = \frac{-\left(\alpha + |\xi|^2\right) |\xi|^2 \pm |\xi| \sqrt{\left(\alpha + |\xi|^2\right)^2 |\xi|^2 - 4\left(1 + |\xi|^2\right)^2}}{2\left(1 + |\xi|^2\right)}. \tag{4.7}$$

The solution of the problem (4.4)-(4.5) with $g = 0$ is given by

$$\hat{u}(\xi, t) = \hat{G}(\xi, t)\hat{u}_1(\xi) + \hat{H}(\xi, t)\hat{u}_0(\xi), \tag{4.8}$$

where

$$\hat{G}(\xi,t)=\frac{1}{\lambda_+(\xi)-\lambda_-(\xi)}(e^{\lambda_+(\xi)t}-e^{\lambda_-(\xi)t}),\tag{4.9}$$

and

$$\hat{H}(\xi,t)=\frac{1}{\lambda_+(\xi)-\lambda_-(\xi)}(\lambda_+(\xi)e^{\lambda_-(\xi)t}-\lambda_-(\xi)e^{\lambda_+(\xi)t}).\tag{4.10}$$

We set $h(r):=(\alpha+r^2)\,r-2\,(1+r^2)\,,\ r\in[0,+\infty)\,.$ So, we need to study the sign of the function h. We claim that $h(r)$ possesses a unique positive zero $r_0>0$. In fact

$$h'(r)=3r^2-4r+\alpha,r\geq 0.$$

Clearly, if $4-3\alpha<0$, then $h'>0$, so h increasing on $[0,+\infty)$. When $0<\alpha<\frac{4}{3}$, the equation $h'\,(r)=0$ has two solutions $r_1=\frac{2-\sqrt{4-3\alpha}}{3},r_2=\frac{2+\sqrt{4-3\alpha}}{3}$. We note that $h(r)<0$ for $\alpha<1$ and $r<2$.

The sign of h' : we have $h'\geq 0$ for $r\in[0,r_1]\cup[r_2,+\infty)$ and $h'<0$ in $(r_1,r_2)\,.$ Then h is increasing on $[0,r_1]\cup[r_2,+\infty)$ and decreasing on $(r_1,r_2)\,.$ Observe that $h(r)<0$ for $\alpha\leq 1$ and $r<2$, while $h(r)>0$ for $\alpha>1$ and $r\geq 2$. In particular $h(r)<0$ for $r<r_2<2$. So by continuity r_0 is unique.

Now it remains the case $1<\alpha\leq\frac{4}{3}$ and $r<2$. We have

$$\begin{aligned}h\,(r_1)&=\left(\alpha+r_1^2\right)r_1-2\left(1+r_1^2\right)\\&=\left(\alpha+\left(\frac{2-\sqrt{4-3\alpha}}{3}\right)^2\right)\left(\frac{2-\sqrt{4-3\alpha}}{3}\right)-2\left(1+\left(\frac{2-\sqrt{4-3\alpha}}{3}\right)^2\right)<0.\end{aligned}$$

Then by monotonicity, we have $-2\leq h\,(r)\leq h\,(r_1)<0$ for any $0\leq r\leq r_1$ and $h\,(r_2)\leq h\,(r)\leq h\,(r_1)<0$ on $(r_1,r_2)\,.$ By a straight calculation, we have

$$h(0)=-2\text{ and }\lim_{r\to+\infty}h\,(r)=+\infty.$$

So by continuity, there exists a unique positive number r_0 such that $h(r_0)=0$.

By the Duhamel principle, we obtain the solution formula to (4.1)-(4.2)

$$u(t) = G(t) * u_1 + H(t) * u_0 + \int_0^t G(t-\tau) * (I - \partial_{xx})^{-1} \partial_{xx} f(u(\tau)) d\tau, \tag{4.11}$$

where

$$G(x,t) = \mathcal{F}^{-1}[\hat{G}(\xi,t)](x), \tag{4.12}$$

and

$$H(x,t) = \mathcal{F}^{-1}[\hat{H}(\xi,t)](x), \tag{4.13}$$

with symbols

$$\hat{G}(\xi,t) = \begin{cases} \frac{2\left(1+|\xi|^2\right)}{|\xi|\sqrt{w(\xi)}} \exp\left(-\frac{\left(\alpha+|\xi|^2\right)|\xi|^2}{2\left(1+|\xi|^2\right)}\right) t \sinh \frac{|\xi|\sqrt{w(\xi)}}{2\left(1+|\xi|^2\right)} t, |\xi| \geq r_0, \\ \frac{2\left(1+|\xi|^2\right)}{|\xi|\sqrt{|w(\xi)|}} \exp\left(-\frac{\left(\alpha+|\xi|^2\right)|\xi|^2}{2\left(1+|\xi|^2\right)}\right) t \sin \frac{|\xi|\sqrt{|w(\xi)|}}{2\left(1+|\xi|^2\right)} t, |\xi| \leq r_0, \end{cases}, \tag{4.14}$$

and

$$\hat{H}(\xi,t) = \begin{cases} \exp\left(-\frac{\left(\alpha+|\xi|^2\right)|\xi|^2}{2\left(1+|\xi|^2\right)}\right) t \left[\cosh \frac{|\xi|\sqrt{w(\xi)}}{2\left(1+|\xi|^2\right)} t + \frac{\left(\alpha+|\xi|^2\right)|\xi|^2}{|\xi|\sqrt{w(\xi)}} \sinh \frac{|\xi|\sqrt{w(\xi)}}{2\left(1+|\xi|^2\right)} t\right], |\xi| \geq r_0 \\ \exp\left(-\frac{\left(\alpha+|\xi|^2\right)|\xi|^2}{2\left(1+|\xi|^2\right)}\right) t \left[\cos \frac{|\xi|\sqrt{|w(\xi)|}}{2\left(1+|\xi|^2\right)} t + \frac{\left(\alpha+|\xi|^2\right)|\xi|^2}{|\xi|\sqrt{|w(\xi)|}} \sin \frac{|\xi|\sqrt{|w(\xi)|}}{2\left(1+|\xi|^2\right)} t\right], |\xi| < r_0. \end{cases}, \tag{4.15}$$

and where

$$w(\xi) = \left(\alpha + |\xi|^2\right)^2 |\xi|^2 - 4\left(1 + |\xi|^2\right)^2.$$

We give now the definition of mild solution to the problem (4.1)-(4.2).

Definition 4.1 *A function $u \in C([0,T]; H^1)$ with $\Lambda^{-1} u_t \in C([0,T]; H^1)$ is called a mild solution to the problem (4.1)-(4.2) over $[0,T]$ if and only if $u(0) = u_0 \in H^1$, $u_t(0) = u_1 \in H^{-1}$ and for all $0 < t \leq T$ the integral equation (4.11) holds. Moreover, if*

$$T_{\max} = \sup\{T > 0 : u = u(x,t) \text{ exists on } [0,T]\} < \infty,$$

then $u(x,t)$ is called the local mild solution of the problem (4.1)-(4.2). If $T_{\max} = \infty$, then $u(x,t)$ is called the global mild solution of the problem (4.1)-(4.2).

We next turn to the derivation of the energy identity. By applying to both sides of (4.1) the operator $A^{-2} = (-\partial_x^2)^{-1}$ we have an equivalent problem

$$\begin{cases} A^{-2}u_{tt} + u + u_{tt} - (u_{xx} + f(u)) + \alpha u_t - u_{xxt} = 0, x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \mathbb{R}, \end{cases} . \tag{4.16}$$

Multiplying the equation (4.16) by u_t and integrating over $\mathbb{R}$ we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|A^{-1}u_t\right\|^2 + \frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}\|u_x\|^2 - \int_{-\infty}^{+\infty} f(u)u_t dx + \alpha\|u_t\|^2 + \|u_{xt}\|^2 = 0. \tag{4.17}$$

From definition of $f(u)$, we have

$$\begin{aligned} -\int_{-\infty}^{+\infty} f(u)u_t dx &= -\int_{-\infty}^{+\infty} |u|^{p-2} u u_t \ln|u|\, dx \\ &= -\frac{1}{p}\int_{-\infty}^{+\infty} \frac{d}{dt}\left(|u|^p\right)\ln|u|\, dx \\ &= -\frac{1}{p}\frac{d}{dt}\int_{-\infty}^{+\infty} |u|^p \ln|u|\, dx + \frac{1}{p}\int_{-\infty}^{+\infty} |u|^{p-2} u u_t dx \\ &= -\frac{1}{p}\frac{d}{dt}\int_{-\infty}^{+\infty} |u|^p \ln|u|\, dx + \frac{1}{p^2}\frac{d}{dt}\int_{-\infty}^{+\infty} |u|^p\, dx. \end{aligned} \tag{4.18}$$

Using (4.18) into (4.17) we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\Lambda^{-1}u_t\right\|^2 + \|u_t\|^2 + \|u\|_{H^1}^2 - \frac{2}{p}\int_{-\infty}^{+\infty} |u|^p \ln|u|\, dx + \frac{2}{p^2}\|u\|_p^p\right) + \alpha\|u_t\|^2 + \|u_{xt}\|^2 = 0. \tag{4.19}$$

We define as the energy of the problem (4.1)-(4.2) the quantity

$$E(t) =: \frac{1}{2}\left\|\Lambda^{-1}u_t\right\|^2 + \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{p}\int_{-\infty}^{+\infty} |u|^p \ln|u|\, dx + \frac{1}{p^2}\|u\|_p^p.$$

So equation (4.19) becomes

$$\frac{d}{dt}E(t) + \alpha \|u_t\|^2 + \|u_{xt}\|^2 = 0. \tag{4.20}$$

Consequently

$$E(t) + \alpha \int_0^t \|u_\tau\|^2 \, d\tau + \int_0^t \|u_{x\tau}\|^2 \, d\tau = E(0). \tag{4.21}$$

Moreover, we introduce some notations to be used in this chapter: the potential functional J defined by

$$J(u) = \frac{1}{2} \|u\|_{H^1}^2 - \frac{1}{p} \int_{-\infty}^{+\infty} |u|^p \ln |u| \, dx + \frac{1}{p^2} \|u\|_p^p ,$$

the Nehari functional I

$$I(u) := \|u\|_{H^1}^2 - \int_{-\infty}^{+\infty} |u|^p \ln |u| \, dx, \tag{4.22}$$

and the depth of potential well

$$d = \inf_{u \in \mathcal{N}} J(u),$$

where $\mathcal{N} = \{u \in H^1(\mathbb{R}) / I(u) = 0, u \neq 0\}$. Notice that

$$J(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|_{H^1}^2 + \frac{1}{p} I(u) + \frac{1}{p^2} \|u\|_p^p ,$$

and

$$E(t) = \frac{1}{2} \left\|\Lambda^{-1} u_t\right\|^2 + \frac{1}{2} \|u_t\|^2 + J(u).$$

We also define stable and unstable sets

$$\begin{aligned} W &= \left\{u \in H^1,\ I(u) > 0,\ J(u) < d\right\} \cup \{0\} , \\ V &= \left\{u \in H^1,\ I(u) < 0,\ J(u) < d\right\} . \end{aligned}$$

In addition, we define

$$\begin{aligned} W' &= \left\{u \in H^1, I(u) > 0\right\} \cup \{0\} \\ V' &= \left\{u \in H^1, I(u) < 0\right\}. \end{aligned}$$

we give some lemmas, which will be needed in our proofs of the main results.

Lemma 4.1 *Suppose $u \in H^1$, the following statements hold:*

i) If $0 < \|u\|_{H^1} < r_*$ then $I(u) > 0$

ii) If $I(u) < 0$ then $\|u\|_{H^1} > r_*$

iii) If $I(u) = 0, u \neq 0$, i.e., $u \in \mathcal{N}$, then $\|u\|_{H^1} \geq r_*$,

where $r_* = \left(\frac{1}{C_*^{p+1}}\right)^{\frac{1}{p}}$ and $C_* = \sup\limits_{u \in H^1 \setminus \{0\}} \frac{\|u\|_{p+1}}{\|u\|_{H^1}}$.

Proof.

i) Assume $0 < \|u\|_{H^1} < r_*$, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} |u|^p \ln|u| dx &< \|u\|_{p+1}^{p+1} \leq C_*^{p+1} \|u\|_{H^1}^{p+1} \\ &= C_*^{p+1} \|u\|_{H^1}^{p-1} \|u\|_{H^1}^2 < \|u\|_{H^1}^2, \end{aligned}$$

where we have used the inequality $\ln y < y$, for $y > 0$ and the Sobolev embedding theorem. So $I(u) > 0$.

ii) From $I(u) < 0$, and the Sobolev embedding theorem, we have

$$\|u\|_{H^1}^2 < \int_{-\infty}^{+\infty} |u|^p \ln|u| dx < C_*^{p+1} \|u\|_{H^1}^{p-1} \|u\|_{H^1}^2,$$

from which it follows that $\|u\|_{H^1} > r_*$.

iii) Note that $I(u) = 0$ and $u \neq 0$, give

$$\|u\|_{H^1}^2 = \int_{-\infty}^{+\infty} |u|^p \ln |u| dx < C_*^{p+1} \|u\|_{H^1}^p \|u\|_{H^1}^2 ,$$

since we have $\|u\|_{H^1} \neq 0$, it yields $\|u\|_{H^1} > r_*$.

■

Lemma 4.2 *The depth of potential d satisfies*

$$d \geq \left(\frac{1}{2} - \frac{1}{p}\right) r_*^2 = \left(\frac{1}{2} - \frac{1}{p}\right) \left(\frac{1}{C_*^{p+1}}\right)^{\frac{2}{p}} .$$

Proof. From definition of d and the last lemma, it follows that $u \in \mathcal{N}$, we find

$$J(u) \geq \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|_{H^1}^2 ,$$

so we obtain

$$d \geq \left(\frac{1}{2} - \frac{1}{p}\right) \left(\frac{1}{C_*^{p+1}}\right)^{\frac{2}{p}} .$$

■

The following lemma is a consequence of (4.20)

Lemma 4.3 *If u is a solution of problem (4.1)-(4.2), then the energy $E(t)$ is a non-increasing function with respect to t.*

Lemma 4.4 *Let $u \in H^1(\mathbb{R})$ satisfying $I(u) < 0$. Then there exists a $\lambda^* \in (0,1)$ such that $I(\lambda^* u) = 0$.*

Proof. Let us define the function

$$\phi(\lambda) := \lambda^{p-2} \int_\Omega |u|^p \log |\lambda u| \, dx, \quad \lambda \in (0, \infty).$$

We have for any $\lambda > 0$

$$\begin{aligned} I(\lambda u) &= \lambda^2 \|u\|_{H^1}^2 - \int_\Omega |\lambda u|^p \log |\lambda u| \, dx \\ &= \lambda^2 (\|u\|_{H^1}^2 - \lambda^{p-2} \int_\Omega |u|^p \log |\lambda u| \, dx) \\ &= \lambda^2 (\|u\|_{H^1}^2 - \phi(\lambda)). \end{aligned} \tag{4.23}$$

Since $I(u) < 0$, by (4.23) and Lemma 4.2 we have

$$\phi(1) > \|u\|_{H^1}^2 \geq r_*^2. \tag{4.24}$$

On the other hand, since $p \geq 2$, we have (as $\lambda \to 0^+$)

$$\begin{aligned} \phi(\lambda) &= \lambda^{p-2} \int_\Omega |u|^p \log |\lambda u| \, dx \\ &= \lambda^{p-2} \log \lambda \, \|u\|_p^p + \lambda^{p-2} \int_{-\infty}^{+\infty} |u|^p \log |u| \, dx \to \begin{cases} -\infty \text{ if } p = 2 \\ 0 \text{ if } p > 2 \end{cases}, \end{aligned}$$

which together with (4.24), implies that there exists a $\lambda^* \in (0,1)$ such that $\phi(\lambda^*) = \|u\|_{H^1}^2$ and $I(\lambda^* u) = 0$. ■

Lemma 4.5 *Let $u \in H^1(\mathbb{R})$ satisfying $I(u) < 0$. Then*

$$I(u) < p(J(u) - d). \tag{4.25}$$

Proof. First from the last lemma there exists a $\lambda^* \in (0,1)$ such that $I(\lambda^* u) = 0$. We consider the function

$$g(\lambda) := pJ(\lambda u) - I(\lambda u), \; \lambda > 0.$$

An easy computation shows that

$$g(\lambda) = \frac{p-2}{2}\lambda^2 \|u\|_{H^1}^2 + \frac{\lambda^p}{p} \|u\|_p^p .$$

Then we obtain

$$\begin{aligned} g'(\lambda) &= (p-2)\lambda \|u\|_{H^1}^2 + \lambda^{p-1} \|u\|_p^p \\ &> (p-2)\lambda \|u\|_{H^1}^2 \\ &> (p-2)\lambda r_0^2 \geq 0, \end{aligned}$$

which, together with $p > 2$, implies that $g(\lambda)$ is strictly increasing for $\lambda > 0$. Hence by $\lambda^* \in (0,1)$ we have $g(1) > g(\lambda^*)$ namely

$$\begin{aligned} pJ(u) - I(u) &> pJ(\lambda^* u) - I(\lambda^* u) \\ &= pJ(\lambda^* u) \geq pd, \end{aligned}$$

where in the last inequality we have used that $\lambda^* u \in \mathcal{N}$ and $d = \inf_{\phi \in \mathcal{N}} J(\phi)$ which immediately gives (4.25). ■

4.3 Linear estimates

We shall investigate some properties of the solution operators $G(t)$ and $H(t)$ in the Sobolev space. The estimates of the solution operators are crucial to establish the local well-posedness

Lemma 4.6 *The operator solutions $G(t)$ and $H(t)$ be given in* (4.9)–(4.10), *fulfil the* *following differential equations*

$$\partial_t \hat{H}(t) = -|\xi|^2 \hat{G}(t), \tag{4.26}$$

$$\partial_t \hat{G} = \hat{H} - \frac{\left(\alpha + |\xi|^2\right)|\xi|^2}{\left(1 + |\xi|^2\right)} \hat{G}, \tag{4.27}$$

$$\partial_{tt} \hat{G} = -|\xi|^2 \hat{G}(t) - \frac{\left(\alpha + |\xi|^2\right)|\xi|^2}{\left(1 + |\xi|^2\right)} \hat{G}_t. \tag{4.28}$$

Proof. The proof follows the plan of that of Lemma 3.1 in [62] and we highlight here the main steps. Equation (4.26) follows from (4.9) and (4.10). Equation (4.27) derived by combining the expression derivative of $\hat{G}$ with respect to t

$$\partial_t \hat{G} = 2 \exp \frac{-\left(\alpha + |\xi|^2\right)|\xi|^2}{2\left(1 + |\xi|^2\right)} t \cosh \frac{|\xi| \sqrt{w(\xi)}}{2\left(1 + |\xi|^2\right)} t - \hat{H},$$

with the expressions (4.14) and (4.15). The last statement of the lemma is a consequence of (4.26) and (4.27). ∎

Based on the above said preparations, we are able to prove estimates for solution operators

Lemma 4.7 *For the operators $G(t)$ and $H(t)$ defined in* (4.12)-(4.13), *the following estimates hold*

$$\|G(t)\varphi\|_{H^1} + \left\|\Lambda^{-1}\partial_t G(t)\varphi\right\| \leq C(t+1)\left\|\Lambda^{-1}\varphi\right\|, \tag{4.29}$$

$$\|H(t)\varphi\|_{H^s} + \left\|\Lambda^{-k}\partial_t H(t)\varphi\right\|_{H^s} \leq C(t+1)\|\varphi\|_{H^s} \ \textit{for } k = 0, 1, \tag{4.30}$$

$$\left\|\Lambda^{-k}\left(I - \partial_{xx}\right)^{-1}\partial_{xx} G(t)\varphi\right\|_{H^s} \leq C(t+1)\|\varphi\|_{\|\varphi\|_{H^{s-2-k}}} \quad \textit{for } k = 0, 1, \tag{4.31}$$

$$\|\partial_t G(t)\varphi\|_{L^2(0,t;H^s)} \leq C(t+1)^{\frac{3}{2}}\left\|\Lambda^{-1}\varphi\right\|_{H^s}, \tag{4.32}$$

for all $t \geq 0$ and $s \in \mathbb{R}$.

Proof. The argument is similar to the one for Lemma 3.2 in [62]. To derive estimates for the solution operators we need to estimate the operators on each separately region lower and higher frequencies.

For $|\xi| \leq 2r_0$, observing that $\frac{\sinh z}{z}$ is bounded on any finite interval. Recalling (4.14), we obtain

$$|G(t,\xi)| \leq Ct \exp\left(-\frac{\left(\alpha+|\xi|^2\right)|\xi|^2}{2\left(1+|\xi|^2\right)}\right) t \leq Ct. \tag{4.33}$$

For $|\xi| > 2r_0$, from (4.9), $G(t,\xi)$ can be rewritten as

$$G(t,\xi) = \frac{1+|\xi|^2}{|\xi|\sqrt{w(\xi)}} e^{\lambda_+(\xi)t}\left(1-e^{-\frac{|\xi|\sqrt{w(\xi)}}{1+|\xi|^2}t}\right),$$

where

$$\lambda_+(\xi) = -\frac{\left(\alpha+|\xi|^2\right)|\xi|^2 - |\xi|\sqrt{w(\xi)}}{2\left(1+|\xi|^2\right)} < 0.$$

Consequently

$$|G(t,\xi)| \leq \frac{\left(1+|\xi|^2\right)e^{\lambda_+(\xi)t}}{|\xi|\sqrt{w(\xi)}} \leq \frac{1+|\xi|^2}{|\xi|\sqrt{w(\xi)}}. \tag{4.34}$$

From estimates (4.33) and (4.34), we infer for any $s \in \mathbb{R}$

$$\begin{aligned} \|G(t)\varphi\|_{H^s}^2 &\leq \int_{|\xi|\leq 2r_0}\left(1+|\xi|^2\right)^s |G(t,\xi)|^2|\hat{\varphi}|^2 d\xi + \int_{|\xi|>2r_0}\left(1+|\xi|^2\right)^s|G(t,\xi)|^2|\hat{\varphi}|^2 d\xi \\ &\leq Ct^2\int_{|\xi|\leq 2r_0}\left(1+|\xi|^2\right)^s|\xi|^2\left||\xi|^{-1}\hat{\varphi}\right|^2 d\xi + \int_{|\xi|>2r_0}\frac{\left(1+|\xi|^2\right)^{s+2}}{w(\xi)}\left||\xi|^{-1}\hat{\varphi}\right|^2 d\xi \\ &\leq C(t+1)^2\left\|\Lambda^{-1}\varphi\right\|_{H^{s-1}}^2. \end{aligned} \tag{4.35}$$

We note from (4.27) that the inequality (4.29) can be derived by giving the estimate of $H(\xi,t)$.

Observing the expression (4.1), for $|\xi| \leq r_0$, it is easily to see

$$\left|\hat{H}(\xi,t)\right| \leq C(t+1).$$

For $|\xi| > r_0$, $\hat{H}(\xi,t)$ can be rewritten as

$$\hat{H}(\xi,t) = e^{\lambda_+(\xi)t}\left(1+\frac{\left(\alpha+|\xi|^2\right)|\xi|^2-|\xi|\sqrt{w(\xi)}}{2|\xi|\sqrt{w(\xi)}}\left(1-e^{-\frac{|\xi|\sqrt{w(\xi)}}{1+|\xi|^2}t}\right)\right),$$

where $\lambda_+(\xi)$ is given by (4.7). From the inequality

$$1 - e^{-z} \leq z, \forall z \geq 0,$$

and using the fact

$$\begin{aligned} \frac{\left(\left(\alpha+|\xi|^2\right)|\xi| - \sqrt{w(\xi)}\right)}{2\sqrt{w(\xi)}} \frac{|\xi|\sqrt{w(\xi)}}{1+|\xi|^2} &= \frac{|\xi|\left(\left(\alpha+|\xi|^2\right)|\xi| - \sqrt{w(\xi)}\right)}{2\left(1+|\xi|^2\right)\left(\left(\alpha+|\xi|^2\right)|\xi| + \sqrt{w(\xi)}\right)} \\ &= \frac{2|\xi|\left(1+|\xi|^2\right)^2}{\left(1+|\xi|^2\right)\left(\left(\alpha+|\xi|^2\right)|\xi| + \sqrt{w(\xi)}\right)} \leq C_\alpha, \end{aligned}$$

we deduce

$$\left|\hat{H}(\xi,t)\right| \leq C(t+1).$$

Consequently,

$$\begin{aligned} \|H(t)\varphi\|_{H^s}^2 &\leq \int_{|\xi|\leq r_0} \left(1+|\xi|^2\right)^s \left|\hat{H}(t,\xi)\right|^2 |\hat{\varphi}|^2 \, d\xi + \int_{|\xi|>r_0} \left(1+|\xi|^2\right)^s \left|\hat{H}(t,\xi)\right|^2 |\hat{\varphi}|^2 \, d\xi \\ &\leq C(t+1)^2 \|\varphi\|_{H^s}^2. \end{aligned} \tag{4.36}$$

In view of (4.26)-(4.33), it follows that

$$\left\|\Lambda^{-k}\partial_t H(t)\varphi\right\|_{H^s} = \left\|\left(1+|\xi|^2\right)^{\frac{s}{2}} |\xi|^{2-k} \hat{G}\hat{\varphi}\right\| \leq C(t+1)\|\varphi\|_{H^s}.$$

Thus the estimate (4.30) is obtained.

We now move to complete the estimate (4.30). Taking $s = 1$ in (4.35) and $s = 0$ in (4.36), and using (4.30), we conclude that

$$\begin{aligned} \left\|\Lambda^{-1}\partial_t G(t)\varphi\right\| &= \left\|F^{-1}\left|\xi\right|^{-1}\hat{G}_t(\xi,t)\hat{\varphi}\right\| = \left\|\left|\xi\right|^{-1}\hat{G}_t(\xi,t)\hat{\varphi}\right\| \\ &\leq \left\|\left|\xi\right|^{-1}\hat{H}\hat{\varphi}\right\| + \left\|\left(\frac{\left(\alpha+\left|\xi\right|^2\right)\left|\xi\right|}{\left(1+\left|\xi\right|^2\right)}\hat{G}\right)\hat{\varphi}\right\| \\ &\leq C\left(t+1\right)\left\|\Lambda^{-1}\varphi\right\|. \end{aligned}$$

The last estimate, combined with the estimate (4.35) with $s = 1$, leads to the result (4.29). Estimate (4.31) can be easily obtained by considering the estimates of $G(\xi, t)$, in fact, we have

$$\begin{aligned} \left\|\Lambda^{-k}\left(I-\partial_{xx}\right)^{-1}\partial_{xx}G(t)\varphi\right\|_{H^s}^2 &\leq Ct^2\int_{|\xi|\leq 2r_0}\left(1+\left|\xi\right|^2\right)^s\left|\xi\right|^{-2k}\frac{\left|\xi\right|^4}{\left(1+\left|\xi\right|^2\right)^2}\left|\hat{\varphi}\right|^2 d\xi \\ &\quad + \int_{|\xi|>2r_0}\frac{\left(1+\left|\xi\right|^2\right)^s}{w(\xi)}\left|\xi\right|^{-2k}\frac{\left|\xi\right|^4}{\left(1+\left|\xi\right|^2\right)^2}\left|\hat{\varphi}\right|^2 d\xi \\ &\leq C\left(t+1\right)^2\left\|\varphi\right\|_{H^{s-2-k}}^2. \end{aligned}$$

Let us now turn to prove the the last statement of the lemma.

For $|\xi| \leq 2r_0$, we have

$$\partial_t\hat{G}(\xi,t) = \exp\left(-\frac{\left(\alpha+\left|\xi\right|^2\right)\left|\xi\right|^2}{2\left(1+\left|\xi\right|^2\right)}\right)t\left[\cos\frac{\left|\xi\right|\sqrt{\left|w\left(\xi\right)\right|}}{2\left(1+\left|\xi\right|^2\right)}t - \frac{\left(\alpha+\left|\xi\right|^2\right)\left|\xi\right|}{\sqrt{\left|w\left(\xi\right)\right|}}\sin\frac{\left|\xi\right|\sqrt{\left|w\left(\xi\right)\right|}}{2\left(1+\left|\xi\right|^2\right)}t\right]. \tag{4.37}$$

It's not hard to see that

$$\begin{aligned}
&\int_0^t \int_{|\xi|\le 2r_0} \left(1+|\xi|^2\right)^s \left|\partial_t \hat{G}(\xi,\tau)\right|^2 |\hat{\varphi}|^2 \, d\xi d\tau \\
\le\ & C(t+1)^2 \int_{|\xi|\le 2r_0} \left(1+|\xi|^2\right)^s \left| \frac{\left(\alpha+|\xi|^2\right)|\xi|^2}{|\xi|\sqrt{|w(\xi)|}} \sin \frac{|\xi|\sqrt{|w(\xi)|}}{2\left(1+|\xi|^2\right)} t \right|^2 |\hat{\varphi}|^2 \, d\xi \\
\le\ & C(t+1)^2 \int_{|\xi|\le 2r_0} \left(1+|\xi|^2\right)^s |\xi|^4 |\hat{\varphi}|^2 \, d\xi \\
\le\ & C(t+1)^2 \left\|\Lambda^{-1}\varphi\right\|_{H^s}^2 . \qquad (4.38)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\partial_t \hat{G}(\xi,t) &= \left(\frac{-\left(\alpha+|\xi|^2\right)|\xi|^2 + |\xi|\sqrt{w(\xi)}}{2|\xi|\sqrt{w(\xi)}} \right) e^{\lambda_+(\xi)t} + \frac{\left(\alpha+|\xi|^2\right)|\xi|^2 + |\xi|\sqrt{w(\xi)}}{2|\xi|\sqrt{w(\xi)}} e^{\lambda_-(\xi)t} \\
&= \frac{1}{2}\left(1 - \frac{\left(\alpha+|\xi|^2\right)|\xi|}{\sqrt{w(\xi)}}\right) e^{\lambda_+(\xi)t} + \frac{1}{2}\left(1 + \frac{\left(\alpha+|\xi|^2\right)|\xi|}{\sqrt{w(\xi)}}\right) e^{\lambda_-(\xi)t}.
\end{aligned}$$

$$\lambda_\pm(\xi) = \frac{-\left(\alpha+|\xi|^2\right)|\xi|^2 \pm |\xi|\sqrt{w(\xi)}}{2\left(1+|\xi|^2\right)}$$

Then, we have

$$\begin{aligned}
&\int_0^t \int_{|\xi|\ge 2r_0} \left(1+|\xi|^2\right)^s \left|\partial_t \hat{G}(\xi,\tau)\right|^2 |\hat{\varphi}|^2 \, d\xi d\tau \\
\le\ & \frac{1}{4}\int_0^t \int_{|\xi|\ge 2r_0} \left(1+|\xi|^2\right)^s \left(1 - \frac{\left(\alpha+|\xi|^2\right)|\xi|}{\sqrt{w(\xi)}}\right)^2 e^{2\lambda_+(\xi)t} |\hat{\varphi}|^2 \, d\xi d\tau \\
& + \frac{1}{4}\int_0^t \int_{|\xi|\ge 2r_0} \left(1+|\xi|^2\right)^s \left(1 + \frac{\left(\alpha+|\xi|^2\right)|\xi|}{\sqrt{w(\xi)}}\right)^2 e^{2\lambda_-(\xi)t} |\hat{\varphi}|^2 \, d\xi d\tau.
\end{aligned}$$

Integrating the resulting inequality by parts with respect to τ, it simplifies to

$$\begin{aligned}
&\int_0^t \int_{|\xi|\geq 2r_0} \left(1+|\xi|^2\right)^s \left|\partial_t \hat{G}(\xi,\tau)\right|^2 |\hat{\varphi}|^2 \, d\xi d\tau \\
\leq\ & \frac{1}{4}\int_{|\xi|\geq 2r_0} \left(1+|\xi|^2\right)^{s+1} \frac{\left(\alpha+|\xi|^2\right)|\xi| - \sqrt{w(\xi)}}{|\xi|\, w(\xi)} \left(1-e^{2\lambda_+(\xi)t}\right) |\hat{\varphi}|^2 \, d\xi \\
&+\frac{1}{4}\int_{|\xi|\geq 2r_0} \left(1+|\xi|^2\right)^{s+1} \frac{\left(\alpha+|\xi|^2\right)|\xi| + \sqrt{w(\xi)}}{|\xi|\, w(\xi)} \left(1-e^{2\lambda_+(\xi)t}\right) |\hat{\varphi}|^2 \, d\xi \\
\leq\ & \frac{1}{4}\int_{|\xi|\geq 2r_0} \left(1+|\xi|^2\right)^{s+1} \frac{4\left(1+|\xi|^2\right)^2 |\xi|}{w(\xi)\left(\left(\alpha+|\xi|^2\right)|\xi| + \sqrt{w(\xi)}\right)} \left||\xi|^{-1}\hat{\varphi}\right|^2 d\xi \\
&+\frac{1}{4}\int_{|\xi|\geq 2r_0} \left(1+|\xi|^2\right)^{s+1} \frac{2\left(\alpha+|\xi|^2\right)|\xi|^2}{w(\xi)} \left||\xi|^{-1}\hat{\varphi}\right|^2 d\xi \qquad (4.39)\\
\leq\ & C\left\|\Lambda^{-1}\varphi\right\|_{H^s}^2,
\end{aligned}$$

which together with (4.38) implies that

$$\int_0^t \int_{|\xi|\leq 2r_0} \left(1+|\xi|^2\right)^s \left|\partial_t \hat{G}(\xi,\tau)\right|^2 |\hat{\varphi}|^2 \, d\xi d\tau \leq C\,(t+1)^2 \left\|(-\Delta)^{-\frac{1}{2}}\varphi\right\|_{H^s}^2.$$

The proof of of Lemma 4.7 is complete. ■

Lemma 4.8 *Let the operators $G(t)$ and $H(t)$ be given in (4.12)–(4.13). Then, for any $\varepsilon \in (0,t)$, $t>0$, there holds*

$$\begin{aligned}
\left\|\Lambda^l \partial_{tt} H(t)\varphi\right\|_{L^2(0,t;L^2)} &\leq C\,(t+1)^{\frac{3}{2}} \|\varphi\|_{H^1}, l=1,2, &(4.40)\\
\left\|\Lambda^l \partial_{tt} G(t)\varphi\right\|_{L^2(\varepsilon,t;L^2)} &\leq C\left((t+1)^{\frac{3}{2}}+\varepsilon\right)\left\|\Lambda^{-1}\varphi\right\|, &(4.41)\\
\left\|\Lambda^l \left(I-\partial_{xx}\right)^{-1}\partial_{xx}\partial_t G(t)\varphi\right\|_{L^2(0,t;L^2)} &\leq C\left((t+1)^{\frac{3}{2}}\right)\|\varphi\|_{H^{-1}}. &(4.42)
\end{aligned}$$

Proof. For the first statement, it follows from the equality (4.26) that

$$\begin{aligned}
\left\|\Lambda^l \partial_{tt} H(t)\varphi\right\|^2 &= \left\||\xi|^{2-l}\,\partial_t \hat{G}(\xi,t)\hat{\varphi}\right\|^2 \\
&\leq \int_{|\xi|\leq 2r_0} |\xi|^{4-2l}\left|\partial_t \hat{G}(\xi,t)\right|^2 |\hat{\varphi}|^2 \, d\xi + \int_{|\xi|>2r_0} |\xi|^{4-2l}\left|\partial_t \hat{G}(\xi,t)\right|^2 |\hat{\varphi}|^2 \, d\xi.
\end{aligned}$$

By virtue of the expression (4.37) and a straightforward calculation, we deduce that

$$\begin{aligned}
&\left(\int_0^t \int_{|\xi|\leq 2r_0} |\xi|^{4-2l} \left|\partial_t \hat{G}(\xi,t)\right|^2 |\hat{\varphi}|^2 \, d\xi d\tau\right)^{\frac{1}{2}} \\
\leq\ & \left(\int_0^t \int_{|\xi|\leq 2r_0} |\xi|^{4-2l} \left|\hat{H}(\xi,t)\right|^2 |\hat{\varphi}|^2 \, d\xi d\tau + \int_0^t \int_{|\xi|\leq 2r_0} |\xi|^{6-2l} \left|\hat{G}(\xi,t)\right|^2 |\hat{\varphi}|^2 \, d\xi d\tau\right)^{\frac{1}{2}} \\
\leq\ & (t+1)^{\frac{3}{2}} \|\varphi\|_{H^1} .
\end{aligned}$$

Taking $s = 2-l$ in (4.39), we obtain

$$\left(\int_0^t \int_{|\xi|>2r_0} |\xi|^{4-2l} \left|\partial_t \hat{G}(\xi,t)\right|^2 |\hat{\varphi}|^2 \, d\xi d\tau\right)^{\frac{1}{2}} \leq C \|\varphi\|_{H^{1-l}} \leq C \|\varphi\|_{H^1} ,$$

which completes the proof of the estimate (4.40).

The second estimate (4.41), follows from

$$\partial_{tt}\hat{G} = -|\xi|^2 \hat{G}(t) - \frac{\left(\alpha + |\xi|^2\right)|\xi|^2}{\left(1+|\xi|^2\right)} \hat{G}_t$$

$$\begin{aligned}
\left\|\Lambda^l \partial_{tt} G(t)\varphi\right\| &= \left\||\xi|^{-l} \partial_{tt}\hat{G}(\xi,t)\hat{\varphi}\right\| \\
&\leq \left\||\xi|^{2-l} \hat{G}(\xi,t)\hat{\varphi}\right\| + \left\||\xi|^{2-l} \frac{\left(\alpha+|\xi|^2\right)}{\left(1+|\xi|^2\right)} \hat{G}_t(\xi,t)\hat{\varphi}\right\| . \qquad (4.43)
\end{aligned}$$

By taking $s = 2-l$ in (4.35), we have

$$\left(\int_0^t \left\||\xi|^{2-l} \hat{G}(\xi,\tau)\hat{\varphi}\right\|^2 d\tau\right)^{\frac{1}{2}} \leq C(t+1)^{\frac{3}{2}} \left\|\Lambda^{-1}\varphi\right\|_{H^{1-l}} \leq C(t+1)^{\frac{3}{2}} \left\|\Lambda^{-1}\varphi\right\| .$$

Concerning the second term in the right-hand side of (4.43), we have

$$\begin{aligned}
\left(\int_0^t \int_{|\xi|\leq 2r_0} |\xi|^{4-2l} \frac{\left(\alpha+|\xi|^2\right)^2}{\left(1+|\xi|^2\right)^2} \left|\hat{G}_t(\xi,t)\right|^2 |\hat{\varphi}|^2 \, d\xi d\tau\right)^{\frac{1}{2}} &\leq C(t+1)^{\frac{3}{2}} \left\|\Lambda^{-1}\varphi\right\|_{H^{1-l}} \\
&\leq C(t+1)^{\frac{3}{2}} \left\|\Lambda^{-1}\varphi\right\| .
\end{aligned}$$

For $|\xi| > 2r_0$, with the help of the the two following inequalities

$$\exp \lambda_+ (\xi) t \leq \frac{2\left(1+|\xi|^2\right) t^{-1}}{\left(\alpha+|\xi|^2\right)|\xi|^2 - |\xi| \sqrt{w(\xi)}}, \ t > 0$$

and

$$\exp \lambda_- (\xi) t \leq \frac{2\left(1+|\xi|^2\right) t^{-1}}{\left(\alpha+|\xi|^2\right)|\xi|^2 + |\xi| \sqrt{w(\xi)}}, \ t > 0$$

it easy to see that for any $t > 0$

$$\begin{aligned}
&\left(\int_\varepsilon^t \int_{|\xi|>2r_0} |\xi|^{4-2l} \frac{\left(\alpha+|\xi|^2\right)^2}{\left(1+|\xi|^2\right)^2} \left|\hat{G}_t(\xi,\tau)\right|^2 |\hat{\varphi}|^2 \, d\xi d\tau\right)^{\frac{1}{2}} \\
\leq \ &\left(\int_\varepsilon^t \tau^{-2} \int_{|\xi|>2r_0} |\xi|^{4-2l} \frac{\left(\alpha+|\xi|^2\right)^2}{\left(1+|\xi|^2\right)^2} \left(1 - \frac{\left(\alpha+|\xi|^2\right)|\xi|}{\sqrt{w(\xi)}}\right)^2 e^{2\lambda_+(\xi)t} |\hat{\varphi}|^2 \, d\xi d\tau\right)^{\frac{1}{2}} \\
&+ \left(\int_\varepsilon^t \tau^{-2} \int_{|\xi|>2r_0} |\xi|^{4-2l} \frac{\left(\alpha+|\xi|^2\right)^2}{\left(1+|\xi|^2\right)^2} \left(1 + \frac{\left(\alpha+|\xi|^2\right)|\xi|}{\sqrt{w(\xi)}}\right)^2 e^{2\lambda_-(\xi)t} |\hat{\varphi}|^2 \, d\xi d\tau\right)^{\frac{1}{2}} \\
\leq \ &\left(\int_\varepsilon^t \tau^{-2} \int_{|\xi|>2r_0} |\xi|^{4-2l} \frac{\left(\alpha+|\xi|^2\right)^2}{\left(1+|\xi|^2\right)^2} \frac{\left|\sqrt{w(\xi)} - \left(\alpha+|\xi|^2\right)|\xi|\right|^2}{w(\xi)} e^{2\lambda_+(\xi)t} |\hat{\varphi}|^2 \, d\xi d\tau\right)^{\frac{1}{2}} \\
&+ \left(\int_\varepsilon^t \tau^{-2} \int_{|\xi|>2r_0} |\xi|^{4-2l} \frac{\left(\alpha+|\xi|^2\right)^2}{\left(1+|\xi|^2\right)^2} \frac{\left(\sqrt{w(\xi)} + \left(\alpha+|\xi|^2\right)|\xi|\right)^2}{w(\xi)} e^{2\lambda_-(\xi)t} |\hat{\varphi}|^2 \, d\xi d\tau\right)^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
&\leq \left(\int_{\varepsilon}^{t} \tau^{-2} \int_{|\xi|>2r_0} |\xi|^{4-2l} \left(\alpha+|\xi|^2\right)^2 \frac{\left(\sqrt{w(\xi)} - \left(\alpha+|\xi|^2\right)|\xi|\right)^2}{w(\xi)\left(\left(\alpha+|\xi|^2\right)|\xi|^2 - |\xi|\sqrt{w(\xi)}\right)^2} |\hat{\varphi}|^2 \, d\xi d\tau \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{\varepsilon}^{t} \tau^{-2} \int_{|\xi|>2r_0} |\xi|^{4-2l} \left(\alpha+|\xi|^2\right)^2 \frac{\left(\sqrt{w(\xi)} + \left(\alpha+|\xi|^2\right)|\xi|\right)^2}{\left(\left(\alpha+|\xi|^2\right)|\xi|^2 + |\xi|\sqrt{w(\xi)}\right)^2 w(\xi)} |\hat{\varphi}|^2 \, d\xi d\tau \right)^{\frac{1}{2}} \\
&\leq \left(\int_{\varepsilon}^{t} \tau^{-2} \int_{|\xi|>2r_0} |\xi|^{4-2l} \frac{\left(\alpha+|\xi|^2\right)^2}{w(\xi)} \frac{1}{|\xi|^2} |\hat{\varphi}|^2 \, d\xi d\tau \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{\varepsilon}^{t} \tau^{-2} \int_{|\xi|>2r_0} |\xi|^{4-2l} \frac{\left(\alpha+|\xi|^2\right)^2}{w(\xi)} \frac{1}{|\xi|^2} |\hat{\varphi}|^2 \, d\xi d\tau \right)^{\frac{1}{2}} \\
&\leq \left(\int_{\varepsilon}^{t} \tau^{-2} \int_{|\xi|>2r_0} |\xi|^{4-2l} \frac{\left(\alpha+|\xi|^2\right)^2}{w(\xi)} \frac{1}{|\xi|^2} |\hat{\varphi}|^2 \, d\xi d\tau \right)^{\frac{1}{2}} \\
&\leq C\varepsilon^{-\frac{1}{2}} \left\| \Lambda^{-1}\varphi \right\|_{H^{1-l}} \leq C\varepsilon^{-\frac{1}{2}} \left\| \Lambda^{-1}\varphi \right\|.
\end{aligned}$$

Hence, for $l = 1, 2$, and $\tau \in (0, t)$, we find

$$\left(\int_{\varepsilon}^{t} \left\| |\xi|^{2-l} \frac{\left(\alpha+|\xi|^2\right)}{\left(1+|\xi|^2\right)} \hat{G}_t(\xi, t)\hat{\varphi} \right\|^2 d\tau \right)^{\frac{1}{2}} \leq C\left((t+1)^{\frac{3}{2}} + \varepsilon^{-\frac{1}{2}} \right) \left\| (-\Delta)^{-\frac{1}{2}} \varphi \right\|,$$

which immediately implies (4.41) according to (4.43). Following a similar reasoning to third statement, as (4.41), we conclude the proof of lemma 4.8. ■

Lemma 4.9 *Let the operator $G(t)$ be given in (4.12). Then, for $t \geq 0$ and $m = 1$, 2, the following estimate*

$$\left\| \Lambda^{-m} \int_{0}^{t} \left(I - \partial_{xx}\right)^{-1} \partial_{xx}\partial_t G(t-\tau) g(x,\tau)\, d\tau \right\|_{L^2(0,t;L^2)} \leq C(t+1)^{\frac{3}{2}} \left(\int_0^t \|g(x,\tau)\|_{H^{-1}}^2 \, d\tau \right)^{\frac{1}{2}}, \tag{4.44}$$

holds, provided that the terms on the right-hand side of (4.44) are finite.

Proof. It follows from the Minkowski and Hölder inequalities that

$$\begin{aligned}
&\left\|\Lambda^{-m}\int_0^t (I-\partial_{xx})^{-1}\partial_{xx}\partial_t G(t-\tau)g(x,\tau)\,d\tau\right\|_{L^2(0,t;L^2)}\\
=&\left(\int_0^t\left\|\int_0^s \Lambda^{-m}(I-\partial_{xx})^{-1}\partial_{xx}\partial_s G(s-\tau)g(x,\tau)\,d\tau\right\|_{L^2}^2 ds\right)^{\frac{1}{2}}\\
\le&\left(\int_0^t\left(\int_0^s \left\||\xi|^{-m+2}\left(1+|\xi|^2\right)^{-1}\partial_s\hat{G}(s-\tau)\hat{g}(\xi,\tau)\right\|_{L^2} d\tau\right)^2 ds\right)^{\frac{1}{2}}\\
\le&\left(\int_0^t\left(\int_0^s\left(\int_{|\xi|\le 2r_0}|\xi|^{-2m+4}\left(1+|\xi|^2\right)^{-2}\left|\partial_s\hat{G}(s-\tau)\right|^2|\hat{g}(\xi,\tau)|^2 d\xi\right)^{\frac{1}{2}} d\tau\right)^2 ds\right)^{\frac{1}{2}}\\
&+\left(\int_0^t\left(\int_0^s\left(\int_{|\xi|> 2r_0}|\xi|^{-2m+4}\left(1+|\xi|^2\right)^{-2}\left|\partial_s\hat{G}(s-\tau)\right|^2|\hat{g}(\xi,\tau)|^2 d\xi\right)^{\frac{1}{2}} d\tau\right)^2 ds\right)^{\frac{1}{2}}\\
\equiv& I_1+I_2.
\end{aligned}$$

For I_1, observing that

$$\int_{|\xi|\le 2r_0}|\xi|^{-2m+4}\left(1+|\xi|^2\right)^{-2}\left|\partial_s\hat{G}(s-\tau)\right|^2|\hat{g}(\xi,\tau)|^2 d\xi \le C(s-\tau+1)^2\|g(x,\tau)\|_{H^{-2}}^2$$

Therefore, by Hölder's inequality, we obtain

$$\begin{aligned}
I_1 &\le C\left(\int_0^t\left(\int_0^s (s-\tau+1)\|g(x,\tau)\|_{H^{-2}}\,d\tau\right)^2 ds\right)^{\frac{1}{2}}\\
&\le C(t+1)^{\frac{3}{2}}\left(\int_0^t\|g(x,\tau)\|_{H^{-2}}^2\,d\tau\right)^{\frac{1}{2}}. \qquad (4.45)
\end{aligned}$$

In order to estimate I_2, we follow the estimates performed in (4.44) to find that

$$\begin{aligned}
&\int_{|\xi|>2r_0} |\xi|^{-2m+4}\left(1+|\xi|^2\right)^{-2}\left|\partial_s \hat{G}(s-\tau)\right|^2 |\hat{g}(\xi,\tau)|^2 d\xi \\
\leq\ & \frac{1}{4}\int_{|\xi|>2r_0} |\xi|^{-2m+4}\left(1+|\xi|^2\right)^{-2}\frac{\left|\sqrt{w(\xi)}-\left(\alpha+|\xi|^2\right)|\xi|\right|^2}{w(\xi)} e^{2\lambda_+(\xi)(s-\tau)}|\hat{g}(\xi,\tau)|^2 d\xi \\
&+\frac{1}{4}\int_{|\xi|>2r_0} |\xi|^{-2m+4}\left(1+|\xi|^2\right)^{-2}\frac{\left|\sqrt{w(\xi)}+\left(\alpha+|\xi|^2\right)|\xi|\right|^2}{w(\xi)} e^{2\lambda_-(\xi)(s-\tau)}|\hat{g}(\xi,\tau)|^2 d\xi \\
\leq\ & C(s-\tau)^{-1}\|g(x,\tau)\|_{H^{-1}}^2 .
\end{aligned}$$

Then, using the Minkowski's inequality for integrals, we have

$$\begin{aligned}
I_2 &\leq C\left(\int_0^t\left(\int_0^s \tau^{-\frac{1}{2}}\|g(x,s-\tau)\|_{H^{-1}} d\tau\right)^2 ds\right)^{\frac{1}{2}} \\
&= C\left(\int_0^t\left(\int_0^t \tau^{-\frac{1}{2}}\chi_{[0,s]}(\tau)\|g(x,s-\tau)\|_{H^{-1}} d\tau\right)^2 ds\right)^{\frac{1}{2}} \\
&\leq C\int_0^t \tau^{-\frac{1}{2}}\left(\int_0^t \left|\chi_{[0,s]}(\tau)\right|^2\|g(x,s-\tau)\|_{H^{-1}}^2 ds\right)^{\frac{1}{2}} d\tau \\
&= C\int_0^t \tau^{-\frac{1}{2}}\left(\int_\tau^t \|g(x,s-\tau)\|_{H^{-1}}^2 ds\right)^{\frac{1}{2}} d\tau \\
&\leq C\int_0^t (t-\tau)^{-\frac{1}{2}}\left(\int_0^{t-\tau} \|g(x,s)\|_{H^{-1}}^2 ds\right)^{\frac{1}{2}} d\tau \\
&\leq Ct^{1/2}\left(\int_0^t \|g(x,s)\|_{H^{-1}}^2 ds\right)^{\frac{1}{2}}. \qquad (4.46)
\end{aligned}$$

Consequently, estimate (4.44) follows readily from (4.45) and (4.46). ■

4.4 Main resuts and their proofs

Our first result is as follows

Theorem 4.1 ***(Local Existence)*** *Suppose $u_0 \in H^1$, $u_1 \in L^2$ and $\Lambda^{-1}u_1 \in L^2$, then there exists a maximal time $T_{\max}$ which depends only on u_0, u_1 such that the Cauchy* *problem (4.1)-(4.2) admits a unique local mild solution* $u \in C([0, T_{\max}); H^1)$, $\Lambda^{-1}u_t \in C([0, T_{\max}); L^2)$ *with* $\Lambda^{-k}u_{tt} \in C([\varepsilon, T_{\max}); L^2), k = 0, 1$ *satisfying*

$$E(t) + \alpha \int_0^t \|u_\tau\|^2 \, d\tau + \int_0^t \|u_{x\tau}\|^2 \, d\tau = E(0).$$

Moreover if $T_{\max} < \infty$, *then*

$$\lim_{t \to T_{\max}} \sup(\left\|\Lambda^{-1}u_t\right\|^2 + \|u_t\|^2 + \|u\|_{H^1}^2) = +\infty.$$

Consequently

$$\lim_{t \to T_{\max}} \|u\|_{L^q}^2 = +\infty, \text{ for all } 1 < q < p + 1. \tag{4.47}$$

Proof. based on the contraction mapping principle. The Proof is split into the following five steps. ■

Step 1. We choose a closed convex set $X(T)$ to be a ball of radius R in the space $C([0, T]; H^1)$, i.e.

$$X = X(T) = \left\{u \in C([0, T]; H^1) : \|u\|_{X(T)} \leq R\right\},$$

with the norm $\|u\|_X = \max_{t\in[0,T]} \|u(t)\|_{H^1}$ and metric $d(u, v) = \|u - v\|_X$ for all $u, v \in X$. The nonnegative constant R depends on the initial data and will be determined later. Obviously, (X, d) is a complete metric space. Consider the mapping

$$\Phi\left(u\right)(t) = G(t) * u_1 + H(t) * u_0 + \int_0^t \left(I - \partial_{xx}\right)^{-1} \partial_{xx} G(t - \tau) * f(u(\tau)) d\tau. \tag{4.48}$$

The first goal is to show that Φ has a unique fixed point in X if R and T are well chosen. For any $u \in X$, we have

$$\|\Phi(u)\|_{H^1} \leq \|H(t) * u_0\|_{H^1} + \|G(t) * u_1\|_{H^1} + \int_0^t \left\|\left(I - \partial_{xx}\right)^{-1} \partial_{xx} G(t - \tau) * f(u(\tau))\right\|_{H^1} d\tau.$$

Then using estimates (4.29), (4.30) and (4.31) of Lemma 4.7 with $s=1$ and $k=0$, we obtain

$$\begin{aligned} \|\Phi(u)\|_{H^1} &\le C(t+1)\left(\|u_0\|_{H^1}+\left\|\Lambda^{-1}u_1\right\|+\int_0^t \|f(u(\tau))\|_{H^{-1}}\,d\tau\right) \\ &\le C(t+1)\left(\|u_0\|_{H^1}+\left\|\Lambda^{-1}u_1\right\|+\int_0^t \|f(u(\tau))\|_{L^2}\,d\tau\right) \\ &= C(t+1)\left(\|u_0\|_{H^1}+\left\|\Lambda^{-1}u_1\right\|+\int_0^t \left\||u|^{p-2}u\ln|u(\tau)|\right\|_{L^2}d\tau\right), \quad (4.49) \end{aligned}$$

where we have used the Sobolev embedding $L^2 \subset H^{-1}$.

Using the following elementary inequality

$$y^{p-2}|\ln y| \le C\left(1+y^{p-1}\right), \text{ for } y>0, p>2, \ [1] \tag{4.50}$$

to the third term in (4.49), we infer

$$\begin{aligned} \|\Phi(u)\|_{H^1} &\le C(t+1)\left(\|u_0\|_{H^1}+\left\|\Lambda^{-1}u_1\right\|+\int_0^t\left(\int_{-\infty}^{+\infty}|u|^2|u|^{2(p-2)}|\ln|u(\tau)||^2dx\right)^{\frac{1}{2}}d\tau\right) \\ &\le C(t+1)\left(\|u_0\|_{H^1}+\left\|\Lambda^{-1}u_1\right\|+\int_0^t\left(\int_{-\infty}^{+\infty}|u|^2\left(1+|u|^{2(p-1)}\right)dx\right)^{\frac{1}{2}}d\tau\right). \end{aligned}$$

We have

$$\begin{aligned} \int_0^t\left(\int_{-\infty}^{+\infty}|u|^2\left(1+|u|^{2(p-1)}\right)dx\right)^{\frac{1}{2}}d\tau &= \int_0^t\left(\int_{-\infty}^{+\infty}|u|^2dx+\int_{-\infty}^{+\infty}|u|^2|u|^{2(p-1)}dx\right)^{\frac{1}{2}}d\tau \\ &\le C\int_0^t\left(\|u\|_2+\left(\int_{-\infty}^{+\infty}|u|^2|u|^{2(p-1)}dx\right)^{\frac{1}{2}}\right)d\tau. \end{aligned}$$

By Holder's inequality, we obtain

$$\begin{aligned} \left(\int_{-\infty}^{+\infty}|u|^2|u|^{2(p-1)}dx\right)^{\frac{1}{2}} &\le \left(\int_{-\infty}^{+\infty}|u|^{2(p-1)}\right)^{\frac{1}{2(p-1)}}\left(\int_{-\infty}^{+\infty}|u|^{2\frac{(p-1)^2}{p-2}}\right)^{\frac{p-2}{2(p-1)}} \\ &= \|u\|_{L^{2(p-1)}}\|u\|^{p-1}_{L^{2\frac{(p-1)^2}{p-2}}}. \end{aligned}$$

Hence, we obtain

$$\int_0^t \left(\int_{-\infty}^{+\infty} |u|^2 \left(1 + |u|^{2(p-1)}\right) dx \right)^{\frac{1}{2}} d\tau \leq C \int_0^t \left(\|u\|_2 + \|u\|_{L^{2(p-1)}} \|u\|^{p-1}_{L^{2\frac{(p-1)^2}{p-2}}} \right) d\tau$$

Then, by Sobolev embeddings $H^1(\mathbb{R}) \subset L^{2(p-1)}(\mathbb{R})$, $H^1(\mathbb{R}) \subset L^{2\frac{(p-1)^2}{p-2}}(\mathbb{R})$, we find

$$\int_0^t \left(\int_{-\infty}^{+\infty} |u|^{2p} \left|\ln |u(\tau)|\right|^2 dx \right)^{\frac{1}{2}} d\tau \leq C \int_0^t \left(\|u\|_{H^1} + \|u\|^p_{H^1} \right) d\tau.$$

Therefore, we have

$$\|\Phi(u)\|_{H^1} \leq C\,(t+1)\,(\delta + CT\,(R + R^p)),$$

for some positive constant C. Hence, if we choose T and R such that

$$C\,(T+1)\,(\delta + CT\,(R + R^p)) \leq R,$$

then $\Phi u \in X$.

Now we explain Φ is contractive. For any $u, v \in X$, we have

$$\Phi u - \Phi v = \int_0^t (I - \partial_{xx})^{-1} \partial_{xx} G(t-\tau)(f(u(\tau)) - f(v(\tau)))d\tau.$$

Applying Lemma 4.7 with $s = 1$ and $k = 0$, then using the embedding $L^2 \subset H^{-1}$, we have

$$\begin{aligned} \|\Phi u - \Phi v\|_{H^1} &\leq C(1+t) \int_0^t \|f(u(\tau)) - f(v(\tau))\|_{H^{-1}} d\tau \\ &\leq C(1+t) \int_0^t \|f(u(\tau)) - f(v(\tau))\|_{L^2} d\tau. \end{aligned} \tag{4.51}$$

By the mean value theorem, we have

$$
\begin{aligned}
|f(u)-f(v)| &= |f'(\theta u+(1-\theta)v)|\,|u-v| \\
&\leq (1+(p-1)|\ln|\theta u+(1-\theta)v||)\,|\theta u+(1-\theta)v|^{p-2}\,|u-v| \\
&= |\theta u+(1-\theta)v|^{p-2}\,|u-v| \\
&\quad +(p-1)\,|\theta u+(1-\theta)v|^{p-2}\,|\ln|\theta u+(1-\theta)v||\,|u-v|\,,\ \theta\in(0,1)\,,
\end{aligned}
$$

which implies

$$
\begin{aligned}
|f(u)-f(v)| &\leq C\left(|u|^{p-2}+|v|^{p-2}\right)|u-v| + C\left(1+|\theta u+(1-\theta)v|^{p-1}\right)|u-v| \\
&\leq C\left(|u|+|v|\right)^{p-2}|u-v| + C\left(1+|u|^{p-1}+|v|^{p-1}\right)|u-v|\,, \qquad (4.52)
\end{aligned}
$$

where we have used again the inequality (4.50).

Using the estimates (4.51), (4.52), the Hölder inequality and Sobolev embedding theorem, we easily obtain

$$
\begin{aligned}
\int_{-\infty}^{+\infty}|u-v|^2\left(|u|+|v|\right)^{2(p-2)}dx &\leq \left(\int_{-\infty}^{+\infty}|u-v|^{2(p-1)}\,dx\right)^{\frac{1}{(p-1)}}\left(\int_{-\infty}^{+\infty}\left(|u|+|v|\right)^{2(p-1)}dx\right)^{\frac{(p-2)}{(p-1)}} \\
&\leq C\,\|u-v\|_{L^{2(p-1)}}^2\left(\|u\|_{L^{2(p-1)}}^{2(p-2)}+\|v\|_{L^{2(p-1)}}^{2(p-2)}\right) \\
&\leq C\,\|u-v\|_{H^1}^2\left(\|u\|_{H^1}^{2(p-2)}+\|v\|_{H^1}^{2(p-2)}\right). \qquad (4.53)
\end{aligned}
$$

In a similar way, we obtain

$$
\begin{aligned}
\int_{-\infty}^{+\infty}|u-v|^2\left(|u|+|v|\right)^{2(p-1)}dx &\leq \left(\int_{-\infty}^{+\infty}|u-v|^{2(p-1)}\,dx\right)^{\frac{1}{(p-1)}}\left(\int_{-\infty}^{+\infty}\left(|u|+|v|\right)^{2(p-1)+\frac{2(p-1)}{(p-2)}}dx\right)^{\frac{(p-2)}{(p-1)}} \\
&\leq C\,\|u-v\|_{L^{2(p-1)}}^2\left(\|u\|_{L^{\frac{2(p-1)^2}{p-2}}}^{2(p-1)}+\|v\|_{L^{\frac{2(p-1)^2}{p-2}}}^{2(p-1)}\right) \\
&\leq C\,\|u-v\|_{H^1}^2\left(\|u\|_{H^1}^{2(p-1)}+\|v\|_{H^1}^{2(p-1)}\right).
\end{aligned}
$$

Therefore we deduce from the last inequality and (4.53) that

$$\|\Phi u-\Phi v\|_{H^1}\le C(1+t)\int_0^t\|u-v\|_{H^1}\left(1+\|u\|_{H^1}^{p-2}+\|v\|_{H^1}^{p-2}+\|u\|_{H^1}^{p-1}+\|v\|_{H^1}^{p-1}\right)d\tau.$$

We choose T so small enough such that

$$CT(1+T)\left(R^{p-1}+R^{p-2}\right)\le\frac{1}{2},\tag{4.54}$$

holds. So there exists $T_0>0$, such that for any $T\in(0,T_0]$, Φ is a contraction map on X. It follows from the contraction mapping theorem that the equality (4.48) has a fixed point $u\in X(T)$.

To show u is a mild solution of the Cauchy problem (4.1)-(4.2), we need to prove $\Lambda^{-1}u_t\in C([0,T],L^2)$. First, we prove $u_t\in L_2([0,T];H^1)$. Since

$$\begin{aligned}u_t(x,t)&=\partial_tH(t)u_0+\partial_tG(t)u_1-\int_0^tK(t-\tau)\partial_tf(u(\tau))d\tau\\&=\partial_tH(t)u_0+\partial_tG(t)u_1-K(t)f(u_0)-\int_0^tK(t-\tau)\partial_tf(u(\tau))d\tau,\end{aligned}\tag{4.55}$$

we have from Lemma 4.7 and the embedding

$$\begin{aligned}\|u_t\|_{L^2(0,t;H^1)}&\le\left(\int_0^t\|\partial_tH(t)u_0\|_{H^1}^2\,d\tau\right)^{\frac12}+C(1+t)^{\frac32}\left\|\Lambda^{-1}u_1\right\|_{H^1}\\&\quad+\left(\int_0^t\|K(\tau)f(u_0)\|_{H^1}^2\,d\tau\right)^{\frac12}+\left(\int_0^t\left\|\int_0^{t'}K(t'-\tau)\partial_tf(u(\tau))d\tau\right\|_{H^1}^2dt'\right)^{\frac12}\\&\le C(1+t)^{\frac32}\left(\|u_0\|_{H^1}+\left\|\Lambda^{-1}u_1\right\|^2+\|u_0\|_{H^1}^p\right)\\&\quad+\left(\int_0^t\left\|\int_0^{t'}K(t'-\tau)\partial_\tau f(u(\tau))d\tau\right\|_{H^1}^2dt'\right)^{\frac12},\end{aligned}\tag{4.56}$$

It remains to estimate explicitly the last component on the right-hand side of the inequality (4.56). By the Minkowski and Hölder inequalities and taking $s=1$ and $k=0$ in (4.31), we can reach

$$\begin{aligned}\int_0^t \left\| \int_0^{t'} K(t'-\tau)\partial_\tau f(u(\tau))d\tau \right\|_{H^1}^2 dt' &\leq \int_0^t t' \int_0^{t'} \|K(t'-\tau)\partial_\tau f(u(\tau))d\tau\|_{H^1}^2 d\tau dt' \\ &\leq C\int_0^t t' \int_0^{t'} (1+t'-\tau)^2 \left\|u_\tau |u|^{p-2}\ln|u| + |u|^{p-2} u_\tau\right\|_{H^{-1}}^2 d\tau dt'.\end{aligned}$$

We infer from the Hölder inequality and Sobolev embedding theorem that

$$\begin{aligned}\left\|u_\tau |u|^{p-2}\ln|u| + |u|^{p-2} u_\tau\right\|_{H^{-1}}^2 &\leq C\left\|u_\tau |u|^{p-2}\ln|u| + |u|^{p-2} u_\tau\right\|_{L^2}^2 \\ &\leq C\left\|u_\tau |u|^{p-2}\ln|u|\right\|_{L^2}^2 + C\left\||u|^{p-2} u_\tau\right\|_{L^2}^2 \\ &\leq C\|u_\tau\|_{H^1}^2 + \|u_\tau\|_{L^{2(p-1)}}^2 \left(\|u\|_{L^{2(p-1)}}^{2(p-2)} + \|u\|_{L^{2\frac{(p-1)^2}{p-2}}}^{2(p-1)} \right) \\ &\leq C\|u_\tau\|_{H^1}^2 \left(1 + \|u\|_{H^1}^{2(p-2)} + \|u\|_{H^1}^{2(p-1)}\right). \qquad (4.57)\end{aligned}$$

Consequently

$$\begin{aligned}&\int_0^t \left\| \int_0^{t'} K(t'-\tau)\partial_\tau f(u(\tau))d\tau \right\|_{H^1}^2 dt' \\ \leq\ & Ct(1+t)^2 \left(1 + \max_{t\in[0,T]} \|u\|_{H^1}^{2(p-2)} + \max_{t\in[0,T]} \|u\|_{H^1}^{2(p-1)}\right) \int_0^t \int_0^{t'} \|u_\tau\|_{H^1}^2 d\tau dt' \qquad (4.58) \\ \leq\ & Ct(1+t)^2 \left(1 + \|u\|_X^{2(p-2)} + \|u\|_X^{2(p-1)}\right) \int_0^t \|u_\tau\|_{L^2(0,t';H^1)}^2 dt'. \qquad (4.59)\end{aligned}$$

Substituting (4.59) into (4.56), we obtain

$$\begin{aligned}\|u_\tau\|_{L^2(0,t;H^1)} \leq\ & C(1+t)^{\frac{3}{2}}(\|u_0\|_{H^1} + \left\|\Lambda^{-1}u_1\right\|^2 + \|u_0\|_{H^1}^p) \\ & + Ct(1+t)^{\frac{3}{2}} \|u\|_X^{(p-1)} (\int_0^t \|u_\tau\|_{L^2(0,t';H^1)}^2 dt')^{\frac{1}{2}}.\end{aligned}$$

It then follows from Gronwall's inequality that

$$\|u_\tau\|_{L^2(0,t;H^1)}^2 \leq C(T, \left\|\Lambda^{-1}u_1\right\|, \|u_0\|_{H^1}). \qquad (4.60)$$

Now we turn our attention to prove $\Lambda^{-1}u_t \in C([0,T],L^2)$. Using the expression (4.55) and the estimate (4.29), taking $s=0$ in (4.30) and $s=0$, $k=1$ in (4.31), we obtain

$$\left\|\Lambda^{-1}u_t\right\| \leq C(1+t)\left(\left\|\Lambda^{-1}u_1\right\| + \|u_0\|_{H^1} + \|f(u_0)\|_{H^{-3}} + \int_0^t \|\partial_\tau f(u(\tau)\|_{H^{-3}}\, d\tau\right)$$

From the embedding $H^{-1} \hookrightarrow H^{-3}$, and the fact (4.57), it follows that

$$\begin{aligned}\left\|\Lambda^{-1}u_t\right\| &\leq C(1+t)(\left\|\Lambda^{-1}u_1\right\| + \|u_0\|_{H^1} + \|u_0\|_{H^1}^p + \int_0^t \|u_\tau\|_{H^1}^2 \left(1+\|u\|_{H^1}^{2(p-2)} + \|u\|_{H^1}^{2(p-1)}\right) d\tau \\ &\leq C(1+t)(\left\|\Lambda^{-1}u_1\right\|^2 + \|u_0\|_{H^1} + \|u_0\|_{H^1}^p) \\ &\quad + Ct^{\frac{1}{2}}(1+t)\left(1+\|u\|_X^{2(p-2)} + \|u\|_X^{2(p-1)}\right)\|u_t\|_{L^2(0,t;H^1)}^2 .\end{aligned}$$

Therefore we deduce from (4.60) that

$$\left\|\Lambda^{-1}u_t\right\| \leq C(T, \left\|\Lambda^{-1}u_1\right\|, \|u_0\|_{H^1}) \tag{4.61}$$

Step 2. The uniqueness of the solution can be shown by following the same way as that of (4.53) and we omit the details.

Step 3. We claim $\Lambda^{-k}u_{tt} \in C([\varepsilon, T_{\max}); L^2)$, for $k=0,1$. Taking derivative to the both sides of the equality (4.55), we have

$$u_{tt}(x,t) = \partial_{tt}H(t)u_0 + \partial_{tt}G(t)u_1 - \partial_t K(t)f(u_0) - \int_0^t \partial_t K(t-\tau)f(u(\tau))_\tau d\tau \tag{4.62}$$

From Lemmas 4.8, 4.9 and the estimate (4.57), it immediately follows

$$\begin{aligned}\left\|\Lambda^{-k}u_{tt}\right\|_{L^2(\varepsilon,t;L^2)} &\leq C(1+t)^{\frac{3}{2}}(\|u_0\|_{H^1} + \|u_0\|_{H^1}^p) + C(\varepsilon^{-\frac{1}{2}} + (1+t)^{\frac{3}{2}})\left\|\Lambda^{-1}u_1\right\| \\ &\quad + C(1+t)^{\frac{3}{2}}(\int_0^t \|\partial_\tau f(u(\tau)\|_{H^{-1}}^2 d\tau)^{\frac{1}{2}} \\ &\leq C(T, \|u_0\|_{H^1}, \left\|\Lambda^{-1}u_1\right\|)\end{aligned}$$

for $k = 1, 2$ and sufficiently small $\varepsilon > 0, t > 0$.

Step 4. We extend the interval of existence. Noting that (4.54), the local existence time of u merely depends on the initial data. The fact (4.61) means $\|u\|_{H^1} + \|u_t\| + \|\Lambda^{-1}u_t\|$ remains bounded on $[0, T]$. Therefore, by the construction above, the solution can be continued, see also [62] for a similar argument. Hence, we can reach the existence of the maximal time $T_{\max}$ and if $T_{\max} < \infty$ then

$$\lim_{t \to T_{\max}} \sup(\|u_t\| + \left\|\Lambda^{-1}u_t\right\| + \|u\|_{H^1}) = +\infty. \tag{4.63}$$

Step 5. For the last statement, recalling the energy identity (4.21), it holds

$$\|u_t\| + \left\|\Lambda^{-1}u_t\right\|^2 + \|u\|_{H^1}^2 \leq \frac{4}{p}\|u\|_{L^{p+1}}^{p+1} + 2E(0), \quad \text{for all } t \in [0, T_{\max}), \mu > 0$$

which, along with (4.63) it yields

$$\lim_{t \to T_{\max}^-} \|u\|_{L^{p+1}} = +\infty \tag{4.64}$$

So, by the Sobolev embedding theorem, we have

$$\lim_{t \to T_{\max}^-} \|u\|_{H^1} = +\infty$$

Moreover, by using (4.21) again we get

$$\|u\|_{H^1}^2 \leq \frac{4}{p}C_* \|u\|_{L^{p+1}}^{p+1} + 2E(0), \qquad \forall t \in [0, T_{\max})$$

which, together with the Gagliardo–Nirenberg and Young inequalities, and noting that $1 < q < p+1$ implies $0 < \theta < 1$ and $(p+1)\theta < 2$, it follows

$$\begin{aligned}\frac{p}{4}(\|u\|_{H^1}^2 - 2E(0)) &\leq \|u\|_{L^{p+1}}^{p+1} \leq C\|u\|_{L^q}^{(p+1)(1-\theta)}\|u\|_{H^1}^{(p+1)\theta} \\ &\leq \varepsilon\|u\|_{H^1}^2 + C(\varepsilon)\|u\|_{L^q}^{\frac{2(p+1)(1-\theta)}{2-\theta(p+1)}}\end{aligned}$$

Taking ε sufficiently small, then we immediately yield (4.47).

Thus, Theorem 4.1 is proved.

Theorem 4.2 *For any $u_0 \in H^1, u_1 \in L^2$ and $\Lambda^{-1}u_1 \in L^2$, assume that $E(0) < d$ and $u_0 \in W'$, then the problem (4.1)-(4.2) admits a unique global solution $u(t) \in C([0,\infty); H^1)$ with $\Lambda^{-1}u_t \in C([0,\infty); L^2), u_t \in L^2([0,\infty); L^2),$ and $u \in W$ for $0 \leq t < \infty$.*

Befor prove this theorem, we show that the sets W and V are invariant under the flow of (4.1)-(4.2). Based on this, we establish the global existence and finite time blowup of solution respectively. We first start by dealing with the invariant sets of solutions with $E(0) < d$.

Lemma 4.10 *(**Invariant sets**) Suppose that $u_0 \in H^1, u_1 \in L^2$ and $\Lambda^{-1}u_1 \in L^2$ and $E(0) < d$. Then both sets W and V are invariant under the flow of (4.1)-(4.2) respectively, i.e.,*

i) If $u_0(x) \in W'$, then all solutions of problem (4.1)-(4.2) belong to W.

ii) If $u_0(x) \in V'$, then all solutions of problem (4.1)-(4.2) belong to V.

Proof. We only prove the invariance of W, in the same way we can prove the invariance for V. Let $u(t)$ be a solution of problem (4.1)-(4.2) with $u_0 \in W'$, T is the maximal existence time of u. Now we show that $u(t) \in W$ for $t \in (0,T)$. Suppose the contrary, that is there exists $\bar{t} \in (0,T)$ such that $u(\bar{t}) \notin W$. According to the continuity of $I(u(t))$ there exists $t_0 \in (0,\bar{t})$ such that $u(t_0) \in \partial W$. From the definition of W and $i)$ of Lemma 4.1 we get $B_{r_*} \subset W$, where $B_{r_*} = \{u \in H^1 / \|u\|_{H^1} < r_*\}$. Hence $u(t_0) \in \partial W$ reads $I(u(t_0)) = 0$ with $\|u(t_0)\|_{H^1} \neq 0$, that is $u(t_0) \in \mathcal{N}$. By the definition of d we have $J(u(t_0)) \geq d$ which contradicts

$$\frac{1}{2}\left\|\Lambda^{-1}u_t\right\|^2 + \|u_t\|^2 + \alpha\int_0^t \|u_\tau\|^2 d\tau + \int_0^t \|u_{x\tau}\|^2 d\tau + J(u) \leq E(0) < d.$$

Then the proof of Lemma 4.10 is complete. ■

Corollary 4.1 *Suppose $u_0 \in H^1$ and $\Lambda^{-1}u_1 \in L^2$. Assume that $E(0) < 0$ or $E(0) = 0$, $u_0 \not\equiv 0$, then all solutions of problem (4.1)-(4.2) belong to V.*

Now we move to prove the theorem 4.2:

Proof. From the local existence theorem 4.1, problem (4.1)-(4.2) admits a unique local solution $u \in C([0, T_{\max}], H^1)$ with $\Lambda^{-1}u_t \in C([0, T_{\max}], L^2)$, $T_{\max}$ is the maximal existence time of u. Let us prove that $T_{\max} = \infty$. From Lemma 4.11, we have $u \in W'$ for $0 \leq t < T_m$. Hence by

$$\tfrac{1}{2}\left\|\Lambda^{-1}u_t(t)\right\|^2 + \tfrac{1}{2}\|u_t(t)\|^2 + \alpha\int_0^t \|u_\tau\|^2 d\tau + \int_0^t \|u_{x\tau}\|^2 d\tau + \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_{H^1}^2$$
$$+\tfrac{1}{p}I(u) + \tfrac{1}{p^2}\|u\|_p^p = E(0) < d,\ 0 \leq t < T_{\max}$$

which together with $I(u) \geq 0$ gives

$$\begin{aligned}\left\|\Lambda^{-1}u_t(t)\right\|^2 + \|u_t(t)\|^2 + \|u\|_{H^1}^2 &< \frac{p}{p-2}d,\ 0 \leq t < T_{\max},\\ \int_0^t \|u_\tau\|_{H^1}^2 d\tau &\leq \frac{d}{\alpha},\ 0 \leq t < T_{\max}.\end{aligned}$$

Hence by Theorem 4.1, we have $T_{\max} = \infty$. ■

Next, we consider the finite time blow up of solution for the problems (4.1)-(4.2)

Theorem 4.3 *Suppose $u_0 \in H^1$, $u_1 \in L^2$ $\Lambda^{-1}u_0 \in L^2$ and $\Lambda^{-1}u_1 \in L^2$ satisfying $(\Lambda^{-1}u_1, \Lambda^{-1}u_0) + (u_1, u_0) > 0$. Assume that $E(0) < d$ and $I(u_0) < 0$, then the solu-*

tion of the problem (4.1)-(4.2) blows up in finite time, i.e the maximal existence time $T_{\max}$ *of the local solution* u *for problem (4.1)-(4.2) is finite and*

$$\lim_{t\to T_{\max}} (\left\|\Lambda^{-1}u_t\right\|^2 + \|u\|_{H^1}^2 + \|u_t\|^2) = +\infty.$$

Proof. Let u be a local weak solution of problem (4.1)-(4.2), $T_{\max}$ is the maximal existence time of u, by contradiction we will prove $T_{\max} < \infty$. We suppose that $T_{\max} = +\infty$, then $u \in C([0,T);H^1)$ and $\Lambda^{-1}u \in C([0,T);L^2)$, $\Lambda^{-1}u_t \in C([0,T);L^2)$ for any $T > 0$. Define $\phi(t)$ as follows :

$$\phi(t) = \left\|\Lambda^{-1}u\right\|^2 + \|u\|^2 + \alpha\int_0^t \|u(\tau)\|^2 d\tau + \int_0^t \|u_x(\tau)\|^2 d + (T-t)\alpha\|u_0\|^2 + (T-t)\left\|u_0'\right\|^2,$$

for $0 \le t < T$.

Then, through direct calculation, we have

$$\begin{aligned}\dot{\phi}(t) &= 2(\Lambda^{-1}u_t, \Lambda^{-1}u) + 2(u_t,u) + \alpha\|u\|^2 - \alpha\|u_0\|^2 + \|u_x\|^2 - \|u_{x0}\|^2 \\ &= 2(\Lambda^{-1}u_t, \Lambda^{-1}u) + 2(u_t,u) + 2\alpha\int_0^t (u_\tau,u)d\tau + 2\int_0^t (u_{x\tau},u_x)d\tau. \qquad (4.65)\end{aligned}$$

and

$$\ddot{\phi}(t) = 2\left\|\Lambda^{-1}u_t\right\|^2 + 2(\Lambda^{-1}u_{tt}, \Lambda^{-1}u) + 2\|u_t\|^2 + 2(u_{tt},u) + 2\alpha(u_t,u) + 2(u_{xt},u_x). \quad (4.66)$$

We have $u_t \in C([0,T);H^1)$ for any $T > 0$ and

$$\Lambda^{-2}u_{tt} = -u - u_{tt} + u_{xx} + |u|^{p-2}u\ln|u| - \alpha u_t + u_{xxt}, \in C([0,T);H^1).$$

Multiplying the last equation with u and integrating over $\mathbb{R}$, we obtain

$$(\Lambda^{-2}u_{tt},u) = -\|u\|_{H^1}^2 - (u_{tt},u) + \int_{-\infty}^{+\infty} |u|^p \ln|u|\,dx - \alpha(u_t,u) - (u_{xt},u_x). \qquad (4.67)$$

By using the expression (4.67) into (4.66), we infer

$$\ddot{\phi}(t) = 2\left\|\Lambda^{-1}u_t\right\|^2 + 2\left\|u_t\right\|^2 - 2\left\|u\right\|_{H^1}^2 + 2\int_{-\infty}^{+\infty} |u|^p \ln|u|\,dx. \tag{4.68}$$

Further from (4.65), it yields

$$\begin{aligned}\left(\dot{\phi}(t)\right)^2 &= 4(\Lambda^{-1}u_t,\Lambda^{-1}u)^2 + 4(u_t,u)^2 + 8(\Lambda^{-1}u_t,\Lambda^{-1}u)(u_t,u)\\ &\quad +4\left(\int_0^t (u_{x\tau},u_x)d\tau\right)^2 + 4\alpha^2\left(\int_0^t (u_\tau,u)d\tau\right)^2 + 8\alpha\int_0^t (u_\tau,u)d\tau\int_0^t (u_{x\tau},u_x)d\tau\\ &\quad +8\alpha(\Lambda^{-1}u_t,\Lambda^{-1}u)\int_0^t (u_\tau,u)d\tau + 8(\Lambda^{-1}u_t,\Lambda^{-1}u)\int_0^t (u_{x\tau},u_x)d\tau\\ &\quad +8\alpha(u_t,u)\int_0^t (u_\tau,u)d\tau + 8(u_t,u)\int_0^t (u_{x\tau},u_x)d\tau. \end{aligned}\tag{4.69}$$

By using the Cauchy–Schwarz inequality to the members in the right hand side of (4.69), we have

$$\begin{aligned} 4(\Lambda^{-1}u_t,\Lambda^{-1}u)^2 &\le 4\left\|\Lambda^{-1}u_t\right\|^2\left\|\Lambda^{-1}u\right\|^2,\\ 4(u_t,u)^2 &\le 4\left\|u_t\right\|^2\left\|u\right\|^2,\\ 4\alpha^2\left(\int_0^t (u_\tau,u)d\tau\right)^2 &\le 4\alpha^2\int_0^t \left\|u_\tau\right\|^2 d\tau\int_0^t \left\|u\right\|^2 d\tau,\\ 4\left(\int_0^t (u_{x\tau},u_x)d\tau\right)^2 &\le 4\int_0^t \left\|u_{x\tau}\right\|^2 d\tau\int_0^t \left\|u_x\right\|^2 d\tau,\\ 8(\Lambda^{-1}u_t,\Lambda^{-1}u)(u_t,u) &\le 8\left\|\Lambda^{-1}u_t\right\|\left\|\Lambda^{-1}u\right\|\left\|u_t\right\|\left\|u\right\|\\ &\le 4\left(\left\|\Lambda^{-1}u_t\right\|^2\left\|u\right\|^2 + \left\|\Lambda^{-1}u\right\|^2\left\|u_t\right\|^2\right),\\ 8\alpha(\Lambda^{-1}u_t,\Lambda^{-1}u)\int_0^t (u_\tau,u)d\tau &\le 8\alpha\left\|\Lambda^{-1}u_t\right\|\left\|\Lambda^{-1}u\right\|\left(\int_0^t \left\|u_\tau\right\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \left\|u\right\|^2 d\tau\right)^{\frac{1}{2}}\\ &\le 4\left(\alpha\left\|\Lambda^{-1}u_t\right\|^2\int_0^t \left\|u\right\|^2 d\tau + \alpha\left\|\Lambda^{-1}u\right\|^2\int_0^t \left\|u_\tau\right\|^2 d\tau\right). \end{aligned}$$

We also see from the Cauchy–Schwarz inequality that

$$\begin{aligned}
8(\Lambda^{-1}u_t,\Lambda^{-1}u)\int_0^t (u_{x\tau},u_x)d\tau &\le 8\left\|\Lambda^{-1}u_t\right\|\left\|\Lambda^{-1}u\right\|\left(\int_0^t \|u_{x\tau}\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u_x\|^2 d\tau\right)^{\frac{1}{2}}\\
&\le 4\left(\left\|\Lambda^{-1}u_t\right\|^2\int_0^t \|u_x\|^2 d\tau+\left\|\Lambda^{-1}u\right\|^2\int_0^t \|u_{x\tau}\|^2 d\tau\right)\\
8\alpha(u_t,u)\int_0^t (u_\tau,u)d\tau &\le 8\alpha\|u_t\|\|u\|\left(\int_0^t \|u_\tau\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u\|^2 d\tau\right)^{\frac{1}{2}}\\
&\le 4\left(\alpha\|u_t\|^2\int_0^t \|u\|^2 d\tau+\alpha\|u\|^2\int_0^t \|u_t\|^2 d\tau\right).\\
8(u_t,u)\int_0^t (u_{x\tau},u_x)d\tau &\le 8\|u_t\|\|u\|\left(\int_0^t \|u_{x\tau}\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u_x\|^2 d\tau\right)^{\frac{1}{2}}\\
&\le 4\left(\|u_t\|^2\int_0^t \|u_x\|^2 d\tau+\|u\|^2\int_0^t \|u_{xt}\|^2 d\tau\right)\\
8\alpha\int_0^t (u_\tau,u)d\tau\int_0^t (u_{x\tau},u_x)d\tau &\le 8\alpha\left(\int_0^t \|u_\tau\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u_{x\tau}\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u_x\|^2 d\tau\right)^{\frac{1}{2}}\\
&\le 4\left(\alpha\int_0^t \|u\|^2 d\tau\int_0^t \|u_{x\tau}\|^2 d\tau+\alpha\int_0^t \|u_\tau\|^2 d\tau\int_0^t \|u_x\|^2 d\tau\right).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\left(\dot{\phi}(t)\right)^2 &\le 4\left(\left\|\Lambda^{-1}u\right\|^2+\|u\|^2+\alpha\int_0^t \|u\|^2 d\tau+\int_0^t \|u_x\|^2 d\tau\right)\\
&\quad\times\left(\left\|\Lambda^{-1}u_t\right\|^2+\|u_t\|^2+\alpha\int_0^t \|u_t\|^2 d\tau+\int_0^t \|u_{xt}\|^2 d\tau\right)\\
&= 4\phi(t)\left(\left\|\Lambda^{-1}u_t\right\|^2+\|u_t\|^2+\alpha\int_0^t \|u_t\|^2 d\tau+\int_0^t \|u_{xt}\|^2 d\tau\right). \qquad (4.70)
\end{aligned}$$

Consequently, it follows from (4.68) and (4.70) that

$$\begin{aligned}
\phi(t)\ddot{\phi}(t)-\frac{p+2}{4}\dot{\phi}^2(t) &\ge \phi(t)\left(2\left\|\Lambda^{-1}u_t\right\|^2+2\|u_t\|^2-2\|u\|_{H^1}^2-2\|u\|^2+2\int_{-\infty}^{+\infty}|u|^p\ln|u|\,dx\right.\\
&\quad\left.-4\left\|\Lambda^{-1}u_t\right\|^2-4\|u_t\|^2-4\alpha\int_0^t \|u_t\|^2 d\tau-4\int_0^t \|u_{xt}\|^2 d\tau\right)\\
&\ge \phi(t)\left(2\left\|\Lambda^{-1}u_t\right\|^2+2\|u_t\|^2-2I(u)\right.\\
&\quad\left.-(p+2)\left(\left\|\Lambda^{-1}u_t\right\|^2+\|u_t\|^2+\alpha\int_0^t \|u_t\|^2 d\tau+\int_0^t \|u_{xt}\|^2 d\tau\right)\right). (4.71)
\end{aligned}$$

From the energy identity

$$\begin{aligned}&\frac{1}{2}\left(\left\|\Lambda^{-1}u_t\right\|^2+\|u_t\|^2\right)+\left(\frac{1}{2}-\frac{1}{p}\right)\|u\|_{H^1}^2+\frac{1}{p}I(u)+\frac{1}{p^2}\|u\|_p^p\\&+\alpha\int_0^t\|u_t\|^2\,d\tau+\int_0^t\|u_{xt}\|^2\,d\tau=E(0),\end{aligned}$$

we have

$$\begin{aligned}-2I(u)&=p\left(\left\|\Lambda^{-1}u_t\right\|^2+\|u_t\|^2\right)+(p-2)\|u\|_{H^1}^2+\frac{2}{p}\|u\|_p^p\\&\quad+2\alpha p\int_0^t\|u_t\|^2\,d\tau+2p\int_0^t\|u_{xt}\|^2\,d\tau-2pE(0).\end{aligned}$$

Using the last equation into (4.70), we can deduce that

$$\begin{aligned}\phi(t)\ddot{\phi}(t)-\frac{p+2}{4}\dot{\phi}^2(t)&\geq\phi(t)\Big(2\left\|\Lambda^{-1}u_t\right\|^2+2\|u_t\|^2\\&\quad+p\left(\left\|\Lambda^{-1}u_t\right\|^2+\|u_t\|^2\right)+(p-2)\|u\|_{H^1}^2+\frac{2}{p}\|u\|_p^p+2\alpha p\int_0^t\|u_t\|^2\,d\tau\\&\quad+2p\int_0^t\|u_{xt}\|^2\,d\tau-2pE(0)\\&\quad-(p+2)\left(\left\|\Lambda^{-1}u_t\right\|^2+\|u_t\|^2+\alpha\int_0^t\|u_t\|^2\,d\tau+\int_0^t\|u_{xt}\|^2\,d\tau\right)\Big).\end{aligned}$$

The last inequality can be then rewriten as

$$\phi(t)\ddot{\phi}(t)-\frac{p+2}{4}\dot{\phi}^2(t)\geq\phi(t)\psi(t)$$

where $\psi:[0,T]\to\mathbb{R}_+$ is defined by

$$\psi(t)=(p-2)\|u\|_{H^1}^2+\frac{2}{p}\|u\|_p^p+\alpha(p-2)\int_0^t\|u_t\|^2\,d\tau+(p-2)\int_0^t\|u_{xt}\|^2\,d\tau-2pE(0).$$

Clearly, we have from definition of J that

$$2pJ(u) = (p-2)\|u\|_{H^1}^2 + 2I(u) + \frac{2}{p}\|u\|_p^p.$$

Hence

$$2pJ(u) - 2I(u) = (p-2)\|u\|_{H^1}^2 + \frac{2}{p}\|u\|_p^p.$$

Therefore, we obtain

$$\begin{aligned}
\psi(t) &= (p-2)\|u\|_{H^1}^2 + \frac{2}{p}\|u\|_p^p + \alpha(p-2)\int_0^t \|u_t\|^2 d\tau + (p-2)\int_0^t \|u_{xt}\|^2 d\tau - 2pE(0) \\
&= 2pJ(u) - 2I(u) - 2pE(0) + \alpha(p-2)\int_0^t \|u_t\|^2 d\tau + (p-2)\int_0^t \|u_{xt}\|^2 d\tau \\
&= 2p(J(u) - E(0)) - 2I(u) + \alpha(p-2)\int_0^t \|u_t\|^2 d\tau + (p-2)\int_0^t \|u_{xt}\|^2 d\tau \\
&\geq 2p(J(u) - d) - 2I(u) + \alpha(p-2)\int_0^t \|u_t\|^2 d\tau + (p-2)\int_0^t \|u_{xt}\|^2 d\tau > 0.
\end{aligned}$$

By vertu of Lemma 4.5, we have $2p(J(u)-d) - 2I(u) > 0$ which implies

$$\psi(t) > 0, \text{ for all } t \geq 0.$$

Hence , it follows that

$$\phi(t)\ddot{\phi}(t) - \frac{p+2}{4}\dot{\phi}^2(t) \geq 0.$$

By the definition of $\phi(t), \phi(0) > 0$ and by the assumption $(\Lambda^{-1}u_1, \Lambda^{-1}u_0) + (u_1, u_0) > 0$, we have $\phi'(0) > 0$. According to Lemma 3.1, we can find $t_1 \leq T_0 = \frac{4\phi(0)}{(p-2)\phi'(0)}$ such that $\lim_{t\to t_1} \phi(t) = +\infty$, which contradicts our assumption $T_{\max} = \infty$. The proof of Theorem is completed. ■

Theorem 4.4 *Suppose $u_0 \in H^1$, $u_1 \in L^2$ and $\Lambda^{-1}u_1 \in L^2$. Assume that $E(0) = d$, $u_0 \in W'$ then the problem (4.1)-(4.2) admits a unique global solution $u \in C([0,\infty], H^1)$, $\Lambda^{-1}u_t \in C([0,\infty], L^2)$, $u_t \in L^2([0,\infty], L^2)$ and $u \in W$ for $0 \leq t < \infty$.*

Theorem 4.5 *Suppose $u_0 \in H^1$, $u_1 \in L^2$ and $\Lambda^{-1}u_0, \Lambda^{-1}u_1 \in L^2$. Assume that $E(0) = d$ and $I(u_0) < 0$ then the solution of problem (4.1)-(4.2) blows up in finite time i.e, $T_{\max}$ is the maximal existence time of u is finite unless u is a stationary solution of problems (4.1)-(4.2).*

We turn now to the proofs of global existence and blow-up of solution for problem (4.1)-(4.2) in the critical initial condition $E(0) = d$.To commence, we initiate the proof with the following lemma, affirming the invariance of sets W and V.

Lemma 4.11 *Suppose $u_0 \in H^1$, $u_1 \in L^2$ and $\Lambda^{-1}u_1 \in L^2$. Assume that $E(0) = d$, then both sets W and V are invariant under the flow of problem (4.1)-(4.2).*

Proof. We only prove this lemma by considering W (the proof for V is similar). Let u be the unique local solution of problem (4.1)-(4.2) with $E(0) = d$, $u_0 \in W'$, $T_{\max}$ is the maximal existence time of u. We prove $u(t) \in W$ for $0 < t < T_{\max}$, if it is false, then by continuity there exists a $t_0 \in (0, T_{\max})$ such that $u(t_0) \in \partial W$ i.e., $I(u(t_0)) = 0$ and $u(t_0) \neq 0$ i.e, $u(t_0) \in \mathcal{N}$. We have from definition of d, $J(u(t_0)) \geq d$ and

$$\frac{1}{2}\left\|\Lambda^{-1}u_t\right\|^2 + \frac{1}{2}\left\|u_t\right\|^2 + \alpha\int_0^t \left\|u_\tau\right\|^2 d\tau + \int_0^t \left\|u_{x\tau}\right\|^2 d\tau + J(u) = E(0) = d, \quad 0 \leq t \leq T_{\max},$$

we get $\int_0^t \left\|u_\tau\right\|^2 d\tau = 0$ and $\int_0^t \left\|u_{x\tau}\right\|^2 d\tau = 0$, which implies $\|u_t\| = 0$, $0 \leq t < t_0$ and $u_t = 0$ for $(x,t) \in \mathbb{R} \times [0, t_0]$ and consequently $u(t_0) = u_0 \in W'$ which contradicts $u(t_0) \in \mathcal{N}$. ■

Lemma 4.12 *Suppose $u_0 \in H^1$, $u_1 \in L^2$ and $\Lambda^{-1}u_1 \in L^2$. Assume that $E(0) = d$ and u is the unique local solution of problems (4.1)-(4.2), $T_{\max}$ is the maximal existence time of u. Then there exists $t_0 \in [0, T_{\max}]$ such that*

$$\int_0^{t_0} \left\|u_\tau\right\|_{H^1}^2 d\tau > 0,$$

unless u is a stationary solution of problems (4.1)-(4.2).

Proof. We suppose that there don't exist $t_0 \in [0, T_{\max}]$ satisfying $\int_0^{t_0} \|u_\tau\|_{H^1}^2 \, d\tau > 0$, so we get $\int_0^t \|u_\tau\|^2 \, d\tau \equiv 0$, $\forall t \in [0, T_{\max}] \Rightarrow \|u_\tau\| \equiv 0$ for $t \in [0, T_{\max}]$ and $u(t) = u_0$ for $t \in [0, T_{\max}]$ i.e., u is a stationary solution of problem (4.1)-(4.2).

From the local existence theorem, problem (4.1)-(4.2) admits a unique local solution $u \in C([0, T_{\max}], H^1)$ with $\Lambda^{-1}u_t \in C([0, T_{\max}], L^2)$, $T_{\max}$ is the maximal existence time of u. Let us prove that $T_{\max} = \infty$. From Lemma 4.10, we have $u \in W$ for $0 \le t < T_{\max}$. Hence by

$$\begin{gathered}\tfrac{1}{2}\left\|\Lambda^{-1}u_t(t)\right\|^2 + \tfrac{1}{2}\left\|u_t(t)\right\|^2 + \alpha \int_0^t \|u_\tau\|^2 \, d\tau + \int_0^t \|u_{x\tau}\|^2 \, d\tau + \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_{H^1}^2 \\ + \tfrac{1}{p} I(u) + \tfrac{1}{p^2}\|u\|_p^p = E(0) = d, \; 0 \le t < T_{\max},\end{gathered}$$

which together with $I(u) \ge 0$ gives

$$\begin{aligned}\left\|\Lambda^{-1}u_t(t)\right\|^2 + \|u_t(t)\|^2 + \|u\|_{H^1}^2 &\le \frac{p}{p-2}d, \; 0 \le t < T_{\max}, \\ \int_0^t \|u_\tau\|_{H^1}^2 \, d\tau &\le \frac{d}{\alpha}, \; 0 \le t < T_{\max}.\end{aligned}$$

Hence by Theorem 4.1, we have $T_{\max} = \infty$. ■

Proof. 4.5 Let u be the unique local solution of problem (4.1)-(4.2) but be not a stationary solution, $T_{\max}$ is the maximal existence time of u. We prove $T_{\max} < \infty$, the lemma 4.12 implies there exists $t_0 \in [0, T_{\max}]$ such that $\int_0^{t_0} \|u_\tau\|^2 \, d\tau > 0$

$$E(t_0) = E(0) - \alpha \int_0^{t_0} \|u_\tau\|^2 \, d\tau - \int_0^{t_0} \|u_{x\tau}\|^2 \, d\tau < d.$$

On the other hand we have from Lemma 4.10 $I(u(t_0)) < 0$. Therefore according to Theorem 4.3 $T_{\max} < \infty$, when taking $t = t_0$ as an initial value. ■

Corollary 4.2 *Let $u_0 \in H^1$, $u_1 \in L^2$ and $\Lambda^{-1}u_1 \in L^2$. Assume that $E(0) \le d$, $\Lambda^{-1}u_0 \in L^2$ then when $I(u_0) > 0$, the solution of problem (4.1)-(4.2) exists globally and when $I(u_0) < 0$*

the solution of problem (4.1)-(4.2) blows up in finite time.

Conclusion

This thesis examines the generalized Boussinesq equation within two distinct contexts. The first focuses on a bounded domain Ω, specifically exploring two cases of the function f. The second context involves a one-dimensional fifth-order Boussinesq equation featuring logarithmic nonlinearity. In the first case, we establishe the local well-posedness of solutions, present exponential energy decay estimates for global solutions under certain initial data assumptions, and prove the potential occurrence of infinite-time blow-up. For the second case, the existence and uniqueness of local mild solutions in the energy space are established. Additionally, we establish, under specific constraints on initial data, results regarding the existence and uniqueness of global solutions. The inclusion of a nonlinear logarithmic term introduces challenges, such as difficulties in applying Sobolev's logarithmic inequality and the absence of Poincaré's inequality in the interval $(-\infty, \infty)$, which is crucial for handling nonlinear estimates. In light of these complexities, a fundamental question arises: Can the problem exhibit a global solution with subcritical initial energy $E(0) > d$, and is there a possibility of the solution undergoing finite-time blow-up?

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