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Title

Contributions to the stochastic optimal control of Mckean-Vlasov stochastic differential systems via the derivatives with respect to measures with some applications

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Dedicace

I dedicate this work to my $\mathcal{M}$other and my $\mathcal{F}$ather,

To my sisters, and my brothers

Acknowledgments

First of all and foremost, praises and thanks to Allah, for his help and blessings throughout the research work to complete my thesis successfully.

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Symbols and Acronyms

1. $\mathbb{R}$: Set of Real numbers.
2. $\mathbb{N}$: Set of Natural numbers.
3. $\mathbb{R}_+$: Set of Non-negative real numbers.
4. SDE : Stochastic differential equation.
5. BSDE : Backward stochastic differential equation.
6. FBSDEs : Forward-backward stochastic differential equations.
7. FBSDEJs : Forward-Backward stochastic differential equations with jumps.
8. PDE : Partial differential equation.
9. ODE : Ordinary differential equation.
10. a.e. almost every where
11. a.s. almost surely
12. càdlàg continu à droite, limite à gauche
13. càglàd continu à gauche, limite à droite
14. e.g. for example (abbreviation of Latin exempli gratia)
15. i.e,. that is (abbreviation of Latin id est)
16. HJB The Hamilton-Jacobi-Bellman equation
17. $\dfrac{\partial f}{\partial x}$, f_x : The derivatives with respect to x.
18. $\mathbb{P}\otimes dt$: The product measure of $\mathbb{P}$ with the Lebesgue measure dt on $[0, T]$.
19. $E(\cdot)$, $E(\cdot \mid G)$ Expectation; conditional expectation
20. $\sigma(A)$: $\sigma-$algebra generated by A.
21. I_A : Indicator function of the set A.
22. $\mathcal{F}^Y$: The filtration generated by the process Y.

23. $W(\cdot), B(\cdot)$: Brownian motions

24. $\mathcal{F}_t^B$ the natural filtration generated by the brownian motion $B(\cdot)$,

25. $\mathrm{F}_1 \vee \mathrm{F}_2$ denotes the σ-field generated by $\mathrm{F}_1 \cup \mathrm{F}_2$.

26. $(\Omega, \mathcal{F}, \mathbb{P})$ probability space

27. $\{\mathcal{F}_t\}_{t\geq 0}$: filtration

28. $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ filtered probability space.

29. $\mathbb{L}^p(\mathcal{F})$: the space of $\mathbb{R}^n$-valued $\mathcal{F}-$measurable random variables X such that

$$E(\,|X|^p) < \infty.$$

30. $\mathbb{L}^p_{\mathcal{G}}(\Omega, \mathbb{R}^n\ )$: the space of $\mathbb{R}^n$-valued $\mathcal{G}-$measurable random variables X such that

$$E(\,|X|^p) < \infty.$$

31. $\mathbb{L}^p_{\mathcal{F}}([0,T]\,, \mathbb{R}^n)$: the space of all $(\mathcal{F}_t)_{t\geq 0}$-adapted $\mathbb{R}^n$-valued processes X such that

$$E\int_0^T |X(t)|^p\,dt < \infty.$$

32. $\mathbb{L}^\infty_{\mathcal{F}}([0,T]\,, \mathbb{R}^n)$: the space of all $(\mathcal{F}_t)_{t\geq 0}$-adapted $\mathbb{R}^n$-valued processes X essentially bounded processes.

33. $(u(\cdot), \xi(\cdot))$: continous-singular control.

34. $\partial_\mu g$: the derivatives with respect to measur μ.

35. $\mathcal{D}_\zeta g(\mu_0)$: the *Fréchet-derivative* of g at μ_0 in the direction ξ.

Résumé

Cette thèse de doctorat s'inscrit dans le cadre de la théorie de contrôle et l'optimisation stochastique. Le premier chapitre est de nature introductif, qui contient la formulation d'un problème de contrôle optimal stochastique de type mean-field avec quelques concepts et résultats de base qui permettent d'aborder notre travail ; tels que les processus stochastiques, principe du maximum,. . . etc. On s'intéresse aussi dans ce chapitre par les deux méthodes de résolutions.

Dans le deuxième chapitre, on a présenté la méthode de dérivation par rapport a une mesure de probabilité qui a été introduit par LIONS « Lions P.L. Cours au Collège de France : Théorie des jeu à champs moyens. http ://www. college- de-france.fr/ default/EN /all/equ [1] der/ audiovideo. jsp. (2013) ». On s'intéresse aussi dans ce chapitre par les différentes classes de contrôle optimal stochastique.

Dans le troisième chapitre, on a présenté notre première contribution, où on a prouvé les conditions nécessaires d'optimalité pour des classes de contrôle singulier (non regulier) partiellement observés. Les systems sont gouvernés par des équations differentielles stochastique EDSs de type McKean-Vlasov avec un saut de Poisson. Théorème de Girsanov et la dérivée par rapport à une mesure de probabilité au sense de Lions ont été utilisé pour établir notre résultat. Nous appliquons nos résultats pour étudier le problème conditionnelle de selection de portefeuille moyenne-variance avec interventions, où les interventions de change sont destinées à contenir les fluctuations excessives des taux de change.

Dans le quatrième chapitre, on a étudié un problème de contrôle stochastique de second-ordre pour systems de type mean-field. On a présenté notre première contribution, où on a prouvé un principe du maximum de seconde-ordre. Les systems considérés sont gouvernés par des équations differentielles stochastique EDSs de type McKean-Vlasov. Dans ce travail, on a présenté notre deuxième contribution où nous prouvons un nouveau seconde-order principe de maximum stochastique pour une classe de problèmes de contrôle optimal de type Mckean-Vlasov. Le domaine de contrôle est supposé convexe. Les dérivées par rap-

port à la mesure de probabilité et la formule d'Itô associée sont appliquées pour prouver nos principaux résultats. Dans le cinquième chapitre, un principe de maximum stochastique pour un modèle stochastique gouvernées par des équations différentielles Itô-stochastiques contrôlées non linéaires de type champ moyen est démontré. Nous étudions le problème de contrôle optimal stochastique de type mean-field suivant : Minimiser une fonctionnelle de coût de type champ moyen de la forme :

$$J\left(\alpha(\cdot)\right)=E\int_{\mathbb{R}^d}\Phi(\mathrm{y}_\alpha(\tau),\mu^{\mathrm{y}_\alpha(\tau)})\mu(d\mathrm{y}_\alpha),$$

telle que $\mathrm{y}_\alpha(\cdot)$ solution de $t\in[0,\tau]$

$$\begin{cases} d\mathrm{y}_\alpha(t)=\int_{\mathbb{R}^d}\varphi\left(t,\mathrm{y}_\alpha(t),\mu^{\mathrm{y}_\alpha(t)},\alpha(t)\right)\mu(\mathrm{dy}_\alpha)dt \\ \qquad\qquad +\int_{\mathbb{R}^d}\psi\left(t,\mathrm{y}_\alpha(t),\mu^{\mathrm{y}_\alpha(t)},\alpha(t)\right)\mu(\mathrm{dy}_\alpha)\mathrm{d}W(t), \\ \mathrm{y}_\alpha(0)=\mathrm{y}_0. \end{cases}$$

où $\alpha(\cdot)$ est la variable de contrôle donnée dans un sous-ensemble convexe borné, $\mathrm{y}_\alpha(\cdot)$ est la variable d'état contrôlée, $W(\cdot)$ est un mouvement brownien standard, $\mu^{\mathrm{y}_\alpha(t)}$ est la distribution de $\mathrm{y}_\alpha(t)$.

Abstract

This thesis is concerned with stochastic optimal control problems of mean-field type. The central theme is to establish a set of necessary conditions, in the form of stochastic maximum for a different systems. This thesis is structured around five chapters :

The first chapter is essentially a reminder. We presents some concepts and results that allow us to prove our results, such as stochastic processes, conditional expectation, martingales, Itô formulas, different methods of solving of optimal control (maximum principle, which has been introduced by Pontryagin et al and dynamical programming principle, wich has been introduced by Bellman) with some different class of stochastic control, (feedback, singular, impulsional, relaxed, near-optimal, ...etc.

In the second chapter, we present the method of the derivative with respect to probability measure. This new approch of derivatives has been introduced by P.L Lions *"Cours au Collège de France : Théorie des jeu à champs moyens. http ://www. college-de-france.fr/default/EN/all/equ[1]der/audiovideo.jsp. (2013) »*

Recently, in the third chapter of this thesis, we study partially observed optimal stochastic singular control problems of general mean-field with correlated noises between the system and the observation. The control variable has two components, the first being absolutely continuous and the second is a bounded variation, non decreasing continuous on the right with left limits. The dynamic system is governed by Itô-type controlled stochastic differential equation with jumps. The coefficients of the dynamic depend on the state process as well as of its probability law and the continuous control variable.In this work, we formulate this problem mathematically as a combined stochastic continuous control and irregular control problem. We study partially observed optimal stochastic intervention control problem for systems governed by mean-field SDEs with correlated noisy between the system and the observation, allowing both classical and intervention control.

In the fourth chapter, we establish a second-order stochastic maximum principle for optimal stochastic control of stochastic differential equations of general mean-field type.

The coefficients of the system are nonlinear and depend on the state process as well as of its probability law. The control variable is allowed to enter into both drift and diffusion terms. We establish a set of second-order necessary conditions for the optimal control in integral form. The control domain is assumed to be convex. The proof of our main result is based on the the first and second-order derivatives with respect to the probability law and by using a convex perturbation with some appropriate estimates. In the fifth chapter, a maximum principle for stochastic model governed by mean-field nonlinear controlled Itô-stochastic differential equations is proved. We study the following mean-field-type stochastic optimal nonlinear control problem : Minimize a mean-field cost functional

$$J\left(\alpha(\cdot)\right) = E\int_{\mathbb{R}^d} \Phi(\mathrm{y}_\alpha(\tau), \mu^{\mathrm{y}_\alpha(\tau)})\mu(d\mathrm{y}_\alpha),$$

subject to $\mathrm{y}_\alpha(\cdot)$ solution of the (MF-SDE) : $t \in [0, \tau]$

$$\begin{cases} d\mathrm{y}_\alpha(t) = \int_{\mathbb{R}^d} \varphi\left(t, \mathrm{y}_\alpha(t), \mu^{\mathrm{y}_\alpha(t)}, \alpha(t)\right)\mu(\mathrm{dy}_\alpha)dt \\ \qquad\qquad + \int_{\mathbb{R}^d} \psi\left(t, \mathrm{y}_\alpha(t), \mu^{\mathrm{y}_\alpha(t)}, \alpha(t)\right)\mu(\mathrm{dy}_\alpha)\mathrm{d}W(t), \\ \mathrm{y}_\alpha(0) = \mathrm{y}_0. \end{cases}$$

where, $\alpha(\cdot)$ is the control variable valued in a convex bounded subset $\mathbb{U} \subset \mathbb{R}^k$, $\mathrm{y}_\alpha(\cdot)$ is the controlled state variable, $W(\cdot)$ is a standard Brownian motion, $\mu^{\mathrm{y}_\alpha(t)}$ is the distribution of $\mathrm{y}_\alpha(t)$

General Introduction

The McKean-Vlasov type stochastic differential equations are Itô stochastic differential equations, where the coefficients of the state equation depend on the state process as well as its probability law. This type of equations was studied by Kac (1959) as a stochastic model for the plasma Vlasov equation and whose study was initiated by McKean (1966) to provide a rigorous treatment of special nonlinear partial differential equations. This thesis is concerned with stochastic optimal control problems of mean-field type. The central theme is to establish a set of necessary conditions, in the form of stochastic maximum for a different systems.

This thesis is structured around five chapters :

The first chapter is essentially a reminder. We presents some concepts and results that allow us to prove our results, such as stochastic processes, conditional expectation, martingales, Itô formulas, different methods of solving of optimal control (maximum principle, which has been introduced by Pontryagin et al and dynamical programming principle, wich has been introduced by Bellman) with some different class of stochastic control, (feedback, singular, impulsional, relaxed, near-optimal, ...etc.

In the second chapter, we present the method of the derivative with respect to probability measure. This new approch of derivatives has been introduced by P.L Lions *"Cours au Collège de France : Théorie des jeu à champs moyens. http ://www. college-de-france.fr/default/EN/all/equ[1]der/audiovideo.jsp. (2013) »*

Recently, in the third chapter of this thesis, we study partially observed optimal stochastic singular control problems of general mean-field with correlated noises between the system and the observation. The control variable has two components, the first being absolutely continuous and the second is a bounded variation, non decreasing continuous on the right with left limits. The dynamic system is governed by Itô-type controlled stochastic differential equation with jumps. The coefficients of the dynamic depend on the state process as well as of its probability law and the continuous control variable.In this work, we formulate this problem mathematically as a combined stochastic continuous control and irregular control problem. We study partially observed optimal stochastic intervention control problem for systems governed by mean-field SDEs with correlated noisy between the system and the observation, allowing both classical and intervention control of the form : $t \in [0,T]$

$$\begin{cases} \mathrm{d}x^{u,\xi}(t) = f(t, x^{u,\xi}(t), \mathbb{P}_{x^{u,\xi}(t)}, u(t))\mathrm{d}t + \sigma(t, x^{u,\xi}(t), \mathbb{P}_{x^{u,\xi}(t)}, u(t))\mathrm{d}W(t) \\ \qquad + \int_{\Theta} g(t, x^{u,\xi}(t_-), \mathbb{P}_{x^{u,\xi}(t_-)}, u(t), \theta)\widetilde{\eta}(\mathrm{d}\theta, \mathrm{d}t) \\ \qquad + c(t, x^{u,\xi}(t), \mathbb{P}_{x^{u,\xi}(t)}, u(t))\mathrm{d}\widetilde{W}(t) + G(t)\mathrm{d}\xi(t), \\ x^{u,\xi}(0) = x_0, \end{cases} \tag{1}$$

where $\mathbb{P}_{x^{u,\xi}(t)} = \mathcal{P}\circ\left(x^{u,\xi}\right)^{-1}$denotes the law of the random variable $x^{u,\xi}$

We assume that the state process $x^{v,\xi}(\cdot)$ cannot be observed directly, but the controllers can observe a related noisy process $Y(\cdot)$, which is governed by the following equation :

$$\begin{cases} dY(t) &= h(t, x^{v,\xi}(t), v(t))dt + \mathrm{d}\widetilde{W}(t) \\ Y(0) &= 0, \end{cases}$$

We define the $\mathcal{F}_t^Y-$martingale $\rho^v(t)$ which is the solution of the equation

$$\begin{cases} \mathrm{d}\rho^v(t) = \rho^v(t)h\left(t, x^v(t), v(t)\right)\mathrm{d}Y(t), \\ \rho^v(0) = 1. \end{cases}$$

This martingale allowed to define a new probability, denoted by $\mathbb{P}^v$ on the space $(\Omega, \mathcal{F})$, to emphasize the fact that it depend on the control $v(\cdot)$. It is given by the Radon-Nikodym derivative :

$$\left.\frac{\mathrm{d}\mathbb{P}^v}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t^Y} = \rho^v(t).$$

Hence, by Girsanov's theorem and hypothesis (C1) and (C2), $\mathbb{P}^v$ is a new probability measure of density $\rho^v(t)$. The process

$$\widetilde{W}(t) = Y(t) - \int_0^t h(s, x^{v,\xi}(s), v(s))\mathrm{d}s,$$

is a standard Brownian motion independent of $W(\cdot)$ and x_0 on the new probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^v)$.

By using Radon-Nikodym derivative, and the martingale property of $\rho^v(t)$, the cost functional can be written as

$$J(v(\cdot), \xi(\cdot)) = E\left[\int_0^T \rho^v(t) l(t, x^{v,\xi}(t), \mathbb{P}_{x^{u,\xi}(t)}, v(t))\mathrm{d}t + \rho^v(T)\psi(x^{v,\xi}(T), \mathbb{P}_{x^{v,\xi}(T)}) \right.$$
$$\left. + \int_{[0,T]} \rho^v(t) M(t) \mathrm{d}\xi(t)\right].$$

In terms of a classical convex variational techniques, we establish a set of necessary contiditions of optimal singular control in the form of maximum principle. Our main result is proved by applying Girsanov's theorem and the derivatives with respect to probability law in P.L. Lions' sense. An example is given to illustrate our theoretical result. The results obtained in Chapter §3 are all new and are the subject of a first article entitled :

Fatiha Korichi, Samira Boukaf, Mokhtar Hafayed, : Stochastic intervention control of mean-field Poisson-jump-system with noisy observation via L-derivatives with respect to probability law . **Boletim da Sociedade Paranaense de Matemática** Vol 42 , 2024 pp 1-25.

In the fourth chapter, we establish a second-order stochastic maximum principle for optimal stochastic control of stochastic differential equations of general mean-field type. The coefficients of the system are nonlinear and depend on the state process as well as of its probability law. The control variable is allowed to enter into both drift and diffusion terms. We establish a set of second-order necessary conditions for the optimal control in integral form. The control domain is assumed to be convex. The proof of our main result is based on the the first and second-order derivatives with respect to the probability law and by using a convex perturbation with some appropriate estimates.

The systems is governed by nonlinear controlled Itô stochastic differential systems.

$$\begin{cases} dx^u(t) = f\left(t, x^u(t), P_{x^u(t)}, u(t)\right) dt + \sigma\left(t, x^u(t), P_{x^u(t)}, u(t)\right) \mathrm{d}W(t), \\ x^u(0) = x_0. \end{cases}$$

The expected cost to be minimized over the class of admissible controls has the form

$$J\left(u(\cdot)\right) = E\left[h(x^u(T), P_{x^u(t)}) + \int_0^T \ell(t, x^u(t), P_{x^u(t)}, u(t))\mathrm{d}t\right].$$

Our control problem under studied provides also an interesting models in many applications such as economics and mathematical finance. This result extends the results obtained in "*Zhang H., Zhang X. : Pointwise second-order necessary conditions for stochastic optimal controls, Part I : The case of convex control constraint, SIAM J. Control Optim. 53(4), 2267-2296 (2015)*" to a class of continuous-singular stochastic control with jumps under partial pbservation. The results obtained in Chapter §4 are all new and are the subject of a second article entitled :

Samira Boukaf & **Fatiha Korichi** & Mokhtar Hafayed,& Muthukumar Palanisamy. *On pointwise second-order maximum principle for optimal stochastic controls of general mean-field type. Asian Journal of Control*, Doi : 10.1002/asjc.3271, Vol 26 (2) pp 790-802 (2024)

In the fifth chapter, a maximum principle for stochastic model governed by mean-field nonlinear controlled Itô-stochastic differential equations is proved. We study the following mean-field-type stochastic optimal nonlinear control problem : Minimize a mean-field cost functional

$$J\left(\alpha(\cdot)\right) = E\int_{\mathbb{R}^d} \Phi(\mathrm{y}_\alpha(\tau), \mu^{\mathrm{y}_\alpha(\tau)})\mu(d\mathrm{y}_\alpha),$$

subject to $\mathrm{y}_\alpha(\cdot)$ solution of the (MF-SDE) : $t \in [0,\tau]$

$$\begin{cases} d\mathrm{y}_\alpha(t) = \int_{\mathbb{R}^d} \varphi\left(t, \mathrm{y}_\alpha(t), \mu^{\mathrm{y}_\alpha(t)}, \alpha(t)\right) \mu(\mathrm{dy}_\alpha) dt \\ \qquad\qquad + \int_{\mathbb{R}^d} \psi\left(t, \mathrm{y}_\alpha(t), \mu^{\mathrm{y}_\alpha(t)}, \alpha(t)\right) \mu(\mathrm{dy}_\alpha) \mathrm{d}W(t), \\ \mathrm{y}_\alpha(0) = \mathrm{y}_0. \end{cases}$$

In the above, $\alpha(\cdot)$ is the control variable valued in a convex bounded subset $\mathbb{U} \subset \mathbb{R}^k$, $\mathrm{y}_\alpha(\cdot)$ is the controlled state variable, $W(\cdot)$ is a standard Brownian motion, $\mu^{\mathrm{y}_\alpha(t)}$ is the distribution of $\mathrm{y}_\alpha(t)$ and Φ, φ and ψ are a given maps. The coefficients of our model are nonlinear and depend explicitly on the control variable, the state process as well as of its probability distribution. The control region is assumed to be bounded and convex. Our main result is derived by applying the Lions's partial-derivatives with respect to random measures in Wasserstein space. The associated Itô-formula and convex-variation approach are applied to establish the optimal control. The results obtained in Chapter §5 are all new and are the subject of a third article entitled :

Fatiha Korichi & Mokhtar Hafayed, *Lions's partial derivatives with respect to probability measures for general mean-field stochastic control problem.* Doi 10.22124/jmm.2024.27136.2390. Journal of Mathematical Modeling. (2024), Vol. 12, No. 3, pp. 517–532. (2024)

Chapitre 1

Stochastic processes and preliminary

Optimal control theory can be described as the study of strategies to optimally influence a system x with dynamics evolving over time according to a differential equation. The influence on the system is modeled as a vector of parameters, u, called the control. It is allowed to take values in some set U, which is known as the action space. For a control to be optimal, it should minimize a cost functional (or maximize a reward functional), which depends on the whole trajectory of the system x and the control u over some time interval $[0, T]$. The infimum of the cost functional is known as the value function (as a function of the initial time and state). This minimization problem is infinite dimensional, since we are minimizing a functional over the space of functions $u(t), t \in [0, T]$. Optimal control theory essentially consists of different methods of reducing the problem to a less transparent, but more manageable problem.

1.1 Formulation of stochastic optimal control problem

It is well-known that control theory was founded by *N. Wiener in 1948*. After that, this theory was greatly extended to various complicated settings and widely used in sciences and technologies. Clearly, control means a suitable manner for people to change the dynamics of a system under consideration. Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ be a given filtered probability space.

1.1.1 Stochastic process

Let $\mathbb{T}$ be a nonempty index set and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. A family $\{X(t) : t \in \mathbb{T}\}$ of random variables from $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}^n$ is called a stochastic process. For any $w \in \Omega$ the map $t \mapsto X(t, w)$ is called a sample path.

1.1.2 Natural fitration

Let $X = (X_t, t \geq 0)$ a stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The natural filtration of X , denoted by $\mathcal{F}_t^X$, is defined by $\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$. Also, we called the filtaration generated by X.

1.1.3 Brownian motion

The stochastic process $(W(t),\, t \geq 0)$ is a brownian motion (standard) iff :

1. $\mathbb{P}[W(0) = 0] = 1$.
2. $t \rightarrow W(t, w)$ is continuous.$\mathbb{P}-p.s.$
3. $\forall s \leq t$, $W(t) - W(s)$ is normally distributed ; center with variation $(t-s)$ i.e $W(t) - W(s) \sim \mathcal{N}(0, t-s)$.
4. $\forall n$, $\forall\ 0 \leq t_0 \leq t_1 \leq ... \leq t_n$, the variables $\left(W_{t_n} - W_{t_{n-1}}, ..., W_{t_1 -} W_{t_0}, W_{t_0}\right)$ are independents.The following result gives special case of the Itô formula for jump diffusions.

1.1.4 Integration by parts formula

Suppose that the processes $x_i(t)$ are given by : for $i = 1, 2,\ t \in [0, T]$:

$$\begin{cases} dx_i(t) = f\left(t, x_i(t)\right) dt + \sigma\left(t, x_i(t)\right) dW(t) \\ x_i(0) = 0. \end{cases}$$

Then we get

$$\begin{aligned} E\left(x_1(T)x_2(T)\right) = E&\left[\int_0^T x_1(t)dx_2(t) + \int_0^T x_2(t)dx_1(t)\right] \\ &+ E\int_0^T \sigma^{\intercal}\left(t, x_1(t)\right)\sigma\left(t, x_2(t)\right) dt. \end{aligned}$$

In this section, we present two mathematical formulations (strong and weak formulations) of stochastic optimal control problems in the following two subsections, respectively.

1.1.5 Strong formulation

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ be a given filtered probability space satisfying the usual condition, on which an d-dimensional standard Brownian motion $W(\cdot)$ is defined, consider the following controlled stochastic differential equation :

$$\begin{cases} dx(t) &= f(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), \\ x(0) &= x_0 \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where

$$f : [0, T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}^n,$$
$$\sigma : [0, T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}^{n\times d},$$

and $x(\cdot)$ is the variable of state.

The function $u(\cdot)$ is called the control representing the action of the decision-makers (controller). At any time instant the controller has some information (as specified by the information field $\{\mathcal{F}_t\}_{t\in[0,T]}$) of what has happened up to that moment, but not able to foretell what is going to happen afterwards due to the uncertainty of the system (as a consequence, for any t the controller cannot exercise his/her decision $u(t)$ before the time t really comes), This nonanticipative restriction in mathematical terms can be expressed as "$u(\cdot)$ is $\{\mathcal{F}_t\}_{t\in[0,T]}-$adapted".

The control $u(\cdot)$ is an element of the set

$$\mathcal{U}[0,T] = \{u(\cdot) : [0,T] \times \Omega \longrightarrow \mathbb{A} \text{ such that } u(\cdot) \text{ is } \{\mathcal{F}_t\}_{t\in[0,T]} - \text{adapted}\}.$$

We introduce the cost functional as follows

$$J(u(\cdot)) \doteq E\left[\int_0^T l(t, x(t), u(t))dt + g(x(T))\right], \tag{1.2}$$

where

$$l : [0,T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R},$$

$$g : \mathbb{R}^n \longrightarrow \mathbb{R}.$$

Definition 1.1. Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ be given satisfying the usual conditions and let $W(t)$ be a given d-dimensional standard $\{\mathcal{F}_t\}_{t\in[0,T]}$-Brownian motion.

A control $u(\cdot)$ is called an admissible control, and $(x(\cdot), u(\cdot))$ an admissible pair, if

i) $u(\cdot) \in \mathcal{U}[0,T]$; $x(\cdot)$ is the unique solution of equation (1.1) ;

ii) $l(\cdot, x(\cdot), u(\cdot)) \in \mathbb{L}^1_{\mathcal{F}}([0,T];\mathbb{R})$ and $g(x(T)) \in \mathbb{L}^1_{\mathcal{F}_T}(\Omega;\mathbb{R})$.

The set of all admissible controls is denoted by $\mathcal{U}([0,T])$. Our stochastic optimal control problem under strong formulation can be stated as follows :

Problem 1.1 Minimize (1.2) over $\mathcal{U}([0,T])$. The goal is to find $u^*(\cdot) \in \mathcal{U}([0,T])$, such that

$$J(u^*(\cdot)) \;=\; \inf_{u(\cdot)\in\mathcal{U}([0,T])} J(u(\cdot)). \tag{1.3}$$

For any $u^*(\cdot) \in \mathcal{U}^s\left([0,T]\right)$ satisfying (1.3) is called an strong optimal control. The corresponding state process $x^*(\cdot)$ and the state control pair $(x^*(\cdot), u^*(\cdot))$ are called an strong optimal state process and an strong optimal pair, respectively.

1.1.6 Weak formulation

In stochastic control problems, there exists for the optimal control problem another formulation of a more mathematical aspect, it is the weak formulation of the stochastic optimal control problem. Unlike in the strong formulation the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ on which we define the Brownian motion $W\left(\cdot\right)$ are all fixed, but it is not the case in the weak formulation, where we consider them as a parts of the control.

Definition 1.1.2. A 6-tuple $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}, W\left(\cdot\right), u\left(\cdot\right)\right)$ is called weak-admissible control and $(x(\cdot), u(\cdot))$ an weak admissible pair, if

1. $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ is a filtered probability space satisfying the usual conditions;
2. $W\left(\cdot\right)$ is an d-dimensional standard Brownian motion defined on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$;
3. $u(\cdot)$ is an $\{\mathcal{F}_t\}_{t\in[0,T]}-$adapted process on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in U;
4. $x(\cdot)$ is the unique solution of equation (1.1),
5. $l(\cdot, x(\cdot), u(\cdot)) \in \mathbb{L}^1_{\mathcal{F}}\left([0,T]\,; \mathbb{R}\right)$ and $g(x(T)) \in \mathbb{L}^1_{\mathcal{F}}\left(\Omega; \mathbb{R}\right)$.

The set of all weak admissible controls is denoted by $\mathcal{U}^w\left([0,T]\right)$. Sometimes, might write $u(\cdot)) \in \mathcal{U}^w\left([0,T]\right)$ instead of

$$\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}, W\left(\cdot\right), u\left(\cdot\right)\right) \in \mathcal{U}^w\left([0,T]\right).$$

Our stochastic optimal control problem under weak formulation can be formulated as follows :

Problem 1.1.2. The objective is to minimize the cost functional given by equation (1.2) over the of admissible controls $\mathcal{U}^w([0,T])$.

Namely, one seeks $v^*(\cdot) = \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}, W(\cdot), u(\cdot)\right) \in \mathcal{U}^w([0,T])$ such that

$$J(v^*(\cdot)) = \inf_{v(\cdot)\in\mathcal{U}^w([0,T])} J(v(\cdot)).$$

1.2 Methods to solving optimal control problem

In optimal control problems, two major tools for studing optimal control are Pontryagin's maximum principle and Bellman's dynamic programming method.

1.2.1 The Dynamic Programming (*Bellman* Principle)

We present an approach to solving optimal control problems, namely, the method of dynamic programming. Dynamic programming, originated by R. Bellman (*Bellman, R. : Dynamic programming, Princeton Univ. Press., (1957)*) is a mathematical technique for making a sequence of interrelated decisions, which can be applied to many optimization problems (including optimal control problems). The basic idea of this method applied to optimal controls is to consider a family of optimal control problems with different initial times and states, to establish relationships among these problems via the so-called Hamilton-Jacobi-Bellman equation (HJB, for short), which is a nonlinear first-order (in the deterministic case) or second-order (in the stochastic case) partial differential equation. If the HJB equation is solvable (either analytically or numerically), then one can obtain an optimal feedback control by taking the maximize/minimize of the Hamiltonian or generalized Hamiltonian involved in the HJB equation. This is the so-called verification technique. Note that this approach actually gives solutions to the whole family of problems (with different initial times and states).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, satisfying the usual conditions, $T > 0$ a finite time, and W a d-dimensional Brownian motion defined on the filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]}\right)$.

The Bellman dynamic programming principle. We consider the following stochastic differential equation

$$dx(s) \;=\; f(s, x(s), u(s))ds + \sigma(s, x(s), u(s))dW(s), \;\; s \in [0, T]\,. \tag{1.4}$$

The control $u = u(s)_{0\leq s\leq T}$ is a progressively measurable process valued in the control set U, a subset of $\mathbb{R}^k$, satisfies a square integrability condition. We denote by $\mathcal{U}\left([t, T]\right)$ the set of control processes u.

Conditions. To ensure the existence of the solution to SDE-(1.4), the Borelian functions

$$f : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$$

$$\sigma : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^{n\times d}$$

satisfy the following conditions :

$$|f(t, x, u) - f(t, y, u)| + |\sigma(t, x, u) - \sigma(t, y, u)| \leq C\,|x - y|\,,$$

$$|f(t, x, u)| + |\sigma(t, x, u)| \leq C\,[1 + |x|]\,,$$

for some constant $C > 0$. We define the gain function as follows :

$$J(t, x, u) \;=\; E\left[\int_t^T l(s, x(s), u(s))ds + g(x\,(T))\right], \tag{1.5}$$

where

$$l : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R},$$

$$g : \mathbb{R}^n \longrightarrow \mathbb{R},$$

be given functions. We have to impose integrability conditions on f and g in order for the above expectation to be well-defined, e.g. a lower boundedness or quadratic growth condition. The objective is to maximize this gain function. We introduce the so-called value function :

$$V(t,x) \quad = \quad \sup_{u\in\mathcal{U}([t,T])} J(t,x,u), \tag{1.6}$$

where $x(t) = x$ is the initial state given at time t. For an initial state (t,x), we say that $u^* \in \mathcal{U}([t,T])$ is an optimal control if

$$V(t,x) = J(t,x,u^*).$$

Theorem 1.1.1 Let $(t,x) \in [0,T] \times \mathbb{R}^n$ be given. Then we have for $t \leq t+h \leq T$

$$V(t,x) = \sup_{u\in\mathcal{U}([t,T])} E\left[\int_t^{t+h} l(s,x(s),u(s))dt + V(t+h,x(t+h))\right],. \tag{1.7}$$

Proof. The proof of the dynamic programming principle is technical and has been studied by different methods, we refer the reader to Yong and Zhou [120].

The Hamilton-Jacobi-Bellman equation. The HJB equation is the infinitesimal version of the dynamic programming principle. It is formally derived by assuming that the value function is $C^{1,2}([0,T]\times\mathbb{R}^n)$, applying Itô's formula to $V(s,x^{t,x}(s))$ between $s=t$ and $s=t+h$, and then sending h to zero into (1.6). The classical HJB equation associated to the stochastic control problem (1.6) is

$$-V_t(t,x) - \sup_{u\in U}\left[\mathcal{L}^u V(t,x) + l(t,x,u)\right] \quad = \quad 0, \text{ on } [0,T]\times\mathbb{R}^n, \tag{1.8}$$

where $\mathcal{L}^u$ is the second-order infinitesimal generator associated to the diffusion x with control u

$$\mathcal{L}^u V \quad = \quad f(x,u).D_x V + \frac{1}{2} tr\left(\sigma(x,u)\,\sigma^\intercal(x,u)\,D_x^2 V\right).$$

This partial differential equation (PDE) is often written also as :

$$-V_t(t,x) - H(t,x,D_xV(t,x),D_x^2V(t,x)) \;=\; 0, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n, \tag{1.9}$$

where for $(t, x, \Psi, Q) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}_n$ ($\mathcal{S}_n$ is the set of symmetric $n \times n$ matrices) :

$$H(t,x,\Psi,Q) \;=\; \sup_{u \in U} \left[f(t,x,u).\Psi + \frac{1}{2} tr\left(\sigma\sigma^{\intercal}(t,x,u)\, Q\right) + l(t,x,u) \right]. \tag{1.10}$$

The function H is sometimes called Hamiltonian of the associated control problem, and the PDE (1.8) or (1.9) is the dynamic programming or HJB equation.
There is also an a priori terminal condition :

$$V(T,x) \;=\; g(x), \; \forall x \in \mathbb{R}^n,$$

which results from the very definition of the value function V.

The classical verification approach The classical verification approach consists in finding a smooth solution to the HJB equation, and to check that this candidate, under suitable sufficient conditions, coincides with the value function. This result is usually called a verification theorem and provides as a byproduct an optimal control. It relies mainly on Itô's formula. The assertions of a verification theorem may slightly vary from problem to problem, depending on the required sufficient technical conditions. These conditions should actually be adapted to the context of the considered problem. In the above context, a verification theorem is roughly stated as follows :

Theorem 1.1.2. Let W be a $C^{1,2}$ function on $[0,T] \times \mathbb{R}^n$ and continuous in T, with suitable growth condition. Suppose that for all $(t,x) \in [0,T] \times \mathbb{R}^n$, there exists $u^*(t,x)$

mesurable, valued in U such that W solves the HJB equation :

$$\begin{aligned} 0 &= -W_t(t,x) - \sup_{u\in U}\left[\mathcal{L}^u W(t,x) + l(t,x,u)\right] \\ &= -W_t(t,x) - \mathcal{L}^{u^*(t,x)} W(t,x) - l(t,x,u^*(t,x)), \text{ on } [0,T]\times\mathbb{R}^n, \end{aligned}$$

together with the terminal condition $W(T,\cdot) = g$ on $\mathbb{R}^n$, and the stochastic differential equation :

$$dx(s) = f(s,x(s),u^*(s,x(s)))ds + \sigma(s,x(s),u^*(s,x(s)))dW(t),$$

admits a unique solution x^*, given an initial condition $x(t) = x$. Then, $W = V$ and $u^*(s,x^*)$ is an optimal control for $V(t,x)$.

A proof of this verification theorem can be found in book, by Yong & Zhou [120].

1.2.2 The pontryagin type stochastic maximum principle

The pioneering works on the stochastic maximum principle were written by Kushner [69, 70]. Since then there have been a lot of works on this subject, among them, in particular, those by Bensoussan [14], Peng [107], and so on. The stochastic maximum principle gives some necessary conditions for optimality for a stochastic optimal control problem. The original version of Pontryagin's maximum principle was first introduced for deterministic control problems in the 1960's by Pontryagin et al. (*Pontryagin,L.S., Boltyanski,V.G., Gamkrelidze, R.V., Mischenko, E.F.)* as in classical calculus of variation. The basic idea is to perturbe an optimal control and to use some sort of Taylor expansion of the state trajectory around the optimal control, by sending the perturbation to zero, one obtains some inequality, and by duality.

The deterministic maximum principle. As an illustration, we present here how the maximum principle for a deterministic control problem is derived. In this setting, the state of the system is given by the ordinary differential equation (ODE) of the form

$$\begin{cases} dx(t) &= f(t,x(t),u(t))dt, \ t \in [0,T], \\ x(0) &= x_0, \end{cases} \tag{1.11}$$

where

$$f : [0,T] \times \mathbb{R} \times \mathcal{A} \longrightarrow \mathbb{R},$$

and the action space $\mathcal{A}$ is some subset of $\mathbb{R}$. The objective is to minimize some cost function of the form :

$$J(u(\cdot)) \ = \ \int_0^T l(t,x(t),u(t)) + g(x(T)), \tag{1.12}$$

where

$$l : [0,T] \times \mathbb{R} \times \mathcal{A} \longrightarrow \mathbb{R},$$

$$g : \mathbb{R} \longrightarrow \mathbb{R}.$$

That is, the function l inflicts a running cost and the function g inflicts a terminal cost. We now assume that there exists a control $u^*(t)$ which is optimal, i.e.

$$J(u^*(\cdot)) \ = \ \inf_u J(u(\cdot)).$$

We denote by $x^*(t)$ the solution to (1.11) with the optimal control $u^*(t)$. We are going to derive necessary conditions for optimality, for this we make small perturbation of the optimal control. Therefore we introduce a so-called spike variation, i.e. a control which is equal to u^* except on some small time interval :

$$u^{\varepsilon}(t) = \begin{cases} v & \text{for } \tau - \varepsilon \leq t \leq \tau, \\ u^*(t) & \text{otherwise.} \end{cases} \tag{1.13}$$

We denote by $x^{\varepsilon}(t)$ the solution to (1.11) with the control $u^{\varepsilon}(t)$. We set that $x^{*}(t)$ and $x^{\varepsilon}(t)$ are equal up to $t=\tau-\varepsilon$ and that

$$\begin{aligned} x^{\varepsilon}(\tau)-x^{*}(\tau) &= (f(\tau,x^{\varepsilon}(\tau),v)-f(\tau,x^{*}(\tau),u^{*}(\tau)))\varepsilon+o(\varepsilon) \\ &= (f(\tau,x^{*}(\tau),v)-f(\tau,x^{*}(\tau),u^{*}(\tau)))\varepsilon+o(\varepsilon), \end{aligned} \tag{1.14}$$

where the second equality holds since $x^{\varepsilon}(\tau)-x^{*}(\tau)$ is of order ε. We look at the Taylor expansion of the state with respect to ε. Let

$$z(t)=\frac{\partial}{\partial\varepsilon}x^{\varepsilon}(t)\mid_{\varepsilon=0},$$

i.e. the Taylor expansion of $x^{\varepsilon}(t)$ is

$$x^{\varepsilon}(t) = x^{*}(t)+z(t)\varepsilon+o(\varepsilon). \tag{1.15}$$

Then, by (1.14)

$$z(\tau) = f(\tau,x^{*}(\tau),v)-f(\tau,x^{*}(\tau),u^{*}(\tau)). \tag{1.16}$$

Moreover, we can derive the following differential equation for $z(t)$.

$$\begin{aligned} dz(t) &= \frac{\partial}{\partial\varepsilon}dx^{\varepsilon}(t)\mid_{\varepsilon=0} \\ &= \frac{\partial}{\partial\varepsilon}f(t,x^{\varepsilon}(t),u^{\varepsilon}(t))dt\mid_{\varepsilon=0} \\ &= f_x(t,x^{\varepsilon}(t),u^{\varepsilon}(t))\frac{\partial}{\partial\varepsilon}x^{\varepsilon}(t)dt\mid_{\varepsilon=0} \\ &= f_x(t,x^{*}(t),u^{*}(t))z(t)dt, \end{aligned}$$

where f_x denotes the derivative of f with respect to x. If we for the moment assume that

$l = 0$, the optimality of $u^*(t)$ leads to the inequality

$$\begin{aligned} 0 \leq \frac{\partial}{\partial \varepsilon} J(u^\varepsilon)\Big|_{\varepsilon=0} &= \frac{\partial}{\partial \varepsilon} g\left(x^\varepsilon(T)\right)|_{\varepsilon=0} \\ &= g_x\left(x^\varepsilon(T)\right) \frac{\partial}{\partial \varepsilon} x^\varepsilon(T)\,|_{\varepsilon=0} \\ &= g_x\left(x^*(T)\right) z(T). \end{aligned}$$

We shall use duality to obtain a more explicit necessary condition from this. To this end we introduce the adjoint equation :

$$\begin{cases} d\Psi(t) &= -f_x(t, x^*(t), u^*(t))\Psi(t)dt, t \in [0, T], \\ \Psi(T) &= g_x(x^*(T)). \end{cases}$$

Then it follows that

$$d(\Psi(t)z(t)) = 0,$$

i.e. $\Psi(t)z(t)) =$ constant. By the terminal condition for the adjoint equation we have

$$\Psi(t)z(t) = g_x(x^*(T))z(T) \geq 0, \text{ for all } 0 \leq t \leq T.$$

In particular, by (1.16)

$$\Psi(\tau)\left(f(\tau, x^*(\tau), v) - f(\tau, x^*(\tau), u^*(\tau))\right) \geq 0.$$

Since τ was chosen arbitrarily, this is equivalent to

$$\Psi(t) f(t, x^*(t), u^*(t)) = \inf_{v \in \mathcal{U}} \Psi(t) f(t, x^*(t), v), \text{ for all } 0 \leq t \leq T.$$

By repeating the calculations above for this two-dimensional system, one can derive the necessary condition

$$H(t, x^*(t), u^*(t), \Psi(t)) = \inf_{v \in \mathcal{U}} H(t, x^*(t), v, \Psi(t)) \text{ for all } 0 \leq t \leq T, \tag{1.17}$$

where H is the so-called Hamiltonian (sometimes defined with a minus sign which turns the minimum condition above into a maximum condition) :

$$H(x, u, \Psi) = l(x, u) + \Psi f(x, u),$$

and the adjoint equation is given by

$$\begin{cases} d\Psi(t) = -(l_x(t, x^*(t), u^*(t)) + f_x(t, x^*(t), u^*(t))\Psi(t))dt, \\ \Psi(T) = g_x(x^*(T)). \end{cases} \tag{1.18}$$

The minimum condition (1.17) together with the adjoint equation (1.18) specifies the Hamiltonian system for our control problem.

The stochastic maximum principle. Stochastic control is the extension of optimal control to problems where it is of importance to take into account some uncertainty in the system. One possibility is then to replace the differential equation by an SDE :

$$dx(t) = f(t, x(t), u(t))dt + \sigma(t, x(t))dW(t), t \in [0, T], \tag{1.19}$$

where f and σ are deterministic functions and the last term is an Itô integral with respect to a Brownian motion W defined on a probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$.
More generally, the diffusion coefficient σ may has an explicit dependence on the control : $t \in [0, T]$.

$$dx(t) = f(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), \tag{1.20}$$

The cost function for the stochastic case is the expected value of the cost function (1.12), i.e. we want to minimize

$$J(u(\cdot)) = E\left[\int_0^T l(t,x(t),u(t)) + g(x(T))\right].$$

For the case (1.19) the adjoint equation is given by the following Backward SDE :

$$\begin{cases} -d\Psi(t) &= \{f_x(t,x^*(t),u^*(t))\Psi(t) + \sigma_x(t,x^*(t))Q(t) \\ & \quad +(l_x(t,x^*(t),u^*(t))\}dt - Q(t)dW(t), \\ \Psi(T) &= g_x(x^*(T)). \end{cases} \tag{1.21}$$

A solution to this backward SDE is a pair $(\Psi(t), Q(t))$ which fulfills (1.21). The Hamiltonian is

$$H(x,u,\Psi(t),Q(t)) = l(t,x,u) + \Psi(t)f(t,x,u) + Q(t)\sigma(t,x),$$

and the maximum principle reads for all $0 \leq t \leq T$,

$$H(t,x^*(t),u^*(t),\Psi(t),Q(t)) = \inf_{u\in\mathcal{U}} H(t,x^*(t),u,\Psi(t),Q(t)) \text{ a.s.} \tag{1.22}$$

Noting that there is also third case : if the state is given by (1.20) but the action space $\mathcal{A}$ is assumed to be convex, it is possible to derive the maximum principle in a local form. This is accomplished by using a convex perturbation of the control instead of a spike variation, see Bensoussan 1983 [14]. The necessary condition for optimality is then given by the following : for all $0 \leq t \leq T$

$$E\int_0^T H_u(t,x^*(t),u^*(t),\Psi^*(t),Q^*(t))\,(u-u^*(t))\,dt \geq 0.$$

Chapitre 2

Derivatives on the Wasserstein space and class of controls

2.1 Kantorovich Distance Between Probability Measures

The *Monge-Kantorovich* Distance is a metric between two probability measures on a metric space. The *Monge-Kantorovich distance* has its origins in the mathematical theory of mass transportation. In 1781, *Monge* first proposed the mathematical problem of optimizing the cost of moving a pile of soil from a given starting configuration to a given ending configuration. In his original formulation the problem was highly nonlinear, thus extremely difficult. In 1942, *Kantorovich* introduced another simple and relaxed version of this problem.

The Kantorovich metric arises in very different contexts and under different names. In statistical applications it was known as the Wasserstein distance and more recently it appeared with the development of fractal geometry and its applications to computer graphics under the name of *Hutchinson distance, see* [22].

To be more precise, we assume that probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ is rich enough in the sense that for every $\mu \in \mathbb{X}_2\left(\mathbb{R}^d\right)$, there is a random variable $\vartheta \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ such that $\mu = \mathcal{P}_\vartheta$.

2.2 L-derivatives with respect to probability measures

Now, we recall briefly the innovative notion of L-derivatives with respect to probability distribution over Wasserstein spaces, which was studied by Lions [94], and Cardaliaguet [27] and the pioneering work by Cardaliaguet et. al. [24] in their study of the so-called master equation in mean field game systems.

The main idea is to identify a distribution $\mu \in \mathbb{X}_2\left(\mathbb{R}^d\right)$ with a random variables $\vartheta \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ so that $\mu = \mathcal{P}_\vartheta$.

Let $\mathbb{X}_2\left(\mathbb{R}^d\right)$ be the space of all probability measures μ on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$ with finite second moment, i.e, $\int_{\mathbb{R}^d} |x|^2 \mu(\mathrm{d}x) < +\infty$, endowed with the following Wasserstein metric $\mathbb{D}_2(\cdot,\cdot)$; for $\mu, \nu \in \mathbb{X}_2\left(\mathbb{R}^d\right)$,

$$\mathbb{D}_2(\mu,\nu) = \inf_{\delta(\cdot,\cdot)\in\mathbb{X}_2(\mathbb{R}^{2d})} \left\{ \left[\int_{\mathbb{R}^{2d}} |x-y|^2 \delta\left(\mathrm{d}x, \mathrm{d}y\right) \right]^{\frac{1}{2}} \right\},$$

where $\delta\left(\cdot,\cdot\right) \in \mathbb{X}_2\left(\mathbb{R}^{2d}\right)$, $\delta\left(A, \mathbb{R}^d\right) = \mu\left(A\right)$, $\delta\left(\mathbb{R}^d, B\right) = v\left(B\right)$.

This distance is just the *Monge-Kankorovich distance* when $p = 2$. Moreover, it has been shown that $(\mathbb{X}_2(\mathbb{R}^n), \mathbb{D}\left(\cdot,\cdot\right))$ is a complete metric space.

Example : For example, if $\mu_1 = \delta_{x_1}$ and $\mu_2 = \delta_{x_2}$ be two degenerate Dirac measures located at points x_1 and x_2 (respect.,) in $\mathbb{R}$, then we have

$$\mathbb{D}_2\left(\mu_1, \mu_2\right) = \left|x_1 - x_2\right|.$$

2.3 Lift function

Definition 2.3.1 (Lift function) Let Φ be a given function such that $\Phi : \mathbb{X}_2\left(\mathbb{R}^d\right) \to \mathbb{R}$. We define the lift function $\widetilde{\Phi} : \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right) \to \mathbb{R}$ such that

$$\widetilde{\Phi}\left(Z\right) = \Phi\left(\mathcal{P}_Z\right), Z \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right).$$

Clearly, the lift function $\widetilde{\Phi}$ of Φ, depends only on the law of random variable $Z \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ and is independent of the choice of the representative Z.

A function $f : \mathbb{X}_2\left(\mathbb{R}^d\right) \to \mathbb{R}$ is said to be differentiable at $\mu_0 \in \mathbb{X}_2\left(\mathbb{R}^d\right)$ if there exists $Z_0 \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ with $\mu_0 = \mathcal{P}_{Z_0} \in \mathbb{X}_2\left(\mathbb{R}^d\right)$ such that its lift function $\widetilde{f}$ is *Fréchet differentiable* at Z_0. More precisely, there exists a continuous linear functional $D\widetilde{f}\left(\cdot\right) : \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right) \to \mathbb{R}$ such that

$$\begin{aligned}\widetilde{f}\left(Z_0+\tau\right) - \widetilde{f}\left(Z_0\right) &= \left\langle D\widetilde{f}\left(Z_0\right), \tau\right\rangle + o\left(\|\tau\|_2\right) \\ &= D_\tau f\left(\mu_0\right) + o\left(\|\tau\|_2\right),\end{aligned} \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ is the dual product on $\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$, and we will refer to $D_\tau f\left(\mu_0\right)$ as the Fréchet derivative of f at μ_0 in the direction τ.

In this case, we have

$$\begin{aligned}D_\tau f\left(\mu_0\right) &= \left\langle D\widetilde{f}\left(Z_0\right), \tau\right\rangle \\ &= \frac{d}{dt}\widetilde{f}\left(Z_0+t\tau\right)\Big|_{t=0}, \text{ with } \mu_0 = \mathcal{P}_{Z_0}.\end{aligned}$$

So,

$$D_\tau f\left(\mathcal{P}_{Z_0}\right) = \frac{d}{dt}\left[\widetilde{f}\left(Z_0+t\tau\right)\right]\Big|_{t=0}. \tag{2.2}$$

From (2.2), then we obtain the following form of the Taylor expansion

$$f\left(\mathcal{P}_{Z}\right)-f\left(\mathcal{P}_{Z_{0}}\right)=D_{\alpha} f\left(\mathcal{P}_{Z}\right)+\mathcal{E}(\tau), \tag{2.3}$$

where $\mathcal{E}(\tau)$ is of order $o\left(\|\tau\|_{2}\right)$ with $o\left(\|\tau\|_{2}\right) \rightarrow 0$ for $\tau(\cdot) \in \mathbb{L}^{2}\left(\mathcal{F} ; \mathbb{R}^{d}\right)$.

Riesz representation theorem Let H ba a Hilbert space Let f ba a continuous linear functional $f \in H^{*}$, then there exists a unique $y \in H$ such that

$$f(x)=\langle y, x\rangle,$$

for any $x \in H$, Moreover, $\|y\|=\|f\|$.

By using the *Riesz' representation theorem*, there is a unique random variable Z_{0} in the Hilbert space $\mathbb{L}^{2}\left(\mathcal{F} ; \mathbb{R}^{d}\right)$ such that

$$\left\langle D \widetilde{f}(Z), \tau\right\rangle=\left(Z_{0}, \tau\right)_{2}=E\left[\left(Z_{0}, \tau\right)_{2}\right],$$

where $\tau(\cdot) \in \mathbb{L}^{2}\left(\mathcal{F} ; \mathbb{R}^{d}\right)$.

It was shown, see the works of Lions [94], see also Cardaliaguet [27], Buckdahn, Li, and Ma [?], that there exists a Boral function $\psi\left[\mu_{0}\right]: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$, depending only on the law $\mu_{0}=\mathcal{P}_{Z}$ but not on the particular choice of the representative Z such that

$$Z_{0}=\psi\left[\mu_{0}\right](Z).$$

Thus, we can write (2.1) as $\forall \vartheta \in \mathbb{L}^{2}(\mathcal{F} ; \mathbb{R}^{d})$.

$$f\left(\mathcal{P}_{\vartheta}\right)-f\left(\mathcal{P}_{Z}\right)=\left(\psi\left[\mu_{0}\right](Z), \vartheta-Z\right)_{2}+o\left(\|\vartheta-Z\|_{2}\right).$$

We denote $\partial_{\mu} f\left(\mathcal{P}_{Z}, y\right)=\psi\left[\mu_{0}\right](y),\ y \in \mathbb{R}^{d}$. We note that for each $\mu \in \mathbb{X}_{2}\left(\mathbb{R}^{d}\right)$, $\partial_{\mu} f\left(\mathcal{P}_{Z}, \cdot\right)=\psi\left[\mathcal{P}_{Z}\right](\cdot)$ is only defined in a $\mathcal{P}_{Z}(d x)-a.e$ sense, where $\mu=\mathcal{P}_{Z}$.

2.4 Space of differentiable functions in $\mathbb{X}_2\left(\mathbb{R}^d\right)$

Space of differentiable functions in $\mathbb{X}_2\left(\mathbb{R}^d\right)$.

Definition 2.4.1. We say that the function $f \in \mathbb{C}_b^{1,1}\left(\mathbb{X}_2\left(\mathbb{R}^d\right)\right)$ if for all $\vartheta \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$, there exists a $\mathcal{P}_\vartheta$-modification of $\partial_\mu f\left(\mathcal{P}_\vartheta,\cdot\right)$ such that $\partial_\mu f : \mathbb{X}_2\left(\mathbb{R}^d\right)\times\mathbb{R}^d \to \mathbb{R}^d$ is bounded and Lipchitz continuous. That is for some $C > 0$, it holds that

(i) $\left|\partial_\mu f\left(\mu,x\right)\right| \leq C$, $\forall \mu \in \mathbb{X}_2\left(\mathbb{R}^d\right)$, $\forall x \in \mathbb{R}^d$;

(ii) $\left|\partial_\mu f\left(\mu_1,x\right) - \partial_\mu f\left(\mu_2,y\right)\right| \leq C\left(\mathbb{D}_2(\mu_1,\mu_2) + |x-y|\right)$, $\forall\ \mu_1,\mu_2 \in \mathbb{X}_2\left(\mathbb{R}^d\right)$, $\forall\ x,y \in \mathbb{R}^d$.

Second-order derivatives with respect to probability law : We present a second order derivatives with respect to measure of probability.

Let $g \in \mathbb{C}_b^{1,1}(\Gamma_2(\mathbb{R}^n))$ and consider the mapping $\left(\partial_\mu g\left(\cdot,\cdot\right)_1, \partial_\mu g\left(\cdot,\cdot\right)_2, ..., \partial_\mu g\left(\cdot,\cdot\right)_n\right)^\top$: $\Gamma_2(\mathbb{R}^n)\times\mathbb{R}^n \to \mathbb{R}^n$.

Definition 2.4.2. We say that the function $g \in \mathbb{C}_b^{2,1}(\mathbb{X}_2(\mathbb{R}^n))$ if $g \in \mathbb{C}_b^{1,1}(\mathbb{X}_2(\mathbb{R}^n))$ such that $\partial_\mu g(\cdot,x) : \mathbb{X}_2(\mathbb{R}^n) \to \mathbb{R}^n$

(1) $\partial_\mu g(\cdot,y)_i \in \mathbb{C}_b^{1,1}(\mathbb{X}_2(\mathbb{R}^n))$, $\forall y \in \mathbb{R}^n$ and $i \in \{1,2,...,n\}$.

(2) $\partial_\mu g(\mu,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable, for every $\mu \in \mathbb{X}_2(\mathbb{R}^n)$.

(3) The maps

$$\partial_x\partial_\mu g(\cdot,\cdot) : \mathbb{X}_2(\mathbb{R}^n)\times\mathbb{R}^n \to \mathbb{R}^n\otimes\mathbb{R}^n$$

and

$$\partial_\mu^2 g(P_{x_0},y,z) : \mathbb{X}_2(\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n\otimes\mathbb{R}^n$$

are bounded and Lipschitz continuous, where

$$\partial_\mu^2 g(P_{x_0},y,z) = \partial_\mu\left[\partial_\mu g(\cdot,y)\right](P_{x_0},z).$$

2.5 Control classes

Let $(\Omega,\mathcal{F},\mathcal{F}_{t\geq 0},P)$ be a complete filtred probability space.

1. Admissible control An admissible control is $\mathcal{F}_t$-adapted process $u(t)$ with values in a borelian $A \subset \mathbb{R}^n$

$$\mathcal{U} := \{u(\cdot) : [0,T] \times \Omega \to A : u(t) \text{ is } \mathcal{F}_t\text{-adapted}\}. \tag{2.4}$$

2. Optimal control The optimal control problem consists to minimize a cost functional $J(u)$ over the set of admissible control $\mathcal{U}$. We say that the control $u^*(\cdot)$ is an optimal control if

$$J(u^*(t)) \leq J(u(t)), \text{ for all } u(\cdot) \in \mathcal{U}.$$

3. Near-optimal control Let $\varepsilon > 0$, a control $u^\varepsilon(\cdot)$ is a near-optimal control (or ε-optimal) if for all control $u(\cdot) \in \mathcal{U}$ we have

$$J(u^\varepsilon(t)) \leq J(u(t)) + \varepsilon. \tag{2.5}$$

See for some applications.

4. Singular control. An admissible control is a pair $(u(\cdot), \xi(\cdot))$ of measurable $\mathbb{A}_1 \times \mathbb{A}_2-$valued, $\mathcal{F}_t-$adapted processes, such that $\xi(\cdot)$ is of bounded variation, non-decreasing continuous on the left with right limits and $\xi(0_-) = 0$. Since $d\xi(t)$ may be singular with respect to Lebesgue measure dt, we call $\xi(\cdot)$ the singular part of the control and the process $u(\cdot)$ its absolutely continuous part.

5. Feedback control : We say that $u(\cdot)$ is a feedback control if $u(\cdot)$ depends on the state variable $X(\cdot)$.

If $\mathcal{F}_t^X$ the natural filtration generated by the process X, then $u(\cdot)$ is a feedback control if $u(\cdot)$ is $\mathcal{F}_t^X-$adapted.

6. Robust control. In the problems formulated above, the dynamics of the control system is assumed to be known and fixed. Robust control theory is a method to measure the performance changes of a control system with changing system parameters. This is of course important in engineering systems, and it has recently been used in finance in relation with the theory of risk measure.

Indeed, it is proved that a coherent risk measure for an uncertain payoff $x(T)$ at time T is represented by :

$$\rho(-X(t)) = \sup_{Q\in\mathcal{M}} E^{Q}(X(T)),$$

where $\mathcal{M}$ is a set of absolutly continuous probability measures with respect to the original probability P.

7. Partial observation control problem It is assumed so far that the controller completely observes the state system. In many real applications, he is only able to observe partially the state via other variables (called observed variable) and there is noise in the observation system. For example in financial models, one may observe the asset price but not completely its rate of return and/or its volatility, and the portfolio investment is based only on the asset price information. This may be formulated in a general form as follows : we have a controlled (unobserved) process governed by the following SDE :

$$dx\left(t\right) = f\left(t, x\left(t\right), y(t), u\left(t\right)\right) dt + \sigma\left(t, x\left(t\right), y\left(t\right), u\left(t\right)\right) dW\left(t\right),$$

and $y\left(t\right)$ an observation process defined by

$$dy\left(t\right) = h\left(t, x\left(t\right), u\left(t\right)\right) dW\left(t\right),$$

where $B\left(t\right)$ is another Brownian motion, eventually correlated with $W\left(t\right)$. The control $u(t)$ is adapted with respect to the filtration generated by the observation F_t^Y and the cost functional to optimize is :

$$J\left(u\left(\cdot\right)\right) = E\left[h\left(x\left(T\right), y(T)\right) + \int_0^T g\left(t, x\left(t\right), y(t), u\left(t\right)\right) dt\right].$$

8. Ergodic control Some stochastic systems may exhibit over a long period a stationary behavior characterized by an invariant measure. This measure, if it does exists, is obtained by the average of the states over a long time. An ergodic control problem consists in optimizing over the long term some criterion taking into account this invariant measure. (See Pham [106], Borkar [16]). The cost functional is given by

$$\lim \sup_{T \to +\infty} \frac{1}{T} E \int_0^T f(x(t), u(t)) dt.$$

9. Random horizon In classicla problem, the time horizon is fixed until a deterministic terminal time T. In some real applications, the time horizon may be random, the cost functional is given by the following :

$$J(u(\cdot)) = E\left[h(x(\tau)) + \int_0^\tau g(t, x(t), y(t), u(t)) dt\right],$$

where τ s a finite random time.

10. Relaxed control The idea is to compactify the space of controls $\mathcal{U}$ by extending the definition of controls to include the space of probability measures on U. The set of relaxed controls $\mu_t(du)\, dt$, where μ_t is a probability measure, is the closure under weak* topology of the measures $\delta_{u(t)}(du)dt$ corresponding to usual, or strict, controls. This notion of relaxed control is introduced for deterministic optimal control problems in Young (*Young, L.C. Lectures on the calculus of variations and optimal control theory, W.B. Saunders Co., 1969.*) (See Borkar [16]).

11. Impulsive control. (Impulse control). Here one is allowed to reset the trajectory at stopping times τ_i from $X_{\tau_{i-}}$ (the value immediately before i) to a new (non-anticipative) value X_{τ_i}, resp., with an associated cost $M\left(X_{\tau_{i-}}, X_{\tau_i}\right)$. The purpose of the controller is to minimizes the cost functional :

$$
\begin{aligned}
J(u(\cdot)) = E \int_0^T & \exp\left[-\int_0^t C(X(s),u(s))ds\right] K(X(t),u(t)) \\
& + \sum_{\tau_i < T} \exp\left[-\int_0^{\tau_i} C(X(s),u(s))ds\right] M(X_\tau, X_{\tau_{i-}}) \\
& + \exp\left[-\int_0^{\tau_i} C(X(s),u(s))ds\right] h(X(T)).
\end{aligned}
$$

In this model, we should assume that $M(X_\tau, X_{\tau_{i-}}) > \delta$ for some $\delta > 0$ to avoid infinitely many jumps in a finite time interval.

Some recent examples and applications on control classes can be found in [16], [60], [106] and [120].

Chapitre 3

Stochastic intervention control of mean-field jump system with noisy observation via L-derivatives with application to finance

3.1 Introduction

In this chapter, we study stochastic optimal intervention control of mean-field jump system with noisy observation via L-derivatives on Wasserstein space of probability measures We derive the necessary conditions of optimality for partially observed optimal intervention control problems of mean-field type. The coefficients depend on the state of the solution process as well as of its probability distribution and the control variable. The proof of our main results are obtained by applying L-derivatives in the sense of Lions. In our control problem, there are two models of jumps for the state process, the inaccessible ones which come from the Poission process and the predictable ones which come from the intervention control Finally, we apply our result to study conditional mean-variance portfolio selection problem with interventions, where the foreign exchange interventions are intended to contain excessive fluctuations in foreign exchange rates and to stabilize them.

Since the development of nonlinear filtering theory, stochastic control problems under partial observation have received much attention and became a powerful tool in many fields with important applications, such as finance and economics, etc. In many situations, the states of the systems cannot be completely observed; however, some other processes related to the unobservable states can be observed. Such subjects have been discussed by many authors, such as Wang, Wu and Xiong [111], Wang, Zhang, and Zhang [115], Wang, Wu and Xiong [113], Bensoussan and Yam [18], Wang, Shi and Meng [112], Lakhdari, Miloudi and Hafayed [75], Miloudi et al [101], Abada, Hafayed and Meherrem [2].

General mean-field type stochastic differential equations (SDEs) are Itô's stochastic differential equations, where the coefficients of the state equation depend on the time variable, the state of the solution process as well as of its probability law. In his course at Collége de France [94], (refer to Cardaliaguet [27] for the written version) P.L. Lions introduced and studied the innovative notion of new derivatives with respect to measure over Wasserstein spaces. Strongly motivated by these works, Buckdahn et al, [19] proved the necessary conditions for general mean-field systems. Stochastic maximum principles for general mean-field models were later studied in [101, 91, 38].

Stochastic irregular (singular or impulse) control problems have received considerable attention in the literature. There are numerous papers by different authors investigating the stochastic optimal singular or impulse control problems, e.g., Cadenillas and Haussmann [25], Dufour and Miller [30], Hafayed and Abbas [42], Zhang [79], Jeanblanc-Piqué [80], Korn [74], Wu and Zhang [?]. An extensive list of recent references to singular control problem, with some applications in finance and economics can be found in [42, 65, 78, 81]. Optimal control problems for SDEs with jump processes have been investigated by many authors, see for instance, [21, 23, 83, 84]. A good account and an extensive list of references on jump processes can be founded in [82] for a comprehensive theoretical study of the topic.

In the present chapter, we study a new mean-field type intervention control problem. We establish a new set of necessary conditions of optimal intervention control for general mean-field jump systems. Our mean-field dynamic is governed by SDEs with a random measures and an independent Brownian motion, with noisy observation. The coefficients of our mean-field dynamic depend nonlinearly on both the state process as well as of its probability law. The control domain is assumed to be convex. The L-derivatives with respect to probability measure and the associate Itô-formula are applied to prove our main results. Noting that our general mean-field partially observed control problem occur naturally in the probabilistic analysis of financial optimization problems. Our model of partially observed intervention control problem play an important role in different fields of economics and finance, as conditional mean variance portfolio selection problem with discrete movement in incomplete market. Also, optimal consumption and portfolio problem under proportional transaction costs. Moreover, the exchange rate under uncertainty, where government has two means of influencing the foreign exchange rate of its own currency :

1. At all times t the government can choose the domestic interest rate.
2. At selected times τ_i the government, or bank can intervene in the foreign exchange market by selling or buying large amounts of foreign currency.

In our model of mean-field control problem, there are two types of jumps for the state processes, the inaccessible ones which come from the Poission process and the predictable ones which come from the intervention control.

As an illustration, by applying our result, conditional mean-variance portfolio selection problem with interventions with incomplete market is discussed. In financial markets three important objectives of interventions : to influence the level of the exchange rate, to dampen exchange rate volatility or supply liquidity to foreign exchange markets ; and to influence the amount of foreign reserves. Banks intervene in foreign exchange markets in order to achieve a variety of overall economic objectives, such as controlling inflation, maintaining competitiveness or maintaining financial stability.

The rest of this chapter is organized as follows. Sect. 2 begins with a formulation of the partially observed control problem of general mean-field differential equations with Poisson jump processes. We give the notations and definitions of the L-derivatives on the Wasserstein space via P.L. Lions sense and assumptions used throughout the work. In Sect. 3, we prove the necessary conditions of optimality which are our main results. Conditional mean-variance portfolio selection problem with interventions is also given in Sect. 4. At the end of this work, some discussions with concluding remarks and future developments are presented in the last Section.

3.2 Formulation of the problem and preliminaries

Spaces and notations. Let T is a fixed terminal time and $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ be a complete filtered probability space on which are defined two independent standard one-dimensional Brownian motions $W(\cdot)$ and $Y(\cdot)$. Let $\mathbb{R}^n$ is a n-dimensional Euclidean space, $\mathbb{R}^{n\times d}$ the collection of $n \times d$ matrices. Let $k(\cdot)$ be a stationary $\mathcal{F}_t$-Poisson point process with the characteristic measure $m\left(\mathrm{d}\theta\right)$. We denote by $\eta\left(d\theta, dt\right)$ the counting measure or Poisson measure defined on $\Theta \times \mathbb{R}_+$, where Θ is a fixed nonempty subset of $\mathbb{R}$ with its Borel σ-field $\mathcal{B}\left(\Theta\right)$ and set $\widetilde{\eta}\left(\mathrm{d}\theta, \mathrm{d}t\right) = \eta\left(\mathrm{d}\theta, \mathrm{d}t\right) - m\left(\mathrm{d}\theta\right)\mathrm{d}t$ satisfying $\int_\Theta (1 \wedge |\theta|^2) m\left(\mathrm{d}\theta\right) < \infty$ and $m\left(\Theta\right) < +\infty$.

Let $\mathcal{F}_t^W$, $\mathcal{F}_t^Y$ and $\mathcal{F}_t^\eta$ be the natural filtration generated by $W(\cdot)$, $Y(\cdot)$ and $\eta(\cdot,\cdot)$ respectively. We assume that $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^Y \vee \mathcal{F}_t^\eta \vee \mathcal{N}$, where $\mathcal{N}$ denotes the totality of $\mathcal{P}$-null sets. We denote by $\langle \cdot, \cdot \rangle$ (resp. $|\cdot|$) the scalar product (resp., norm), $E\left(\cdot\right)$ denotes the expectation on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$. Throughout this work, we denote by $\mathbb{L}^2\left(\mathcal{F}_t; \mathbb{R}^n\right)$ the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variable X, such that $E(|X|^2) < +\infty$ and by $\mathcal{M}^2\left([0,T]; \mathbb{R}\right)$: the space of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted measurable process $g(\cdot)$, such that $E\int_0^T\int_\Theta |g(t,\theta)|^2 m(\mathrm{d}\theta)\,\mathrm{d}t < +\infty$. Let $\mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ be the Hilbert space with inner product $(X,Y)_2 = E[X.Y]$,where $X,Y \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ and the norm $\|X\|_2^2 = (X,X)_2$. Let $\mathbb{X}_2(\mathbb{R}^d)$ be the space of all probability measures μ on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ with finite second moment, i.e, $\int_{\mathbb{R}^d}|x|^2\mu(\mathrm{d}x) < +\infty$, endowed with the following Wasserstein metric $\mathbb{D}_2(\cdot,\cdot)$; for $\mu,\nu \in \mathbb{X}_2(\mathbb{R}^d)$,

$$\mathbb{D}_2(\mu,\nu) = \inf_{\delta(\cdot,\cdot)\in\mathbb{X}_2(\mathbb{R}^{2d})}\left\{\left[\int_{\mathbb{R}^{2d}}|x-y|^2\delta(\mathrm{d}x,\mathrm{d}y)\right]^{\frac{1}{2}}\right\},$$

where $\delta(\cdot,\cdot)\in\mathbb{X}_2(\mathbb{R}^{2d})$, $\delta(A,\mathbb{R}^d) = \mu(A)$, $\delta(\mathbb{R}^d,B) = v(B)$.

3.2.1 Derivatives on the Wasserstein space

We would like to point out that the version of $\partial_\mu f(\mathcal{P}_\vartheta,\cdot)$, $\vartheta \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ indicated in the above definition is unique.

Let $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathcal{F}}_t,\widehat{\mathcal{P}})$ be a copy of the probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathcal{P})$. For any pair of random variable $(\vartheta_1,\vartheta_2) \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)\times\mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$, we let $(\widehat{\vartheta}_1,\widehat{\vartheta}_2)$ be an independent copy of $(\vartheta_1,\vartheta_2)$ defined on $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathcal{F}}_t,\widehat{\mathcal{P}})$. We consider the product probability space $(\Omega\times\widehat{\Omega},\mathcal{F}\otimes\widehat{\mathcal{F}},\mathcal{F}_t\otimes\widehat{\mathcal{F}}_t,\mathcal{P}\otimes\widehat{\mathcal{P}})$ and setting $(\widehat{\vartheta}_1,\widehat{\vartheta}_2)(w,\widehat{w}) = (\vartheta_1(\widehat{w}),\vartheta_2(\widehat{w}))$ for any $(w,\widehat{w})\in\Omega\times\widehat{\Omega}$. Let $(\widehat{u}(t),\widehat{x}(t))$ be an independent copy of $(u(t),x(t))$ so that $\mathcal{P}_{x(t)}=\widehat{\mathcal{P}}_{\widehat{x}(t)}$. We denote by $\widehat{E}(\cdot) = \widehat{E}_{\widehat{\mathcal{P}}}(\cdot)$ the expectation under probability measure $\widehat{\mathcal{P}}$ and $\mathcal{P}_X = \mathcal{P}\circ X^{-1}$ denotes the law of the random variable X.

Let $\mathbb{A}_1$ be a closed convex subset of $\mathbb{R}^k$ and $\mathbb{A}_2 := [0,+\infty)^m$.

Definition 3.2.1. An admissible continuous control $u(\cdot)$ is an $\mathcal{F}_t^Y$-adapted process with values in $\mathbb{A}_1$ satisfies $\sup_{t\in[0,T]}(E|u(t)|^n) < \infty$, $n = 2,3,\ldots$. We denote by $\mathcal{U}_1^Y$ the set of the admissible regular control variables.

Definition 3.2.2. An intervention control is a stochastic irregular process $\xi(\cdot)$ of measurable $\mathbb{A}_2-$valued, $\mathcal{F}^Y-$adapted processes, such that the process $\xi(\cdot):[0,T]\times\Omega\to\mathbb{A}_2$ is non-decreasing continuous on the right with left-limits, with bounded variation and

$\xi(0) = 0$. Moreover, $E(|\xi(T)|^p) < \infty$ for any $p \geq 2$. We denote by $\mathcal{U}_2^Y$ the set of the admissible intervention control variables.

Definition 3.2.3. An admissible combined control is a pair $(u(\cdot), \xi(\cdot))$ of measurable $\mathbb{A}_1 \times \mathbb{A}_2-$valued, $\mathcal{F}^Y-$adapted processes, such that the process $u(\cdot) : [0,T] \times \Omega \to \mathbb{A}_1$ is regular process and $\xi(\cdot) : [0,T] \times \Omega \to \mathbb{A}_2$ is an intervention control given by *Definition 2.2.* We denote by $\mathcal{U}_1^Y \times \mathcal{U}_2^Y$ the set of the admissible combined control variables.

3.2.2 Partially observed optimal intervention control Model

In this work, we formulate this problem mathematically as a combined stochastic continuous control and irregular control problem. We study partially observed optimal stochastic intervention control problem for systems governed by mean-field SDEs with correlated noisy between the system and the observation, allowing both classical and intervention control of the form : $t \in [0,T]$

$$\begin{cases} \mathrm{d}x^{u,\xi}(t) = f(t, x^{u,\xi}(t), \mathbb{P}_{x^{u,\xi}(t)}, u(t))\mathrm{d}t + \sigma(t, x^{u,\xi}(t), \mathbb{P}_{x^{u,\xi}(t)}, u(t))\mathrm{d}W(t) \\ \qquad + \int_\Theta g(t, x^{u,\xi}(t_-), \mathbb{P}_{x^{u,\xi}(t_-)}, u(t), \theta)\widetilde{\eta}(\mathrm{d}\theta, \mathrm{d}t) \\ \qquad + c(t, x^{u,\xi}(t), \mathbb{P}_{x^{u,\xi}(t)}, u(t))\mathrm{d}\widetilde{W}(t) + G(t)\mathrm{d}\xi(t), \\ x^{u,\xi}(0) = x_0, \end{cases} \tag{3.1}$$

where $\mathbb{P}_{x^{u,\xi}(t)} = \mathcal{P}\circ\left(x^{u,\xi}\right)^{-1}$denotes the law of the random variable $x^{u,\xi}$. The mappings

$$f : [0,T] \times \mathbb{R}^n \times \mathbb{X}_2(\mathbb{R}^d) \times \mathbb{A}_1 \to \mathbb{R}^n$$

$$\sigma : [0,T] \times \mathbb{R}^n \times \mathbb{X}_2(\mathbb{R}^d) \times \mathbb{A}_1 \to \mathcal{M}(\mathbb{R}^{n\times d})$$

$$c : [0,T] \times \mathbb{R}^n \times \mathbb{X}_2(\mathbb{R}^d) \times \mathbb{A}_1 \to \mathcal{M}(\mathbb{R}^{n\times d})$$

$$g : [0,T] \times \mathbb{R}^n \times \mathbb{X}_2(\mathbb{R}^d) \times \mathbb{A}_1 \times \Theta \to \mathcal{M}(\mathbb{R}^{n\times d})$$

$$G : [0,T] \to \mathbb{R}^n$$

are given deterministic functions.

Suppose that the state processes $x^{u,\xi}(\cdot)$ cannot be observed directly, but the controllers can observe a related noisy process $Y(\cdot)$, which is governed by the following equation

$$\begin{cases} dY(t) &= h(t, x^{u,\xi}(t), u(t))dt + d\widetilde{W}(t) \\ Y(0) &= 0, \end{cases} \tag{3.2}$$

where $h : [0,T] \times \mathbb{R}^n \times \mathbb{A}_1 \to \mathbb{R}^r$, and $\widetilde{W}(\cdot)$ is a stochastic process depending on the control $u(\cdot)$.

Consider the cost functional

$$\begin{aligned} J(u(\cdot), \xi(\cdot)) = E^u \Big[\int_0^T & l(t, x^{u,\xi}(t), \mathbb{P}_{x^{u,\xi}(t)}, u(t))\mathrm{d}t \\ & + \psi(x^{u,\xi}(T), \mathbb{P}_{x^{u,\xi}(t)}) + \int_{[0,T]} M(t)\mathrm{d}\xi(t) \Big]. \end{aligned} \tag{3.3}$$

Where $l : [0,T] \times \mathbb{R}^n \times \mathbb{X}_2(\mathbb{R}) \times \mathbb{A}_1 \to \mathbb{R}$, $\psi : \mathbb{R}^n \times \mathbb{X}_2(\mathbb{R}) \to \mathbb{R}$ and E^u stands for the mathematical expectation on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P}^u)$ defined by

$$E^u(X) = E_{\mathcal{P}^u}(X) = \int_\Omega X(w)\mathrm{d}\mathcal{P}^u(w).$$

In this work, we shall make use of the following standing assumptions.

Assumption (H 3.1) The maps $f, \sigma, c, l : [0,T] \times \mathbb{R} \times \mathbb{X}_2(\mathbb{R}) \times \mathbb{A}_1 \to \mathbb{R}$ and $\psi : \mathbb{R} \times \mathbb{X}_2(\mathbb{R}) \to \mathbb{R}$ are measurable in all variables. Moreover, $f(t,\cdot,\cdot,u)$, $\sigma(t,\cdot,\cdot,u)$, $c(t,\cdot,\cdot,u)$, $l(t,\cdot,\cdot,u)$, $g(t,\cdot,\cdot,u,\theta) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times \mathbb{X}_2(\mathbb{R}), \mathbb{R})$ and $\psi(\cdot,\cdot) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times \mathbb{X}_2(\mathbb{R}), \mathbb{R})$ for all $u \in \mathbb{A}_1$.

Assumption (H 3.2) Let $\varphi(x,\mu) = f(t,x,\mu,u)$, $\sigma(t,x,\mu,u)$, $c(t,x,\mu,u)$, $l(t,x,\mu,u)$, $g(t,x,\mu,u,\theta)$, $\psi(x,\mu)$, the function $\varphi(\cdot,\cdot)$ satisfies the following properties :

(**1**) For fixed $x \in \mathbb{R}$ and $\mu \in \mathbb{X}_2(\mathbb{R})$, the function $\varphi(\cdot,\mu) \in \mathbb{C}_b^1(\mathbb{R})$ and $\varphi(x,\cdot) \in$

$\mathbb{C}_b^{1,1}(\mathbb{X}_2\left(\mathbb{R}^d\right), \mathbb{R})$. All the derivatives φ_x and $\partial_\mu \varphi$, for $\varphi = f, \sigma, c, l, \psi$ are bounded and Lipschitz continuous, with Lipschitz constants independent of $u \in \mathbb{A}_1$. Moreover, there exists a constants $C\left(T, m(\Theta)\right) > 0$ such that

$$\sup_{\theta\in\Theta}\left|g_x\left(t, x, \mu, u, \theta\right)\right| + \sup_{\theta\in\Theta}\left|\partial_\mu g\left(t, x, \mu, u, \theta\right)\right| \leq C.$$

$$\begin{aligned}&\sup_{\theta\in\Theta}\left|g_x\left(t, x, \mu, u, \theta\right) - g_x\left(t, x', \mu', u, \theta\right)\right| + \sup_{\theta\in\Theta}\left|\partial_\mu g\left(t, x, \mu, u, \theta\right) - \partial_\mu g\left(t, x', \mu', u, \theta\right)\right| \\ &\leq C\left[\left|x - x'\right| + \mathbb{D}_2(\mu, \mu')\right].\end{aligned}$$

(**2**) The functions f, σ, c, g and l are continuously differentiable with respect to control variable $u\left(\cdot\right)$, and all their derivatives are continuous and bounded. Moreover, there exists a constants $C = C\left(T, m(\Theta)\right) > 0$ such that

$$\sup_{\theta\in\Theta}\left|g_u\left(t, x, \mu, u, \theta\right)\right| \leq C.$$

The function h is continuously differentiable in x and continuous in v, its derivatives and h are all uniformly bounded which satisfies the following *Novikov's condition* :

$$E\left(\exp\left[\frac{1}{2}\int_0^t \left|h(s, x^{u,\xi}(s), u(s))\right|^2 \mathrm{d}s\right]\right) < \infty. \tag{3.4}$$

Assumption (H 3.3) The functions $G\left(\cdot\right) : [0, T] \times \Omega \to \mathbb{R}$, and $M\left(\cdot\right) : [0, T] \times \Omega \to \mathbb{R}^+$ are continuous and bounded.

Clearly, assumption (H 3.3) allows us to define integrals of the form $\int_{[0,T]} G(t)\mathrm{d}\xi(t)$ and $\int_{[0,T]} M(t)\mathrm{d}\xi(t)$. Moreover, under assumptions (H1), (H2) and (H3), for any $\left(u\left(\cdot\right), \xi(\cdot)\right) \in$

$\mathcal{U}_1^Y \times \mathcal{U}_2^Y$ the mean-field equation (3.1) admits a unique strong solution $x^{u,\xi}(t)$ given by

$$\begin{aligned}x^{u,\xi}(t) = x_0 + \int_0^t f(s, x^{u,\xi}(s), \mathcal{P}[x^{u,\xi}(s)], u(s))\mathrm{d}s + \sigma(s, x^{u,\xi}(s), \mathcal{P}[x^{u,\xi}(s)], u(s))\mathrm{d}W(s) \\ + c(s, x^{u,\xi}(s), \mathcal{P}[x^{u,\xi}(s)], u(s))\mathrm{d}\widetilde{W}(s) \\ + \int_0^t \int_\Theta g(s, x^{u,\xi}(s_-), \mathcal{P}[x^{u,\xi}(s_-)], u(s), \theta)\widetilde{\eta}(\mathrm{d}\theta, \mathrm{d}s) \\ + \int_{[0,T]} G(s)\mathrm{d}\xi(s).\end{aligned}$$

We define the $\mathcal{F}_t^Y-$martingale $\alpha^u(t)$ which is the solution of the equation

$$\begin{cases} \mathrm{d}\alpha^u(t) = \alpha^u(t) h\left(t, x^{u,\xi}(t), u(t)\right)\mathrm{d}Y(t), \\ \alpha^u(0) = 1. \end{cases} \tag{3.5}$$

This martingale allowed to define a new probability $\mathcal{P}^u$ on the space $(\Omega, \mathcal{F})$, to emphasize the fact that it depend on the control $u(\cdot)$. It is given by the *Radon-Nikodym derivative* :

$$\left.\frac{\mathrm{d}\mathcal{P}^u}{\mathrm{d}\mathcal{P}}\right|_{\mathcal{F}_t^Y} = \alpha^u(t). \tag{3.6}$$

From the linear equation (3.5), and by a simple computation, we can get

$$\alpha^u(t) = \exp\left[\int_0^t h(s, x^{u,\xi}(s), u(s))\mathrm{d}Y(s) - \frac{1}{2}\int_0^t \left|h(s, x^{u,\xi}(s), u(s))\right|^2 \mathrm{d}s\right]. \tag{3.7}$$

This type of equations are called *Doléan-Dade's exponential.* We note that $E^u(\varphi(X))$ refers to the expected value of $\Psi(X)$ with respect to the probabilily law $\mathcal{P}^u$. Moreover, since $\mathrm{d}\mathcal{P}^u = \alpha^u(t)\mathrm{d}\mathcal{P}$, we have

$$\begin{aligned} E^u(\varphi(X)) = E_{\mathcal{P}^u}(\varphi(X)) &= \int_\Omega \varphi(X(w))\mathrm{d}\mathcal{P}^u(w), \\ &= \int_\Omega \varphi(X(w))\alpha^u(t)\mathrm{d}\mathcal{P}(w), \\ &= E_{\mathcal{P}}(\alpha^u(t)\varphi(X)) = E\left[\alpha^u(t)\varphi(X)\right]. \end{aligned}$$

Applying Itô's formula, we can prove that $\sup_{t\in[0,T]} E\left(|\alpha^u(t)|^n\right) < +\infty$, $n > 1$. By Girsanov's theorem and assumptions (H 3.1), (H 3.2) and (H3.3), $\mathcal{P}^u$ is a new probability measure of density $\alpha^u(t)$. The process

$$\widetilde{W}(t) = Y(t) - \int_0^t h(s, x^{u,\xi}(s), u(s))\mathrm{d}s,$$

is a standard Brownian motion independent of $B(\cdot)$ and x_0 on the new probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P}^u)$.

By Radon-Nikodym derivative (3.6), with the martingale property of $\alpha^u(t)$, the cost functional (3.3) can be written as

$$\begin{aligned} J(u(\cdot), \xi(\cdot)) = E\Bigg[\int_0^T \alpha^u(t) l(t, x^{u,\xi}(t), \mathbb{P}_{x^{u,\xi}(t)}, u(t))\mathrm{d}t + \alpha^u(T)\psi(x^{u,\xi}(T), \mathcal{P}[x^{u,\xi}(T)]) \\ + \int_{[0,T]} \alpha^u(t) M(t)\mathrm{d}\xi(t)\Bigg]. \end{aligned} \tag{3.8}$$

The main purpose of this work is to prove stochastic maximum principle, also called necessary optimality conditions for the partially observed optimal control of mean-field Poisson jumps.

Notice that the jumps of a singular control $\xi(\cdot)$ at any time t_j denote by $\Delta\xi(t_j) = \xi(t_j) -$

$\xi(t_{j-})$ and we define the continuous part of the intervention control by

$$|\xi|(t) = \xi(t) - \sum_{0 \leq t_j \leq t} \Delta\xi(t_j).$$

Here $|\xi|(t)$ the process obtained by removing the jumps of $\xi(t)$.

Throughout this work, we distinguish between the jumps caused by the intervention control $\xi(\cdot)$ and the jumps caused by the random Poisson measure at any jumping time t.The jumps of $x^{u,\xi}(t)$ caused by the intervention control $\xi(\cdot)$ by

$$\triangle_{\xi} x^{u,\xi}(t) = G(t)\triangle\xi(t) = G(t)(\xi(t) - \xi(t_-)), \tag{3.9}$$

and the jumps of $x^{u,\xi}(t)$ caused by the Poisson measure of $\widetilde{\eta}(\theta, t)$ by

$$\begin{aligned} \triangle_{\eta} x^{u,\xi}(t) &= \int_{\Theta} g\left(t, x^{u,\xi}(t_-), \mathcal{P}\left[x^{u,\xi}(t_-)\right], u(t_-), \theta\right) \widetilde{\eta}\left(\mathrm{d}\theta, \{t\}\right) \qquad (3.10) \\ &= \begin{cases} g\left(t, x^{u,\xi}(t_-), \mathcal{P}[x^{u,\xi}(t_-)], u(t_-), \theta\right) : \text{if } \xi \text{ has a jump of size } \theta \text{ at time } t. \\ 0 : \text{otherwise,} \end{cases} \end{aligned}$$

where $\widetilde{\eta}\left(\mathrm{d}\theta, \{t\}\right)$ means the jump in the Poisson random measure, occurring at time t.

Finally, the general jump of the state processes $x^{u,\xi}(\cdot)$ at any jumping time t is given by

$$\triangle x^{u,\xi}(t) = x^{u,\xi}(t) - x^{u,\xi}(t_-) = \triangle_{\xi} x^{u,\xi}(t) + \triangle_{\eta} x^{u,\xi}(t). \tag{3.11}$$

3.3 Necessary conditions for optimal intervention control in Wasserstein space

In this section, we prove the necessary conditions of optimality for our partially observed optimal intervention control problem of general mean-field stochstic differential equations with jumps. The proof is based on Girsanov's theorem, the derivatives with respect to probability measure in Wasserstein space and by introducing the variational equations with some estimates of their solutions.

3.3.1 Main results

Hamiltonian. We define the Hamiltonian

$$H : [0,T] \times \mathbb{R} \times \mathbb{X}_2(\mathbb{R}) \times \mathbb{A}_1 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

associated with our control problem by

$$\begin{aligned} & H(t,x,\mu,u,\Phi(t),Q(t),\overline{Q}(t),K(t),R(t,\theta)) \\ & = l(t,x,\mu,u) + f(t,x,\mu,u)\Phi(t) + \sigma(t,x,\mu,u)Q(t) \\ & + c(t,x,\mu,u)\overline{Q}(t) + h(t,x,u)K(t) + \int_\Theta g(t,x,\mu,u,\theta)R(t,\theta)m(\mathrm{d}\theta). \end{aligned} \tag{3.12}$$

Adjoint equations. We are now ready to introduce two new adjoint equations that will be the building blocks of the stochastic maximum principle and

$$\begin{cases} -\mathrm{d}\Phi(t) & = \Big[f_x(t)\Phi(t) + \widehat{E}\left[\partial_\mu \widehat{f}(t)\widehat{\Phi}(t)\right] + \sigma_x(t)Q(t) + \widehat{E}\left[\partial_\mu\widehat{\sigma}(t)\widehat{Q}(t)\right] \\ & \quad + c_x(t)\overline{Q}(t) + \widehat{E}\left[\partial_\mu\widehat{c}(t)\widehat{\overline{Q}}(t)\right] + l_x(t) + \widehat{E}\left[\partial_\mu\widehat{l}(t)\right] \\ & \quad + \int_\Theta \left[g_x(t,\theta)R(t,\theta) + \widehat{E}\left[\partial_\mu\widehat{g}(t,\theta)\widehat{R}(t,\theta)\right]\right]m(\mathrm{d}\theta) + h_x(t)K(t)\Big]\mathrm{d}t \\ & \quad - Q(t)\mathrm{d}W(t) - \overline{Q}(t)\mathrm{d}\widetilde{W}(t) - \int_\Theta R(t,\theta)\widetilde{\eta}(\mathrm{d}\theta,\mathrm{d}t), \\ \Phi(T) & = \psi_x(x(T),\mathcal{P}[x(T)]) + \widehat{E}\left[\partial_\mu\psi(\widehat{x}(T),\mathcal{P}[x(T)];x(T))\right]. \end{cases} \tag{3.13}$$

and

$$\begin{cases} -\mathrm{d}y(t) & = l(t)dt - z(t)\mathrm{d}W(t) - K(t)\mathrm{d}\widetilde{W}(t) - \int_\Theta R(t,\theta)\widetilde{\eta}(\mathrm{d}\theta,\mathrm{d}t), \\ y(T) & = \psi(x(T),\mathcal{P}[x(t)]), \end{cases} \tag{3.14}$$

Clearly, under assumptions (H 3.1) and (H 3.2), it is easy to prove that BSDEs (3.14) and (3.13) admits a unique strong solutions, given by

$$y(t) = \psi(x(T), \mathcal{P}[x(T)]) + \int_t^T l(s)\mathrm{d}s - \int_t^T z(s)\,\mathrm{d}W(s) - \int_t^T K(s)\,\mathrm{d}\widetilde{W}(s) - \int_t^T \int_\Theta R(s,\theta)\,\widetilde{\eta}(\mathrm{d}\theta, \mathrm{d}s).$$

and

$$\begin{aligned}\Phi(t) &= \psi_x(x(T), \mathcal{P}[x(t)]) + \widehat{E}[\partial_\mu \psi(\widehat{x}(T), \mathcal{P}[x(t)]; x(T))]. \\ &+ \int_t^T \Big[f_x(s)\Phi(s) + \widehat{E}\left[\partial_\mu \widehat{f}(s)\widehat{\Phi}(s)\right] + \sigma_x(s) Q(s) + \widehat{E}\left[\partial_\mu \widehat{\sigma}(s)\widehat{Q}(s)\right] \\ &+ c_x(s)\overline{Q}(s) + \widehat{E}\left[\partial_\mu \widehat{c}(s)\widehat{\overline{Q}}(s)\right] + l_x(s) + \widehat{E}\left[\partial_\mu \widehat{l}(s)\right] \\ &+ \int_\Theta \left[g_x(s,\theta) R(s,\theta) + \widehat{E}\left[\partial_\mu \widehat{g}(s,\theta)\widehat{R}(s,\theta)\right]\right] m(d\theta) + h_x(s) K(s) \Big] \mathrm{d}s \\ &- \int_t^T Q(s)\mathrm{d}W(s) - \int_t^T \overline{Q}(s)\mathrm{d}\widetilde{W}(s) - \int_t^T \int_\Theta R(s,\theta)\,\widetilde{\eta}(\mathrm{d}\theta, \mathrm{d}s).\end{aligned}$$

The main result of this chaptre is stated in the following theorem.

Theorem 3.3.1 Let assumptions (H 3.1) (H 3.2) and (H 3.3) hold. Let $(u^*(\cdot), \xi^*(t), x^*(\cdot))$ be the optimal solution of the control problem (3.1)-(3.3).

Then there exists $(\Phi(\cdot), Q(\cdot), \overline{Q}(\cdot), K(\cdot), R(\cdot, \theta))$ *solution of* (3.13)-(3.14) *such that for any* $(u, \xi) \in \mathbb{A}_1 \times \mathbb{A}_2$, we have $\mathbb{P}-a.s.,\ a.e.t \in [0,T]$,

$$\begin{aligned}0 \leq E^u\left[H_u(t, x^*(t), \mathcal{P}[x^*(t)], u^*(t), \Phi(t), Q(t), \overline{Q}(t), K(t), R(t,\theta))(u(t) - u^*(t)) \mid \mathcal{F}_t^Y\right] \\ + E^u\left[\int_{[0,T]} (M(t) + G(t)\Phi(t))\mathrm{d}(\xi - \xi^*)(t) \mid \mathcal{F}_t^Y\right],\end{aligned} \tag{3.15}$$

where the Hamiltonian function H *is defined by* (3.12).

3.3.2 Proof of main results

Double convex perturbation. To prove our main result, the approach that we use is based on a double perturbation of the optimal control. This perturbation is described as follows :

Let $(u(\cdot), \xi(\cdot)) \in \mathcal{U}_1^Y \times \mathcal{U}_2^Y$, be any given admissible control. Let $\varepsilon \in (0,1)$, and write

$$u^{\varepsilon}(\cdot) = u^*(\cdot) + \varepsilon v(\cdot) \quad \text{where } v(t) = u(t) - u^*(t), \tag{3.16}$$

and

$$\xi^{\varepsilon}(t) = \xi^*(t) + \varepsilon\zeta(t) \text{ where } \zeta(t) = \xi(t) - \xi^*(t), \tag{3.17}$$

where ε a sufficiently small $\varepsilon > 0$. Here $(u^{\varepsilon}(\cdot), \xi^{\varepsilon}(\cdot))$ is the so called convex perturbation of $(u^*(\cdot), \xi^*(\cdot))$ defined as follows : for any $t \in [0,T]$

$$(u^{\varepsilon}(t), \xi^{\varepsilon}(t)) = (u^*(t), \xi^*(t)) + \varepsilon\left[(u(t), \xi(t)) - (u^*(t), \xi^*(t))\right],$$

Denote by $x^{\varepsilon}(\cdot) = x^{u^{\varepsilon},\xi^{\varepsilon}}(\cdot)$ the solution of (3.1) associated with $(u^{\varepsilon}(\cdot), \xi^{\varepsilon}(\cdot))$ and by $\alpha^{\varepsilon}(\cdot)$ the solution of (3.5) corresponding to $u^{\varepsilon}(\cdot)$.

We denote by $x^{\varepsilon}(\cdot), x(\cdot), \alpha^{\varepsilon}(\cdot), \alpha(\cdot)$ the state trajectories of (3.1) and (3.5) corresponding respectively to $u^{\varepsilon}(\cdot)$ and $u(\cdot)$.

Short-hand notation. For simplification, we introduce the short-hand notation

$$\varphi(t) = \varphi\left(t, x^{u,\xi}(t), \mathbb{P}_{x^{u,\xi}(t)}, u(t)\right),$$
$$\varphi^{\varepsilon}(t) = \varphi(t, x^{\varepsilon}(t), \mathcal{P}\left[x^{\varepsilon}(t)\right], u^{\varepsilon}(t)),$$

and

$$g(t,\theta) = g(t, x^{u,\xi}(t_-), \mathcal{P}\left[x^{u,\xi}(t_-)\right], u(t), \theta), \quad h(t) = h\left(t, x^{u,\xi}(t), u(t)\right),$$

$$g^{\varepsilon}(t,\theta) = g(t, x^{\varepsilon}(t_-), \mathcal{P}\left[x^{\varepsilon}(t_-)\right], u^{\varepsilon}(t), \theta), \qquad h^{\varepsilon}(t) = h\left(t, x^{\varepsilon}(t), u^{\varepsilon}(t)\right),$$

where g, h and $\varphi = f, \sigma, c, l$ as well as their partial derivatives with respect to x and u. Also, we will denote for $\varphi = f, \sigma, c, l$ and g :

$$\partial_\mu \varphi(t) = \partial_\mu \varphi\left(t, x(t), \mathcal{P}\left[x(t)\right], u(t); \widehat{x}(t)\right),$$

$$\partial_\mu \widehat{\varphi}(t) = \partial_\mu \varphi\left(t, \widehat{x}(t), \mathcal{P}\left[\widehat{x}(t)\right], \widehat{u}(t); x(t)\right),$$

and

$$\partial_\mu g(t,\theta) = \partial_\mu g\left(t, x(t_-), \mathcal{P}\left[x(t_-)\right], u(t), \theta; \widehat{x}(t)\right),$$

$$\partial_\mu \widehat{g}(t,\theta) = \partial_\mu g\left(t, \widehat{x}(t), \mathcal{P}\left[\widehat{x}(t_-)\right], \widehat{u}(t), \theta; x(t)\right).$$

In order to prove our main result in Theorem 3.3.1, we present some auxiliary results

Lemma 3.3.2 Suppose that assumptions (H 3.1), (H 3.2) and (H 3.3) hold. Then, we have

$$\lim_{\varepsilon\to 0} E\left[\sup_{0\le t\le T} \left|x^\varepsilon(t) - x^*(t)\right|^2\right] = 0. \tag{3.18}$$

Proof Applying standard estimates, the *Burkholder-Davis-Gundy inequality*, and Proposition 5.1 in Bouchard and Elie [21] we have

$$\begin{aligned}
&E\left[\sup_{0\le t\le T} \left|x^\varepsilon(t) - x^*(t)\right|^2\right] \\
&\le E\int_0^t \left|f^\varepsilon(s) - f^*(s)\right|^2 ds + E^u\int_0^t \left|\sigma^\varepsilon(s) - \sigma^*(s)\right|^2 \mathrm{d}s \\
&+ E\int_0^t \left|c^\varepsilon(s) - c^*(s)\right|^2 \mathrm{d}s + E\int_0^t\int_\Theta \left|g^\varepsilon(s,\theta) - g^*(s,\theta)\right|^2 m(\mathrm{d}\theta)\, ds \\
&+ E\left|\int_{[0,t]} G(s)\mathrm{d}\left(\xi^\varepsilon - \xi^*\right)(s)\right|^2,
\end{aligned}$$

According to the Lipschitz conditions on the coefficients f, σ, c and g with respect to x, μ

and u, (assumptions (H 3.2)-(H 3.3)), we obtain the following estimation :

$$\begin{aligned} E\left[\sup_{0\leq t\leq T}\left|x^{\varepsilon}(t)-x^{*}(t)\right|^{2}\right] &\leq C_{T}E\int_{0}^{t}\left[\left|x^{\varepsilon}(s)-x^{*}(s)\right|^{2}+\left|\mathbb{D}_{2}\left(\mathcal{P}\left[x^{\varepsilon}(s)\right],\mathcal{P}\left[x^{*}(s)\right]\right)\right|^{2}\right]\mathrm{d}s \\ &+C_{T}\varepsilon^{2}E\int_{0}^{t}\left|u^{\varepsilon}(s)-u^{*}(s)\right|^{2}\mathrm{d}s \\ &+C_{T}\varepsilon^{2}E\left|\xi^{\varepsilon}(T)-\xi^{*}(T)\right|^{2}. \end{aligned} \tag{3.19}$$

Applying the definition of *Wasserstein metric* $\mathbb{D}_{2}\left(\cdot,\cdot\right)$, we have

$$\begin{aligned} \mathbb{D}_{2}\left(\mathcal{P}[x^{\varepsilon}(s)],\mathcal{P}[x^{*}(s)]\right) &= \inf\left\{\left[E\left|\widetilde{x}^{\varepsilon}(s)-\widetilde{x}^{*}(s)\right|^{2}\right]^{\frac{1}{2}},\right. \\ &\left.\mathcal{P}\left[x^{\varepsilon}(s)\right]=\mathcal{P}\left[\widetilde{x}^{\varepsilon}(s)\right]\text{ and }\mathcal{P}\left[x^{*}(s)\right]=\mathcal{P}\left[\widetilde{x}^{*}(s)\right]\right\} \\ &\leq\left[E\left|x^{\varepsilon}(s)-x^{*}(s)\right|^{2}\right]^{\frac{1}{2}}. \end{aligned} \tag{3.20}$$

for $\widetilde{x}^{\varepsilon}(\cdot),\widetilde{x}^{*}(\cdot)\in\mathbb{L}^{2}\left(\mathcal{F};\mathbb{R}^{d}\right)$, $\mathcal{P}\left[x^{\varepsilon}(s)\right]=\mathcal{P}\left[\widetilde{x}^{\varepsilon}(s)\right]$ and $\mathcal{P}\left[x^{*}(s)\right]=\mathcal{P}\left[\widetilde{x}^{*}(s)\right]$.

From (3.19) and (3.20), we get

$$E\left[\sup_{0\leq t\leq T}\left|x^{\varepsilon}(t)-x^{*}(t)\right|^{2}\right]\leq C_{T}E\int_{0}^{t}\sup_{r\in[0,s]}\left|x^{\varepsilon}(r)-x^{*}(r)\right|^{2}\mathrm{d}s+M_{T}\varepsilon^{2}.$$

Finally, applying Gronwall's inequality, the desired result (3.18) follows immediately by letting ε go to 0. This achieve the proof of Lemma 3.3.2. □

Variational equations. Now, we introduce the following variational equations involved in the stochastic maximum principle for our control problem

$$\begin{cases} \mathrm{d}\mathcal{Z}(t) = \Big[f_x\,(t)\,\mathcal{Z}(t) + \widehat{E}\left[\partial_\mu f\,(t)\,\widehat{\mathcal{Z}}\,(t)\right] + f_u(t)(u(t)-u^*(t))\Big]\,\mathrm{d}t \\ \qquad\quad + \Big[\sigma_x(t)\mathcal{Z}(t) + \widehat{E}\left[\partial_\mu\sigma\,(t)\,\widehat{\mathcal{Z}}\,(t)\right] + \sigma_u(t)(u(t)-u^*(t))\Big]\,\mathrm{d}W(t) \\ \qquad\quad + \Big[c_x(t)\mathcal{Z}(t) + \widehat{E}\left[\partial_\mu c\,(t)\,\widehat{\mathcal{Z}}\,(t)\right] + c_u\,(t)\,(u(t)-u^*(t))\Big]\,\mathrm{d}\widetilde{W}\,(t) \\ \qquad + \int_\Theta \Big[g_x(t,\theta)\mathcal{Z}(t) + \widehat{E}\left[\partial_\mu g\,(t,\theta)\,\widehat{\mathcal{Z}}\,(t)\right] + g_u(t,\theta)(u(t)-u^*(t))\Big]\,\widetilde{\eta}\,(\mathrm{d}\theta,\mathrm{d}t)\,, \\ \qquad + G(t)\mathrm{d}\left(\xi-\xi^*\right)(t), \\ \mathcal{Z}(0) = 0, \end{cases} \tag{3.21}$$

and

$$\begin{cases} \mathrm{d}\alpha_1(t) = & \left[\alpha_1(t)h(t) + \alpha(t)h_x(t)\mathcal{Z}(t) + \alpha(t)h_u(t)(u(t)-u^*(t))\right]\mathrm{d}Y(t), \\ \alpha_1(0) = & 0. \end{cases} \tag{3.22}$$

Under assumptions (H 3.1) and (H 3.2), equations (3.21) and (3.22) admits a unique adapted solutions $\mathcal{Z}(\cdot)$ and $\alpha_1(\cdot)$, respectively.

Lemma 3.3.3 Suppose that assumptions (H 3.1), (H 3.2) and (H 3.3) hold. Then, we have

$$\lim_{\varepsilon\to 0}\sup_{0\le t\le T} E\left|\frac{x^\varepsilon(t)-x(t)}{\varepsilon} - \mathcal{Z}(t)\right|^2 = 0. \tag{3.23}$$

Proof Let $\gamma^\varepsilon(t) = \frac{x^\varepsilon(t)-x^*(t)}{\varepsilon} - \mathcal{Z}(t),\ t\in[0,T]$. To simplify, we will use the following notations, for $\varphi = f,\sigma,c,l$ and g :

$$\varphi_x^{\lambda,\varepsilon}\,(t) = \varphi_x\left(t, x^{\lambda,\varepsilon}\,(t)\,, \mathcal{P}\left[x^\varepsilon(t)\right], u^\varepsilon(t)\right),$$

$$g_x^{\lambda,\varepsilon}\,(t,\theta) = g_x\left(t, x^{\lambda,\varepsilon}\,(t)\,, \mathcal{P}\left[x^\varepsilon(t)\right], u^\varepsilon(t),\theta\right),$$

$$\partial_\mu^{\lambda,\varepsilon}\varphi\,(t) = \partial_\mu\varphi(s, x^\varepsilon(t), \mathcal{P}\left[\widehat{x}^{\lambda,\varepsilon}\,(t)\right], u^\varepsilon(t); \widehat{x}(t)),$$

$$\partial_\mu^{\lambda,\varepsilon} g\,(t,\theta) = \partial_\mu g(t, x^\varepsilon(t), \mathcal{P}\left[\widehat{x}^{\lambda,\varepsilon}\,(t)\right], u^\varepsilon(t),\theta; \widehat{x}(t)),$$

and

$$x^{\lambda,\varepsilon}(s) = x^{*}(s) + \lambda\varepsilon\left(\gamma^{\varepsilon}(s) + \mathcal{Z}(s)\right),$$

$$\widehat{x}^{\lambda,\varepsilon}(s) = x^{*}(s) + \lambda\varepsilon(\widehat{\gamma}^{\varepsilon}(s) + \widehat{\mathcal{Z}}(s)),$$

$$u^{\lambda,\varepsilon}(s) = u^{*}(s) + \lambda\varepsilon v(s).$$

By simple computations, we get

$$\begin{aligned}
\gamma^{\varepsilon}(t) &= \frac{1}{\varepsilon}\int_0^t \left[f^{\varepsilon}(s) - f^{*}(s)\right]\mathrm{d}s + \frac{1}{\varepsilon}\int_0^t \left[\sigma^{\varepsilon}(s) - \sigma^{*}(s)\right]\mathrm{d}W(s) \\
&\quad + \frac{1}{\varepsilon}\int_0^t \left[c^{\varepsilon}(s) - c^{*}(s)\right]\mathrm{d}\widetilde{W}(s) + \frac{1}{\varepsilon}\int_0^t\int_{\Theta}\left[g^{\varepsilon}(s,\theta) - g^{*}(s,\theta)\right]\widetilde{\eta}(\mathrm{d}\theta,\mathrm{d}s) \\
&\quad + \frac{1}{\varepsilon}\int_{[0,t]} G(s)\mathrm{d}\left(\xi^{\varepsilon} - \xi^{*}\right)(s) \\
&\quad - \int_0^t \left[f_x(s)\mathcal{Z}(s) + \widehat{E}\left[\partial_\mu f(s)\widehat{\mathcal{Z}}(s)\right] + f_u(s)\left(u(s) - u^{*}(s)\right)\right]\mathrm{d}s \\
&\quad - \int_0^t \left[\sigma_x(s)\mathcal{Z}(s) + \widehat{E}\left[\partial_\mu \sigma(s)\widehat{\mathcal{Z}}(s)\right] + \sigma_u(s)\left(u(s) - u^{*}(s)\right)\right]\mathrm{d}W(s) \\
&\quad - \int_0^t \left[c_x(s)\mathcal{Z}(s) + \widehat{E}\left[\partial_\mu c(s)\widehat{\mathcal{Z}}(s)\right] + c_u(s)\left(u(s) - u^{*}(s)\right)\right]\mathrm{d}\widetilde{W}(s) \\
&\quad - \int_0^t\int_{\Theta} \left[g_x(s,\theta)\mathcal{Z}(s) + \widehat{E}\left[\partial_\mu g(s,\theta)\widehat{\mathcal{Z}}(s)\right] + g_u(s,\theta)\left(u(s) - u^{*}(s)\right)\right]\widetilde{\eta}(\mathrm{d}\theta, ds) \\
&\quad - \int_{[0,t]} G(s)\mathrm{d}\left(\xi - \xi^{*}\right)(s).
\end{aligned}$$

Now, we decompose $\frac{1}{\varepsilon}\int_0^t \left[f^{\varepsilon}(s) - f^{*}(s)\right]\mathrm{d}s$ into the following parts

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t \left[f^{\varepsilon}(s) - f^{*}(s)\right]\mathrm{d}s \\
&= \frac{1}{\varepsilon}\int_0^t \left[f(s, x^{\varepsilon}(s), \mathcal{P}\left[x^{\varepsilon}(s)\right], u^{\varepsilon}(s)) - f(s, x^{*}(s), \mathcal{P}\left[x^{*}(s)\right], u^{*}(s))\right]\mathrm{d}s \\
&= \frac{1}{\varepsilon}\int_0^t \left[f(s, x^{\varepsilon}(s), \mathcal{P}\left[x^{\varepsilon}(s)\right], u^{\varepsilon}(s)) - f(s, x^{*}(s), \mathcal{P}\left[x^{\varepsilon}(s)\right], u^{\varepsilon}(s))\right]\mathrm{d}s \\
&\quad + \frac{1}{\varepsilon}\int_0^t \left[f(s, x^{*}(s), \mathcal{P}\left[x^{\varepsilon}(s)\right], u^{\varepsilon}(s)) - f(s, x^{*}(s), \mathcal{P}\left[x^{*}(s)\right], u^{\varepsilon}(s))\right]\mathrm{d}s \\
&\quad + \frac{1}{\varepsilon}\int_0^t \left[f(s, x^{*}(s), \mathcal{P}\left[x^{*}(s)\right], u^{\varepsilon}(s)) - f(s, x^{*}(s), \mathcal{P}\left[x^{*}(s)\right], u^{*}(s))\right]\mathrm{d}s.
\end{aligned}$$

We notice that

$$\frac{1}{\varepsilon}\int_0^t \left[f^{\varepsilon}(s)-f(s,x^*(s),\mathcal{P}\left[x^{\varepsilon}(s)\right],u^{\varepsilon}(s))\right]ds=\int_0^t\int_0^1\left[f_x^{\lambda,\varepsilon}(s)\left(\gamma^{\varepsilon}(s)+\mathcal{Z}(s)\right)\right]\mathrm{d}\lambda\mathrm{d}s,$$
$$\frac{1}{\varepsilon}\int_0^t \left[f^{\varepsilon}(s)-f(s,x^{\varepsilon}(s),\mathcal{P}\left[x^*(s)\right],u^{\varepsilon}(s)\right]ds=\int_0^t\int_0^1\widehat{E}\left[\partial_{\mu}^{\lambda,\varepsilon}f(s)\left(\widehat{\gamma}^{\varepsilon}(s)+\widehat{\mathcal{Z}}(s)\right)\right]\mathrm{d}\lambda\mathrm{d}s,$$

and

$$\frac{1}{\varepsilon}\int_0^t\left[f\left(s,x(s),\mathcal{P}\left[x(s)\right],u^{\varepsilon}(s)\right)-f(s,x^*(s),\mathcal{P}\left[x^*(s)\right],u^*(s))\right]\mathrm{d}s$$
$$=\int_0^t\int_0^1\left[f_u\left(s,x(s),\mathcal{P}\left[x(s)\right],u^{\lambda,\varepsilon}(s)\right)\left(u(s)-u^*(s)\right)\right]\mathrm{d}\lambda\mathrm{d}s.$$

By applying similar method developed above, the analogue approachs hold for the coefficients σ, c and g. Moreover, from (3.17), we obtain

$$\frac{1}{\varepsilon}\int_{[0,t]}G(s)\mathrm{d}\left(\xi^{\varepsilon}-\xi^*\right)(s)-\int_{[0,t]}G(s)\mathrm{d}\left(\xi-\xi^*\right)(s)=0.$$

Now, we turn our attention to estimate $\gamma^{\varepsilon}(s)$, then we get

$$
\begin{aligned}
E\left[\sup_{s\in[0,t]}|\gamma^{\varepsilon}(s)|^2\right] &= C(t)\,E\left[\int_0^t\int_0^1\left|f_x^{\lambda,\varepsilon}(s)\,\gamma^{\varepsilon}(s)\right|^2\mathrm{d}\lambda\mathrm{d}s\right.\\
&+\int_0^t\int_0^1\widehat{E}\left|\partial_{\mu}^{\lambda,\varepsilon}f(s)\,\widehat{\gamma}^{\varepsilon}(s)\right|^2\mathrm{d}\lambda\mathrm{d}s\\
&+\int_0^t\int_0^1\left|\sigma_x^{\lambda,\varepsilon}(s)\,\gamma^{\varepsilon}(s)\right|^2\mathrm{d}\lambda\mathrm{d}s\\
&+\int_0^t\int_0^1\widehat{E}\left|\partial_{\mu}^{\lambda,\varepsilon}\sigma(s)\widehat{\gamma}^{\varepsilon}(s)\right|^2\mathrm{d}\lambda\mathrm{d}s\\
&+\int_0^t\int_0^1\left|c_x^{\lambda,\varepsilon}(s)\,\gamma^{\varepsilon}(s)\right|^2\mathrm{d}\lambda\mathrm{d}s\\
&+\int_0^t\int_0^1\widehat{E}\left|\partial_{\mu}^{\lambda,\varepsilon}c(s)\widehat{\gamma}^{\varepsilon}(s))\right|^2\mathrm{d}\lambda\mathrm{d}s\\
&+\int_0^t\int_{\Theta}\int_0^1\left|g_x^{\lambda,\varepsilon}(s,\theta)\,\gamma^{\varepsilon}(s)\right|^2\mathrm{d}\lambda m(\mathrm{d}\theta)\,\mathrm{d}s\\
&\left.+\int_0^t\int_{\Theta}\int_0^1\widehat{E}\left|\partial_{\mu}^{\lambda,\varepsilon}g(s,\theta)\,\widehat{\gamma}^{\varepsilon}(s))\right|^2\mathrm{d}\lambda m(\mathrm{d}\theta)\,\mathrm{d}s\right]\\
&+C(t)\,E\left[\sup_{s\in[0,t]}|\pi^{\varepsilon}(s)|^2\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\pi^{\varepsilon}(t) &= \int_0^t\int_0^1\left[f_x^{\lambda,\varepsilon}(s)-f_x(s)\right]\mathcal{Z}(s)\mathrm{d}\lambda\mathrm{d}s\\
&+\int_0^t\int_0^1\widehat{E}\left[\left(\partial_{\mu}^{\lambda,\varepsilon}f(s)-\partial_{\mu}f(s)\right)\widehat{\mathcal{Z}}(s)\right]\mathrm{d}\lambda\mathrm{d}s\\
&+\int_0^t\int_0^1\left[f_u\left(s,x(s),\mathcal{P}[x(s)],v^{\lambda,\varepsilon}(s)\right)-f_u(s)\right](u(s)-u^*(s))\,\mathrm{d}\lambda\mathrm{d}s\\
&+\int_0^t\int_0^1\left[\sigma_x^{\lambda,\varepsilon}(s)-\sigma_x(s)\right]\mathcal{Z}(s)\mathrm{d}\lambda\mathrm{d}W(s)\\
&+\int_0^t\int_0^1\widehat{E}\left[\left(\partial_{\mu}^{\lambda,\varepsilon}\sigma(s)-\partial_{\mu}\sigma(s)\right)\widehat{\mathcal{Z}}(s)\right]\mathrm{d}\lambda\mathrm{d}W(s)\\
&+\int_0^t\int_0^1\left[\sigma_u\left(s,x(s),\mathcal{P}[x(s)],v^{\lambda,\varepsilon}(s)\right)-\sigma_u(s)\right](u(s)-u^*(s))\,\mathrm{d}\lambda\mathrm{d}W(s)
\end{aligned}
$$

$$+\int_0^t\int_0^1\left[c_x^{\lambda,\varepsilon}(s)-c_x(s)\right]\mathcal{Z}(s)\mathrm{d}\lambda\mathrm{d}\widetilde{W}(s)$$
$$+\int_0^t\int_0^1\widehat{E}\left[\left(\partial_\mu^{\lambda,\varepsilon}c(s)-\partial_\mu c(s)\right)\widehat{\mathcal{Z}}(s)\right]\mathrm{d}\lambda\mathrm{d}\widetilde{W}(s)$$
$$+\int_0^t\int_0^1\left[c_u\left(s,x(s),\mathcal{P}\left[x(s)\right],u^{\lambda,\varepsilon}(s)\right)-c_u(s)\right](u(s)-u^*(s))\mathrm{d}\lambda\mathrm{d}\widetilde{W}(s)$$
$$+\int_0^t\int_\Theta\int_0^1\left[g_x^{\lambda,\varepsilon}(s,\theta)-g_x(s,\theta)\right]\mathcal{Z}(s_-)\mathrm{d}\lambda\widetilde{\eta}(\mathrm{d}\theta,\mathrm{d}s)$$
$$+\int_0^t\int_\Theta\int_0^1\widehat{E}\left[\left(\partial_\mu^{\lambda,\varepsilon}g(s,\theta)-\partial_\mu g(s,\theta)\right)\widehat{\mathcal{Z}}(s_-)\right]\mathrm{d}\lambda\widetilde{\eta}(\mathrm{d}\theta,\mathrm{d}s)$$
$$+\int_0^t\int_\Theta\int_0^1\left[g_u\left(s,x(s),\mathcal{P}\left[x(s)\right],u^{\lambda,\varepsilon}(s),\theta\right)-g_u(s,\theta)\right](u(s)-u^*(s))\mathrm{d}\lambda\widetilde{\eta}(\mathrm{d}\theta,\mathrm{d}s).$$

Now, the derivatives of f,σ,c and g with respect to (x,μ,u) are Lipschitz continuous in (x,μ,u), we get

$$\lim_{\varepsilon\to 0}E\left[\sup_{s\in[0,T]}\left|\pi^\varepsilon(s)\right|^2\right]=0.$$

Note that since the derivatives of the coefficients f,σ,c and γ are bounded with respect to (x,μ,u), we obtain

$$E\left[\sup_{s\in[0,t]}\left|\gamma^\varepsilon(s)\right|^2\right]\leq C(t)\left\{E\int_0^t\left|\gamma^\varepsilon(s)\right|^2ds+E\left[\sup_{s\in[0,t]}\left|\pi^\varepsilon(s)\right|^2\right]\right\}.$$

By applying Gronwall's lemma, we obtain $\forall t\in[0,T]$

$$E\left[\sup_{s\in[0,t]}\left|\gamma^\varepsilon(s)\right|^2\right]\leq C(t)\left\{E\left[\sup_{s\in[0,t]}\left|\pi^\varepsilon(s)\right|^2\right]\exp\left\{\int_0^t C(s)\,ds\right\}\right\}.$$

Finally, the proof of Lemma 3.3.3 is fulfilled by putting $t=T$ and letting ε go to zero. □

Now, we introduce the following lemma which play an important role in computing the variational inequality.

Lemma 3.3.4. Let assumption (H 3.1) hold. Then, we have

$$\lim_{\varepsilon\to 0}\sup_{0\leq t\leq T}E\left|\frac{\alpha^\varepsilon(t)-\alpha^*(t)}{\varepsilon}-\alpha_1(t)\right|^2=0. \tag{3.24}$$

Proof. From the definition of $\alpha^*(\cdot)$ and $\alpha_1(\cdot)$, we obtain

$$\begin{aligned}
\alpha^*(t) + \varepsilon\alpha_1(t) &= \alpha^*(0) + \int_0^t \alpha^*(s)h^*(s)\mathrm{d}Y(s) \\
&+ \varepsilon \int_0^t \left[\alpha_1(s) h^*(s) + \alpha^*(s)h_x(s)\mathcal{Z}(s) + \alpha^*(s)h_u(s)(u(s) - u^*(s))\right] \mathrm{d}Y(s) \\
&= \alpha^*(0) + \varepsilon \int_0^t \alpha_1(s)h^*(s)\mathrm{d}Y(s) \\
&+ \int_0^t \alpha^*(s) h(s, x^*(s) + \varepsilon\mathcal{Z}(s), u^*(s) + \varepsilon v(s))\mathrm{d}Y(s) \\
&- \varepsilon \int_0^t \alpha^*(s) \left[\ell_0^\varepsilon(s)\right] \mathrm{d}Y(s),
\end{aligned}$$

where

$$\begin{aligned}
\ell_0^\varepsilon(s) &= \int_0^1 \left[h_x(s, x^*(s) + \lambda\varepsilon\mathcal{Z}(s), u^*(s) + \lambda\varepsilon v(s)) - h_x(s)\right] \mathcal{Z}(s) d\lambda \\
&+ \int_0^1 \left[h_u(s, x^*(s) + \lambda\varepsilon\mathcal{Z}(s), u^*(s) + \lambda\varepsilon v(s)) - h_u(s)\right] (u(s) - u^*(s)) d\lambda.
\end{aligned}$$

Then, we have

$$
\begin{aligned}
&\alpha^{\varepsilon}(t)-\alpha^{*}(t)-\varepsilon\alpha_{1}(t)\\
&=\int_{0}^{t}\alpha^{\varepsilon}\left(s\right)h^{\varepsilon}\left(t\right)dY(s)-\varepsilon\int_{0}^{t}\alpha_{1}(s)h^{*}(s)\mathrm{d}Y(s)\\
&\quad-\int_{0}^{t}\alpha^{*}(s)h\left(s,x^{*}\left(s\right)+\varepsilon\mathcal{Z}\left(s\right),u^{*}\left(s\right)+\varepsilon v\left(s\right)\right)dY(s)+\varepsilon\int_{0}^{t}\alpha^{*}(s)\ell_{0}^{\varepsilon}\left(s\right)\mathrm{d}Y(s)\\
&=\int_{0}^{t}\left(\alpha^{\varepsilon}\left(s\right)-\alpha^{*}\left(s\right)-\varepsilon\alpha_{1}\left(s\right)\right)h^{\varepsilon}\left(s\right)\mathrm{d}Y\left(s\right)\\
&\quad+\int_{0}^{t}\left(\alpha^{*}\left(s\right)+\varepsilon\alpha_{1}\left(s\right)\right)\left[h^{\varepsilon}(s)-h\left(s,x^{*}(s)+\varepsilon\mathcal{Z}\left(s\right),u^{*}\left(s\right)+\varepsilon v\left(s\right)\right)\right]\mathrm{d}Y\left(s\right)\\
&\quad+\varepsilon\int_{0}^{t}\alpha_{1}\left(s\right)h(s,x^{*}\left(s\right)+\varepsilon\mathcal{Z}\left(s\right),u^{*}\left(s\right)+\varepsilon v\left(s\right))\mathrm{d}Y\left(s\right)\\
&\quad-\varepsilon\int_{0}^{t}\alpha_{1}\left(s\right)h^{*}(s)\mathrm{d}Y\left(s\right)+\varepsilon\int_{0}^{t}\alpha^{*}(s)\ell_{0}^{\varepsilon}\left(s\right)\mathrm{d}Y\left(s\right)\\
&=\int_{0}^{t}\left(\alpha^{\varepsilon}\left(s\right)-\alpha^{*}\left(s\right)-\varepsilon\alpha_{1}\left(s\right)\right)h^{\varepsilon}\left(s\right)\mathrm{d}Y(s)\\
&\quad+\int_{0}^{t}(\alpha^{*}(s)+\varepsilon\alpha_{1}(s))\ell_{1}^{\varepsilon}(s)\mathrm{d}Y(s)+\varepsilon\int_{0}^{t}\alpha_{1}(s)\ell_{2}^{\varepsilon}(s)\mathrm{d}Y(s)\\
&\quad+\varepsilon\int_{0}^{t}\alpha^{*}(s)\ell_{0}^{\varepsilon}(s)\mathrm{d}Y(s),
\end{aligned}
$$

where

$$
\begin{aligned}
\ell_{1}^{\varepsilon}(s)&=h^{\varepsilon}\left(s\right)-h\left(s,x^{*}\left(s\right)+\varepsilon\mathcal{Z}\left(s\right),u^{*}\left(s\right)+\varepsilon v\left(s\right)\right),\\
\ell_{2}^{\varepsilon}(s)&=h(s,x^{*}\left(s\right)+\varepsilon\mathcal{Z}\left(s\right),u^{*}\left(s\right)+\varepsilon v\left(s\right))-h^{*}(s).
\end{aligned}
\tag{3.25}
$$

From (3.25), we have

$$
\begin{aligned}
\ell_{1}^{\varepsilon}(s)=&\int_{0}^{1}\left[h_{x}(s,x^{*}\left(s\right)+\varepsilon\mathcal{Z}\left(s\right)+\lambda(x^{\varepsilon}\left(s\right)-x^{*}\left(s\right)-\varepsilon\mathcal{Z}\left(s\right)),v^{\varepsilon}\left(s\right))\right]\\
&\times\left(x^{\varepsilon}\left(s\right)-x^{*}\left(s\right)-\varepsilon\mathcal{Z}\left(s\right)\right)\mathrm{d}\lambda.
\end{aligned}
$$

By Lemma 3.3.3 , we know that

$$E \int_0^t |(\alpha^*(s) + \varepsilon\alpha_1(s))\ell_1^\varepsilon(s)|^2 \, \mathrm{d}s \leq \varepsilon^2 C(\varepsilon), \tag{3.26}$$

here $C(\varepsilon)$ denotes some nonnegative constant such that $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Moreover, it is easy to see that

$$\sup_{0\leq t\leq T} E\left[\varepsilon \int_0^t \alpha^*(s)\ell_0^\varepsilon(s)\mathrm{d}Y(s)\right]^2 \leq \varepsilon^2 C(\varepsilon), \tag{3.27}$$

and

$$\sup_{0\leq t\leq T} E\left[\varepsilon \int_0^t \alpha_1(s)\ell_2^\varepsilon(s)\mathrm{d}Y(s)\right]^2 \leq \varepsilon^2 C(\varepsilon). \tag{3.28}$$

From (3.26), (3.27) and (3.28), we get

$$\begin{aligned}
&E\left|(\alpha^\varepsilon(t) - \alpha^*(t)) - \varepsilon\alpha_1(t)\right|^2 \\
&\leq C\left[\int_0^t E\left|(\alpha^\varepsilon(s) - \alpha^*(s)) - \varepsilon\alpha_1(s)\right|^2 + E\int_0^t |(\alpha^*(s) + \varepsilon\alpha_1(s))\ell_1^\varepsilon(s)|^2 \, \mathrm{d}s \right.\\
&\quad \left. + \sup_{0\leq s\leq t} E\left(\varepsilon\int_0^t \alpha^*(s)\ell_0^\varepsilon(s)\mathrm{d}Y(s)\right)^2 + \sup_{0\leq s\leq t} E\left(\varepsilon\int_0^t \alpha_1(s)\ell_2^\varepsilon(s)\mathrm{d}Y(s)\right)^2\right] \\
&\leq C\int_0^t E\left|\alpha^\varepsilon(s) - \alpha^*(s) - \varepsilon\alpha_1(s)\right|^2 \mathrm{d}s + C(\varepsilon)\varepsilon^2.
\end{aligned}$$

Finally, by using Gronwall's inequality, the proof of Lemma 3.3.4 is complete. □

Lemma 3.3.5. Let assumption (H 3.1), (H 3.2) and (H 3.3) hold. Then, we have

$$\begin{aligned}
0 \leq E \int_0^T & \Big[\alpha_1(t)\, l(t) + \alpha^*(t)\, l_x(t)\mathcal{Z}(t) + \alpha^*(t)\widehat{E}\left[\partial_\mu l(t)\right]\mathcal{Z}(t) \\
&+ \alpha^*(t) l_u(t)(u(t) - u^*(t))\big]\, dt \\
&+ E\left[\alpha_1(T)\,\psi(x(T), \mathcal{P}[x(T)])\right] + E\left[\alpha^*(T)\,\psi_x(x(T), \mathcal{P}[x(T)])\mathcal{Z}(T)\right] \\
&+ E\Big[\alpha^*(T)\,\widehat{E}\left[\partial_\mu \psi(x(T), \mathcal{P}[x(T)]; \widehat{x}(T))\right]\mathcal{Z}(T)\Big] \\
&+ E\int_{[0,T]} \alpha^*(t)\, M(t)\mathrm{d}(\xi - \xi^*)(t).
\end{aligned} \tag{3.29}$$

Proof. From (3.3), we have

$$\begin{aligned}
0 &\leq \frac{1}{\varepsilon}\left[J\left(u^\varepsilon(t), \xi^\varepsilon(t)\right) - J\left(u^*(t), \xi^*(t)\right)\right] \\
&= \frac{1}{\varepsilon}\left[J\left(u^\varepsilon(t), \xi^\varepsilon(t)\right) - J\left(u^*(t), \xi^\varepsilon(t)\right)\right] \\
&+ \frac{1}{\varepsilon}\left[J\left(u^*, \xi^\varepsilon(t)\right) - J\left(u^*(t), \xi^*(t)\right)\right] \\
&= \mathcal{J}_1 + \mathcal{J}_2.
\end{aligned} \tag{3.30}$$

From (3.8), we get

$$\begin{aligned}
\mathcal{J}_1 &= \frac{1}{\varepsilon}\left[J\left(u^\varepsilon(t), \xi^\varepsilon(t)\right) - J\left(u^*(t), \xi^\varepsilon(t)\right)\right] \\
&= \frac{1}{\varepsilon} E\int_0^T \left[\alpha^\varepsilon(t) l^\varepsilon(t) - \alpha^*(t) l(t)\right] \mathrm{d}t \\
&+ \frac{1}{\varepsilon} E\left[\alpha^\varepsilon(T)\,\psi(x^\varepsilon(T), \mathcal{P}[x^\varepsilon(T)]) - \alpha^*(T)\,\psi(x^*(T), \mathcal{P}[x^*(T)])\right],
\end{aligned} \tag{3.31}$$

and by simple computation, the second term $\mathcal{J}_2$ being

$$\begin{aligned}
\mathcal{J}_2 &= \frac{1}{\varepsilon}\left[J\left(u^*(t), \xi^\varepsilon(t)\right) - J\left(u^*(t), \xi^*(t)\right)\right] \\
&= \frac{1}{\varepsilon}\left[E\int_{[0,T]} \alpha^*(t) M(t)\mathrm{d}\xi^\varepsilon(t) - \int_{[0,T]} \alpha^*(t) M(t)\mathrm{d}\xi^*(t)\right].
\end{aligned} \tag{3.32}$$

Using the Taylor expansion, Lemmas 3.3.3 and Lemma 3.3.4 , we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^{-1}E\left[\alpha^{\varepsilon}(T)\,\psi(x^{\varepsilon}(T),\mathcal{P}\left[x^{\varepsilon}(T)\right])-\alpha^{*}(T)\,\psi(x^{*}(T),\mathcal{P}\left[x^{*}(T)\right])\right]\\
&=E\left[\alpha_1(T)\psi(x(T),\mathcal{P}\left[x(t)\right])+\alpha^{*}(T)\psi_x(x(T),\mathcal{P}\left[x(T)\right])\mathcal{Z}(T)\right]\\
&+E\left[\alpha^{*}(T)\,\widehat{E}\left[\partial_{\mu}\psi(x(T),\mathcal{P}\left[x(t)\right];\widehat{x}(T))\right]\mathcal{Z}(T)\right],
\end{aligned}\tag{3.33}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^{-1}E\int_0^T\left[\alpha^{\varepsilon}(t)l^{\varepsilon}(t)-\alpha^{*}(t)l(t)\right]dt\\
&=E\int_0^T\Big[\alpha_1(t)l(t)+\alpha^{*}(t)l_x(t)\mathcal{Z}(t)+\alpha^{*}(t)\widehat{E}\left[\partial_{\mu}l(t)\right]\widehat{\mathcal{Z}}(t)\\
&+\alpha^{*}(t)l_u(t)(u(t)-u^{*}(t))\Big]dt
\end{aligned}\tag{3.34}$$

From (3.17), and since $\xi^{\varepsilon}(t)-\xi^{*}(t)=\varepsilon(\xi(t)-\xi^{*}(t))$, we get

$$\begin{aligned}
\mathcal{J}_2&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[E\int_{[0,T]}\alpha^{*}(t)M(t)\mathrm{d}\xi^{\varepsilon}(t)-\int_{[0,T]}\alpha^{*}(t)M(t)\mathrm{d}\xi^{*}(t)\right]\\
&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[E\int_{[0,T]}\alpha^{*}(t)M(t)\mathrm{d}(\xi^{\varepsilon}-\xi^{*})(t)\right]\\
&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[E\int_{[0,T]}\varepsilon\alpha^{*}(t)M(t)\mathrm{d}(\xi-\xi^{*})(t)\right]\\
&=E\int_{[0,T]}\alpha^{*}(t)M(t)\mathrm{d}(\xi-\xi^{*})(t).
\end{aligned}\tag{3.35}$$

Substituting (3.33), (3.34) and (3.35) into (3.30), the desired result (3.29) fulfilled immediately. This achieve the proof of Lemma 3.3.5. □

Let $\widetilde{\alpha}(t)=\frac{\alpha_1(t)}{\alpha^{*}(t)}$ then we have

$$\begin{cases}
d\widetilde{\alpha}(t)=\left\{h_x(t)\mathcal{Z}(t)+h_u(t)(u(t)-u^{*}(t))\right\}d\widetilde{W}(t),\\
\widetilde{\alpha}(0)=0,
\end{cases}\tag{3.36}$$

Lemma 3.3.6 Let $\Phi(\cdot)$ and $\mathcal{Z}(\cdot)$ be the solutions of (3.13) and (3.21) respectively. Then we have

$$\begin{aligned}
E^u\left[\Phi(T)\mathcal{Z}(T)\right] &= E^u\int_0^T \Phi(t) f_u(t)(u(t)-u^*(t))\mathrm{d}t + E^u\int_0^T q(t)\sigma_u(t)(u(t)-u^*(t))\mathrm{d}t \\
&+ E^u\int_0^T \overline{q}(t)c_u(t)(u(t)-u^*(t))\mathrm{d}t - E^u\int_0^T \mathcal{Z}(t)\left(l_x(t)+\widehat{E}(\partial_\mu\widehat{l}(t))\right)\mathrm{d}t \\
&+ E^u\int_0^T\int_\Theta R(t,\theta)\, g_u(t,\theta)(u(t)-u^*(t))m(d\theta)\,dt \\
&+ E^u\int_0^T \Phi(t)G(t)\mathrm{d}(\xi-\xi^*)(t),
\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}
E^u\left[y(T)\widetilde{\alpha}(T)\right] &= E^u\int_0^T k(t)\left[h_x(t)\mathcal{Z}(t)+h_u(t)(u(t)-u^*(t))\right]\mathrm{d}t. \\
&- E^u\int_0^T \widetilde{\alpha}(t)\, l(t)\mathrm{d}t.
\end{aligned} \tag{3.38}$$

Proof. By applying Itô's formula to $\Phi(t)\mathcal{Z}(t)$, $y(t)\widetilde{\alpha}(t)$ and taking expectation respectively, where $\mathcal{Z}(0)=0$ and $\widetilde{\alpha}(0)=0$, we obtain

$$\begin{aligned}
&E^u\left[\Phi(T)\mathcal{Z}(T)\right] \\
&= E^u\int_0^T \Phi(t)\,\mathrm{d}\mathcal{Z}(t) + E^u\int_0^T \mathcal{Z}(t)\,\mathrm{d}\Phi(t) \\
&+ E^u\int_0^T Q(t)\left[\sigma_x(t)\mathcal{Z}(t)+\widehat{E}\left[\partial_\mu\sigma(t)\widehat{\mathcal{Z}}(t)\right]+\sigma_u(t)(u(t)-u^*(t))\right]\mathrm{d}t \\
&+ E^u\int_0^T \overline{Q}(t)\left[c_x(t)\mathcal{Z}(t)+\widehat{E}\left[\partial_\mu c(t)\widehat{\mathcal{Z}}(t)\right]+c_u(t)(u(t)-u^*(t))\right]\mathrm{d}t \\
&+ E^u\int_0^T\int_\Theta R(t,\theta)\left[g_x(t,\theta)\mathcal{Z}(t)+\widehat{E}\left[\partial_\mu g(t,\theta)\,\widehat{\mathcal{Z}}(t)\right]+g_u(t,\theta)(u(t)-u^*(t))\right]m(d\theta)\,\mathrm{d}t \\
&= I_1(T)+I_2(T)+I_3(T)+I_4(T).
\end{aligned} \tag{3.39}$$

First, note that

$$
\begin{aligned}
I_1(T) &= E^u \int_0^T \Phi(t)\,\mathrm{d}\mathcal{Z}(t) \\
&= E^u \int_0^T \Phi(t)\left[f_x(t)\mathcal{Z}(t) + \widehat{E}\left[\partial_\mu f(t)\widehat{\mathcal{Z}}(t)\right] + f_u(t)(u(t)-u^*(t))\right]\mathrm{d}t \\
&+ E^u \int_0^T \Phi(t)G(t)\mathrm{d}(\xi-\xi^*)(t), \qquad (3.40)\\
&= E^u \int_0^T \Phi(t)\, f_x(t)\mathcal{Z}(t)dt + E^u \int_0^T \Phi(t)\,\widehat{E}\left[\partial_\mu f(t)\widehat{\mathcal{Z}}(t)\right]\mathrm{d}t \\
&+ E^u \int_0^T \Phi(t)\, f_u(t)(u(t)-u^*(t))dt + E^u \int_0^T \Phi(t)G(t)\mathrm{d}(\xi-\xi^*)(t).
\end{aligned}
$$

We proceed to estimate $I_2(T)$, From equation (3.13), we have

$$
\begin{aligned}
I_2(T) &= E^u \int_0^T \mathcal{Z}(t)\, d\Phi(t) \\
&= -E^u \int_0^T \mathcal{Z}(t)\left[f_x(t)\,\Phi(t) + \widehat{E}\left[\partial_\mu \widehat{f}(t)\,\widehat{\Phi}(t)\right] + \sigma_x(t)\,Q(t)\right. \qquad (3.41)\\
&+ \widehat{E}\left[\partial_\mu \widehat{\sigma}(t)\,\widehat{Q}(t)\right] + c_x(t)\,\overline{Q}(t) + \widehat{E}\left[\partial_\mu \widehat{c}(t)\,\widehat{\overline{Q}}(t)\right] + l_x(t) + \widehat{E}\left[\partial_\mu \widehat{l}(t)\right] \\
&+ \left.\int_\Theta \left[g_x(t,\theta)\,R(t,\theta) + \widehat{E}\left[\partial_\mu \widehat{g}(t,\theta)\,\widehat{R}(t,\theta)\right]\right] m(d\theta) + h_x(t)\,K(t)\right] dt.
\end{aligned}
$$

By simple computation, we have

$$
\begin{aligned}
I_2(T) = & -E^u \int_0^T \mathcal{Z}(t) f_x(t) \Phi(t) dt - E^u \int_0^T \mathcal{Z}(t) \widehat{E}\left[\partial_\mu \widehat{f}(t) \widehat{\Phi}(t)\right] dt \\
& - E^u \int_0^T \mathcal{Z}(t) \sigma_x(t) Q(t) dt - E^u \int_0^T \mathcal{Z}(t) \widehat{E}\left[\partial_\mu \widehat{\sigma}(t) \widehat{Q}(t)\right] dt \\
& - E^u \int_0^T \mathcal{Z}(t) c_x(t) \overline{Q}(t) dt - E^u \int_0^T \mathcal{Z}(t) \widehat{E}\left[\partial_\mu \widehat{c}(t) \widehat{\overline{Q}}(t)\right] dt \\
& - E^u \int_0^T \mathcal{Z}(t) l_x(t) dt - E^u \int_0^T \mathcal{Z}(t) \widehat{E}\left[\partial_\mu \widehat{l}(t)\right] dt \\
& - E^u \int_0^T \int_\Theta \mathcal{Z}(t) g_x(t,\theta) R(t,\theta) m(d\theta) dt \\
& - E^u \int_0^T \int_\Theta \mathcal{Z}(t) \widehat{E}\left[\partial_\mu \widehat{g}(t,\theta) \widehat{R}(t,\theta)\right] m(d\theta) dt \\
& - E^u \int_0^T \mathcal{Z}(t) h_x(t) K(t) dt.
\end{aligned} \tag{3.42}
$$

Similarly, we can obtain

$$
\begin{aligned}
I_3(T) = & E^u \int_0^T Q(t) \left[\sigma_x(t)\mathcal{Z}(t) + \widehat{E}\left[\partial_\mu \sigma(t) \widehat{\mathcal{Z}}(t)\right] + \sigma_u(t)(u(t) - u^*(t))\right] dt \\
& + E^u \int_0^T \overline{Q}(t) \left[c_x(t)\mathcal{Z}(t) + \widehat{E}\left[\partial_\mu c(t) \widehat{\mathcal{Z}}(t)\right] + c_u(t)(u(t) - u^*(t))\right] dt,
\end{aligned} \tag{3.43}
$$

and

$$
I_4(T) = E^u \int_0^T \int_\Theta R(t,\theta) \left[g_x(t,\theta)\mathcal{Z}(t) + \widehat{E}\left[\partial_\mu g(t,\theta) \widehat{\mathcal{Z}}(t)\right] + g_u(t,\theta)(u(t) - u^*(t))\right] m(d\theta) dt. \tag{3.44}
$$

Now, by applying Fubini's theorem, we obtain

$$
E^u \int_0^T \Phi(t) \widehat{E}\left[\partial_\mu \widehat{f}(t) \widehat{\mathcal{Z}}(t)\right] dt = E^u \int_0^T \mathcal{Z}(t) \widehat{E}\left[\partial_\mu f(t) \widehat{\Phi}(t)\right] dt, \tag{3.45}
$$

$$
E^u \int_0^T Q(t) \widehat{E}\left[\partial_\mu \widehat{\sigma}(t) \widehat{\mathcal{Z}}(t)\right] dt = E^u \int_0^T \mathcal{Z}(t) \widehat{E}\left[\partial_\mu \sigma(t) \widehat{Q}(t)\right] dt, \tag{3.46}
$$

$$E^u \int_0^T \overline{Q}(t) \widehat{E}\left[\partial_\mu \widehat{c}(t) \widehat{\mathcal{Z}}(t)\right] dt = E^u \int_0^T \mathcal{Z}(t) \widehat{E}\left[\partial_\mu c(t) \widehat{\overline{Q}}(t)\right] dt, \tag{3.47}$$

and

$$E^u \int_0^T \int_\Theta R(t,\theta) \widehat{E}\left[\partial_\mu \widehat{g}(t,\theta) \widehat{\mathcal{Z}}(t)\right] m(d\theta) dt = E^u \int_0^T \int_\Theta \mathcal{Z}(t) \widehat{E}\left[\partial_\mu g(t,\theta) \widehat{R}(t,\theta)\right] m(d\theta) dt. \tag{3.48}$$

By substituting (3.40), (3.42), (3.43) and (3.44) into (3.39), with the helps of (3.45), (3.46), (3.47) and (3.48) the desired result (3.37) follows immediately.

By applying Itô's formula to $y(t) \widetilde{\alpha}(t)$ and taking expectation, we get

$$\begin{aligned} E^u[y(T) \widetilde{\alpha}(T)] &= E^u \int_0^T y(t) \mathrm{d}\widetilde{\alpha}(t) + E^u \int_0^T \widetilde{\alpha}(t) \mathrm{d}y(t) \\ &+ E^u \int_0^T K(t) \{h_x(t)\mathcal{Z}(t) + h_u(t)(u(t) - u^*(t))\} \mathrm{d}t \\ &= J_1(T) + J_2(T) + J_3(T), \end{aligned} \tag{3.49}$$

where,

$$\begin{aligned} J_1(T) &= E^u \int_0^T y(t) \mathrm{d}\widetilde{\alpha}(t) \\ &= E^u \int_0^T y(t) (h_x(t)\mathcal{Z}(t) + h_u(t)(u(t) - u^*(t))) \mathrm{d}\widetilde{W}(t), \end{aligned} \tag{3.50}$$

is a martingale with zero expectation. Moreover, by a simple computations, we get

$$J_2(T) = E^u \int_0^T \widetilde{\alpha}(t) \mathrm{d}y(t) = -E^u \int_0^T \widetilde{\alpha}(t) l(t) \mathrm{d}t, \tag{3.51}$$

and

$$J_3(T) = E^u \int_0^T K(t) [h_x(t)\mathcal{Z}(t) + h_u(t)(u(t) - u^*(t))] \mathrm{d}t. \tag{3.52}$$

Substituting (3.50), (3.51), (3.52), into (3.49), the desired result (3.38) fulfilled.

Proof of Theorem 3.3.1. From Lemma 3.3.5and based on the fact that

$$y\left(T\right)=\psi(x^{u,\xi}\left(T\right),\mathcal{P}[x^{u,\xi}\left(T\right)]),$$

and

$$\Phi\left(T\right)=\psi_x(x^{u,\xi}\left(T\right),\mathcal{P}[x^{u,\xi}\left(T\right)])+\widehat{E}\left[\partial_\mu\psi(\widehat{x}\left(T\right),\mathcal{P}[\widehat{x}\left(T\right)];x^{u,\xi}(T))\right]$$

we have

$$\begin{aligned}0\leq{}& E\int_0^T\Big[\alpha_1\left(t\right)l(t)+\alpha^*\left(t\right)l_x(t)\mathcal{Z}\left(t\right)+\alpha^*(t)\widehat{E}\left[\partial_\mu l(t)\right]\mathcal{Z}(t)+\alpha^*(t)l_u(t)(u\left(t\right)-u^*(t))\Big]\,\mathrm{d}t\\&+E\left[\alpha_1\left(T\right)y(T)\right]+E\left[\alpha^*\left(T\right)\Phi\left(T\right)\mathcal{Z}\left(T\right)\right]\\&+E\int_{[0,T]}\alpha^*\left(t\right)M(t)\mathrm{d}\left(\xi-\xi^*\right)(t).\end{aligned}\tag{3.53}$$

Since

$$\begin{aligned}E\left[\alpha_1\left(T\right)y(T)\right]&=E\left[\alpha^*\left(T\right)\widetilde{\alpha}\left(T\right)y(T)\right]=E^u\left[\widetilde{\alpha}\left(T\right)y\left(T\right)\right],\\E\left[\alpha^*\left(T\right)\Phi\left(T\right)\mathcal{Z}\left(T\right)\right]&=E^u\left[\Phi\left(T\right)\mathcal{Z}\left(T\right)\right],\\E\int_{[0,T]}\alpha^*\left(t\right)M(t)\mathrm{d}\left(\xi-\xi^*\right)(t)&=E^u\int_{[0,T]}M(t)\mathrm{d}\left(\xi-\xi^*\right)(t).\end{aligned}$$

Finally, by substituting (3.37) and (3.38) of Lemma 3.3.6 into (3.53), we get

$$\begin{aligned}0\leq{}&E\int_0^T\alpha^*(t)\Big[\Phi\left(t\right)f_u(t)+Q\left(t\right)\sigma_u(t)+\overline{Q}\left(t\right)c_u(t)\\&+\int_\Theta R\left(t,\theta\right)g_u(t,\theta)m\left(d\theta\right)+K(t)h_u\left(t\right)+l_u(t)\Big]\left(u\left(t\right)-u^*(t)\right)\mathrm{d}t\\&+E\int_{[0,T]}\alpha^*\left(t\right)\left(M(t)+\Phi(t)G(t)\right)\mathrm{d}\left(\xi-\xi^*\right)(t).\end{aligned}\tag{3.54}$$

This completes the proof of Theorem 3.3.1. □

3.4 Application : Conditional mean-variance portfolio selection problem associated with interventions

In this section, we study a conditional mean-variance portfolio selection problem in incomplete market, where the system is governed by Lévy measure associated with some Gamma process and an independent Brownian motion. The Gamma process is a Lévy process (of bounded variation) $(\Gamma(t))_{t\geq 0}$, with Lévy measure given by

$$\mu(\mathrm{d}x) = \frac{e^{-x}}{x}\chi_{\{x>0\}}\mathrm{d}x. \tag{3.55}$$

It is called *Gamma process* because the probability law of $\Gamma(\cdot)$ is a Gamma distribution with mean t and scale parameter equal to one. The Lévy measure $\mu(\mathrm{d}x)$ dictates how the jumps occur.

Let $(\Gamma(t))_{t\in[0,T]}$ be a $\mathbb{R}$-valued Gamma process, independent of the Brownian motion $W(\cdot)$. Assume that the Lévy measure $\mu(\mathrm{d}x)$ corresponding to the Gamma process $\Gamma(\cdot)$ has a moments of all orders. This implies that $\int_{(-\delta,\delta)^c} e^{|x|}\mu(\mathrm{d}x) < \infty$ for every $\delta > 0$ and $\int_{\mathbb{R}}(x^2 \wedge 1)\mu(\mathrm{d}x) < \infty$. We assume that $\mathcal{F}_t$ is $\mathcal{P}-$ augmentation of the natural filtration $\mathcal{F}_t^{(W,\Gamma)}$ defined as follows

$$\mathcal{F}_t^{(W,\Gamma)} = \mathcal{F}_t^W \vee \sigma\left\{\Gamma(r) : 0 \leq r \leq t\right\} \vee \mathcal{F}_0,$$

where $\mathcal{F}_t^W := \sigma\left\{W(s) : 0 \leq s \leq t\right\}$, $\mathcal{F}_0$ denotes the totality of $\mathcal{P}^u-$null sets and $\mathcal{F}_1 \vee \mathcal{F}_2$ denotes the σ-field generated by $\mathcal{F}_1 \cup \mathcal{F}_2$. We denote by $\Delta\Gamma(\tau_j) = \Gamma(\tau_j) - \Gamma(\tau_{j-})$ the jump size at time τ_j. We denote by $\Gamma^j(t) = \sum_{0\leq s\leq t}\left(\Delta\Gamma(s)\right)^j : j : 1, ..., n$ the power jump processes of $\Gamma\left(\cdot\right)$. By using *Exponential formula* proved in Bertoin [20], we obtain

$$E^u\left(\exp(i\theta\Gamma^j(t))\right) = \exp\left[t\int_0^{+\infty}(\exp(i\theta x^j) - 1)\frac{e^{-x}}{x}\mathrm{d}x\right].$$

Let $\Gamma_0(n)$ Gamma function defined by $\Gamma_0(n) = \int_0^{+\infty} x^{n-1}e^{-x}\mathrm{d}x$, and $\varphi_{\Gamma^j(t)}(t)$: the moment generating function $\varphi_{\Gamma^j(t)}(t) = E^u(\exp(t\Gamma^j(t)))$. Now, based on $\varphi^{(k)}_{\Gamma^j(t)}(0) = E^u\left((\Gamma^j(t))^k\right)$, we deduce

$$E^u\left(\Gamma^j(t)\right) = \varphi^{'}_{\Gamma^j(t)}(0) = t\Gamma_0(j) = (j-1)!t : j : 1, ..., n$$

Now, we proceed to obtain $\mathbb{V}^u_{ar}\left(\Gamma^j(t)\right)$, then we have

$$\begin{aligned}
\mathbb{V}^u_{ar}\left(\Gamma^j(t)\right) &= E^u\left[\left(\Gamma^j(t)\right)^2\right] - \left[E\left(\Gamma^j(t)\right)\right]^2 \\
&= \varphi^{''}_{\Gamma^j(t)}(0) - \left[\varphi^{'}_{\Gamma^j(t)}(0)\right]^2 \\
&= t\int_0^{+\infty} x^{2j-1}e^{-x}\mathrm{d}x \\
&= t\Gamma_0(2j),\ \ j : 1, ..., n,
\end{aligned}$$

Let

$$\mathcal{L}^j(t) = \frac{\Gamma^j(t) - E^u\left(\Gamma^j(t)\right)}{\mathbb{V}^u_{ar}\left(\Gamma^j(t)\right)} = \frac{\sum_{0\leq s\leq t}\left(\Delta\Gamma(s)\right)^j - (j-1)!t}{t\Gamma_0(2j)},\ \ j : 1, ..., n \tag{3.56}$$

then we have $E^u\left(\mathcal{L}^j(t)\right) = 0$ and $\mathbb{V}^u_{ar}\left(\mathcal{L}^j(t)\right) = 1$.

Derivatives with respect to measure in the sense of P.L. Lions. Let $(\Gamma(t))_{t\geq 0}$ be Gamma process with Lévy measure $\mu(\cdot)$ given by (5.32). We give some examples.

1. If $\Phi(\mu) = \int_{\mathbb{R}^n} \varphi(x)\mu(\mathrm{d}x)$ then the derivatives of $\Phi(\mu)$ with respect to measure at z is given by

$$\partial_\mu \Phi(\mu)(z) = \frac{\partial\varphi}{\partial x}(z).$$

2. If $\Phi(\mu) = \int_{\mathbb{R}^n} \varphi(x,\mu)\mu(\mathrm{d}x)$ then the derivatives of $\Phi(\mu)$ with respect to measure at z is given by

$$\partial_\mu \Phi(\mu)(z) = \frac{\partial\varphi}{\partial x}(z,\mu) + \int_{\mathbb{R}^n} \frac{\partial\varphi}{\partial\mu}(x,\mu)(z)\,\mu(\mathrm{d}x).$$

Conditional mean-variance portfolio selection problem with interventions. In this section, we study a conditional mean-variance portfolio selection problem in incomplete market with interventions. As example, foreign exchange interventions are conducted by monetary authorities (Bank or minister of finance) to influence foreign exchange rates by buying and selling currencies in the foreign exchange market.

Suppose that we are given a mathematical market consisting of two investment possibilities :

A risk free security, (bond) where the price $S_0(t)$ evolves according to the ordinary differential equation :

$$\begin{cases} \mathrm{d}S_0(t) = \gamma_0(I_t)S_0(t)\,\mathrm{d}t,\ t \in [0,T], \\ S_0(0) > 0, \end{cases} \tag{3.57}$$

where I_t is a factor process with dynamics governed by a Brownian motion $B(\cdot)$, assumed to be non correlated with the Brownian motion $W(\cdot)$. We shall assume that the natural filtration generated by the observable factor process I_t is equal to the filtration $\mathcal{F}_t^B$ generated by $B(\cdot)$. Notice that the market is incomplete as the agent cannot trade in the factor process. The map $\gamma_0(\cdot) : [0,T] \to \mathbb{R}_+$ is a locally bounded continuous deterministic function.

A risky security (stock), where the price $S_1(t)$ at time t is given by

$$\begin{cases} \mathrm{d}S_1(t) = S_1(t)\left[(\varsigma(I_t) + \gamma_0(I_t))\,\mathrm{d}t + \sigma(I_t)\mathrm{d}W(t)\right] + \mathrm{d}\xi(t) + \sum_{j=1}^{n} \mathcal{L}^j(t), \\ S_1(0) > 0, \end{cases} \tag{3.58}$$

where $\mathcal{L}^j(t)$ is the power jump processes of $\Gamma(\cdot)$ given by (3.56).

Now, in order to ensure that $S_1(t) > 0$ for all $t \in [0,T]$, we assume the functions $\varsigma(\cdot) : [0,T] \to \mathbb{R}$, and $\sigma(\cdot) : [0,T] \to \mathbb{R}$ are bounded continuous deterministic maps such that

$$\varsigma(I_t),\ \sigma(I_t) \neq 0 \text{ and } \varsigma(I_t) - \gamma_0(I_t) > 0,\ \forall t \in [0,T].$$

Let $x(0) = x_0 > 0$ be an initial wealth process. By combining (3.57) and (3.58), we introduce the wealth dynamic

$$\begin{cases} \mathrm{d}x(t) = \gamma_0(I_t)(x(t) - \xi(t))\mathrm{d}t + u(t)\left[\varsigma(I_t)\mathrm{d}t + \sigma(I_t)\mathrm{d}W(t)\right] + \mathrm{d}\xi(t) \\ \qquad\qquad + \sum_{j=1}^{n} \mathcal{L}^j(t), \\ x(0) = x_0. \end{cases} \tag{3.59}$$

where $\gamma_0(I_t)$: is the interest rate, $\varsigma(I_t)$: is the excess rate of return, and $\sigma(I_t)$: the volatility (or the dispersion) of the stock price with $\sigma\left(I_t\right) \geq \varepsilon$ for some $\varepsilon > 0$.are measurable bounded functions of I_t. The process $u = u(t)$ (the regular control process) represents the amount invested in the stock at time t, when the current wealth is $x(t)$ and based on the past partially observations $\mathcal{F}_t^B$ of the factor process, $\xi(t)$ is the intervention control.
The objective of the agent is to minimize over investment strategies a cost functional of the form :

$$J\left(u(\cdot), \xi(\cdot)\right) = E^u\left[\frac{\delta}{2}\mathbb{V}_{ar}^u\left(x(T) - \xi(T) \mid B(T)\right) - E^u(x(T) - \xi(T) \mid B(T))\right], \tag{3.60}$$

for some $\delta > 0$, with a dynamics for the wealth process $x(t)$ controlled by the amount $u(t)$.
If we denote $z(t) = x(t) - \xi(t) - \sum_{j=1}^{\eta} \mathcal{L}^j(t)$, then the dynamic (3.59) has the form :

$$\begin{cases} \mathrm{d}z(t) = \gamma_0(I_t)z(t)\mathrm{d}t + u(t)\left[\varsigma(I_t)\mathrm{d}t + \sigma(I_t)\mathrm{d}W(t)\right], \\ z(0) = x_0. \end{cases} \tag{3.61}$$

and the cost functional $J\left(u(\cdot), \xi(\cdot)\right)$ has the form

$$J\left(u(\cdot), \xi(\cdot)\right) = E^u\left[\frac{\delta}{2}\mathbb{V}_{ar}^u\left(z(T) \mid B(T)\right) - E^u(z(T) \mid B(T))\right], \tag{3.62}$$

where $E^u(z(t) \mid B(t))$ is the conditional expectation and $\mathbb{V}_{ar}^u\left(z(t) \mid B(t)\right)$ is the conditional variance with rspect to $\mathcal{P}^u$. We note that the law of total variance is given by

$$\mathbb{V}_{ar}^{u}\left(z(t)\right)=\mathbb{V}_{ar}^{u}\left(z(t)\mid B(t)\right)+\mathbb{V}_{ar}^{u}\left[E^{u}(z(t)\mid B(t))\right].$$

By applying similar arguments developed in Pham [91], Li and Zhou [77] the optimal intervention control $u^*(t)$ of (3.61)-(3.62) is given in feedback form :

$$\begin{aligned} u^*(t) &= \frac{\varsigma(I_t)}{\sigma^2(I_t)}\left[E^u(z^*(t)\mid B(t))-z^*(t)\right] \\ &\quad+\frac{\varsigma(I_t)}{\sigma^2(I_t)c_t}\left[\frac{1}{2}b_t-a_tE^u\left(z^*(t)\mid B(t)\right)\right], \end{aligned} \tag{3.63}$$

where $z(t)$ is given by Eq-(3.61), and $a_t, b_t\ c_t$ satisfy the linear BSDEs : $t\in[0,T]$

$$\begin{cases} \mathrm{d}a_t=\left[\frac{\varsigma^2(I_t)a_t^2}{\sigma^2(I_t)c_t}-2\gamma_0(I_t)a_t\right]\mathrm{d}t+Z_t^a\mathrm{d}B(t),\ a_T=0. \\ \mathrm{d}b_t=\left[\frac{\varsigma^2(I_t)a_t}{\sigma^2(I_t)c_t}-\gamma_0(I_t)\right]\mathrm{d}t+Z_t^b\mathrm{d}B(t),\ b_T=-1. \\ \mathrm{d}c_t=\left[\frac{\varsigma^2(I_t)}{\sigma^2(I_t)}-2\gamma_0(I_t)\right]c_t\mathrm{d}t+Z_t^c\mathrm{d}B(t),\ c_T=\frac{\delta}{2}. \end{cases} \tag{3.64}$$

The explicit solutions of the above equations are given by

$$\begin{aligned} a_t &\equiv 0,\ \forall t\in[0,T], \\ b_t &= E^u\left[-\exp\int_t^T\gamma_0(I_s)\mathrm{d}s\mid\mathcal{F}_t^B\right]. \\ c_t &= E^u\left[\frac{\delta}{2}\exp\int_t^T(2\gamma_0(I_s)-\frac{\varsigma^2(I_s)}{\sigma^2(I_s)})\mathrm{d}s\mid\mathcal{F}_t^B\right]; \end{aligned} \tag{3.65}$$

Hence, substituting (3.65) into (3.63) yields

$$\begin{aligned} u^*(t) = \frac{\varsigma(I_t)}{\sigma^2(I_t)}&\left[x_0\exp\left(\int_0^t\gamma_0(I_\tau)\mathrm{d}\tau\right)-z^*(t)\right. \\ &\left.+\frac{1}{2}\int_0^t\frac{\varsigma^2(I_t)}{2\sigma^2(I_t)}\frac{|b_s|}{c_s}\exp\left(\int_0^t\gamma_0(I_\tau)\mathrm{d}\tau\right)\mathrm{d}s+\frac{|b_t|}{c_t}\right]. \end{aligned} \tag{3.66}$$

Finally, we deduce that the optimal control of the problem (3.59)-(3.60) is given in feedback form

$$u^*(t) = \frac{\varsigma(I_t)}{\sigma^2(I_t)} \left[x_0 \exp\left(\int_0^t \gamma_0(I_s) \mathrm{d}s \right) - x^*(t) + \xi(t) + \sum_{j=1}^{n} \mathcal{L}^j(t) \right. \tag{3.67}$$
$$\left. + \frac{1}{2} \int_0^t \frac{\varsigma^2(I_t)}{2\sigma^2(I_t)} \frac{|b_s|}{c_s} \exp\left(\int_0^t \gamma_0(I_\tau) \mathrm{d}\tau \right) \mathrm{d}s + \frac{|b_t|}{c_t} \right].$$

Now, let $\xi^*(t)$ be $\mathcal{F}_t^Y-$adapted process satisfies Theorem 3.1, then for any $\xi(\cdot) \in \mathcal{U}_2^Y$ we get

$$E^u \left[\int_{[0,T]} (M(t) + G(t)\Phi(t)) \mathrm{d}\xi^*(t) \mid \mathcal{F}_t^Y \right]$$
$$\leq E^u \left[\int_{[0,T]} (M(t) + G(t)\Phi(t)) \mathrm{d}\xi(t) \mid \mathcal{F}_t^Y \right].$$

We define a subset $\mathcal{E} \subset \Omega \times [0, T]$ such that

$$\mathcal{E} = \left\{ (t, w) \in [0, T] \times \Omega : M(t) + G(t)\Phi(t) > 0 \right\}, \tag{3.68}$$

and let $\xi(\cdot) \in \mathcal{U}_2^Y$ defined by

$$\mathrm{d}\xi(t) = \begin{cases} 0 : \text{if } \ (t, w) \in \mathcal{E}, \\ \mathrm{d}\xi^*(t) : \text{if } \ (t, w) \in \overline{\mathcal{E}}, \end{cases} \tag{3.69}$$

where $\overline{\mathcal{E}}$ is the complement of the set $\mathcal{E}$. We denote by $\boldsymbol{\chi}_{\mathcal{E}}$ the indicator function of $\mathcal{E}$. By a simple computations, we get

$$\begin{aligned}0 &\leq E^u\left[\int_{[0,T]}(M(t)+G(t)\Phi(t))\mathrm{d}\left(\xi(t)-\xi^*(t)\right) \mid \mathcal{F}_t^Y\right]\\ &= E^u\left[\int_{[0,T]}(M(t)+G(t)\Phi(t))\boldsymbol{\chi}_{\mathcal{E}}(t,w)\mathrm{d}\left(-\xi^*\right)(t) \mid \mathcal{F}_t^Y\right]\\ &+ E^u\left[\int_{[0,T]}(M(t)+G(t)\Phi(t))\boldsymbol{\chi}_{\overline{\mathcal{E}}}(t,w)\mathrm{d}\left(\xi^*-\xi^*\right)(t) \mid \mathcal{F}_t^Y\right]\\ &= -E^u\left[\int_{[0,T]}(M(t)+G(t)\Phi(t))\boldsymbol{\chi}_{\mathcal{E}}(t,w)\mathrm{d}\xi^*(t) \mid \mathcal{F}_t^Y\right].\end{aligned}$$

This implies that $\xi^*(\cdot)$ satisfies for any $t \in [0,T]$:

$$E^u\left[\int_{[0,T]}(M(t)+G(t)\Phi(t))\boldsymbol{\chi}_{\mathcal{E}}(t,w)\mathrm{d}\xi^*(t)\right]=0.$$

From (3.68) and (3.69), we can easy shows that the optimal intervention control has the form :

$$\xi^*(t)=\xi(t)+\int_0^t \boldsymbol{\chi}_{\overline{\mathcal{E}}}(s,w)\mathrm{d}s,\ t\in[0,T].$$

Finally, we give the explicit optimal portfolio section strategy for systems governed by Lévy measure associated with some Gamma process in feedback form by :

$$\begin{aligned}u^*(t,x^*) &= \frac{\varsigma(I_t)}{\sigma^2(I_t)}\left[x_0\exp\left(\int_0^t\gamma_0(I_\tau)\mathrm{d}\tau\right)-x^*(t)+\xi(t)+\sum_{j=1}^n\mathcal{L}^j(t)\right.\\ &\left.+\frac{1}{2}\int_0^t\frac{\varsigma^2(I_t)}{2\sigma^2(I_t)}\frac{|b_s|}{c_s}\exp\left(\int_0^t\gamma_0(I_\tau)\mathrm{d}\tau\right)\mathrm{d}s+\frac{|b_t|}{c_t}\right].\\ \xi^*(t) &= \int_0^t\boldsymbol{\chi}_{\overline{\mathcal{E}}}(s,w)ds+\xi(t),\ t\in[0,T].\\ \mathcal{L}^j(t) &= \frac{\sum_{0\leq s\leq t}\left(\Delta\Gamma(s)\right)^j-(j-1)!t}{t\Gamma_0(2j)},\ j:1,...,n.\end{aligned}$$

In this chapter, a new set of general mean-field type necessary conditions for a class of optimal stochastic intervention control problem for partially observed random jumps on Wasserstein space of probability measures has been established. Girsanov's theorem and the L-derivatives with respect to probability law are applied to prove our main result. Conditional mean-variance portfolio selection problem with interventions is investiged. In order to assess the effectiveness of interventions, it is helpful to identify the motives of the governement (or banks) activities in this area. Apparently, there are many problems left unsolved, and one possible problem is to obtain some optimality conditions for partial observed stochastic optimal intervention control for systems governed by general mean-field *backward stochastic differential equations with Lévy* process with moments of all orders with some applications to finance.

Chapitre 4

The pointwise second-order maximum principle for optimal stochastic controls of general mean-field type

4.1 Introduction

In this chapter, we establish a second-order stochastic maximum principle for optimal stochastic control of stochastic differential equations of general mean-field type. The coefficients of the system are nonlinear and depend on the state process as well as of its probability law. The control variable is allowed to enter into both drift and diffusion terms. We establish a set of second-order necessary conditions for the optimal control in integral form. The control domain is assumed to be convex. The proof of our main result is based on the the first and second-order derivatives with respect to the probability law and by using a convex perturbation with some appropriate estimates.

The mean-field stochastic system was introduced by Kac [85] as a stochastic model for the Vlasov-Kinetic equation of plasma and the study of which was initiated by McKean model [86]. Since then, the mean-field theory has found important applications and has become a powerful tool in many fields, such as mathematical finance, economics, optimal control and stochastic mean-field games ; see Huang, Caines, and Malhame [66, 67, 68] and Lasry and Lions [76]. Stochastic differential equations (SDEs) of the mean-field type are Itô's stochastic differential equations, where the coefficients of the state equation depend on the state of the solution process as well as of its probability law. Under partial information, mean-field type maximum principle of optimality for SDEs has been established in Wang et al. [87]. Stochastic optimal control of mean-field jump-diffusion systems with delay has been studied by Meng and Shen [84]. The necessary and sufficient conditions for mean-field SDEs governed by Teugels martingales associated to Lévy process have been studied in [42, 61]. The local first-order maximum principle for optimal singular control for mean-field SDEs has been investigated by Hafayed [48]. First-order necessary conditions for mean-field FBSDEs have been studied by Hafayed et al. [62]. The mean-field maximum principle for SDEs has been established in Buckdahn et al. [12]. Mean-field game has been studied by Lions [88]. The first-order maximum principle for mean-field delay SDE have been investigated in Shen et al. [89]. A general first-order maximum principle for optimal stochastic control has been established in Peng [90]. A Peng's type maximum principle for SDEs of mean-field type was proved by Buckdahn et al., [19] by using second-order derivatives with respect to measures. Forward-backward stochastic differential equations and controlled McKean-Vlasov dynamics have been investigated in Carmona and Delarue [26]. Linear quadratic optimal control problem for conditional mean-field equation with random coefficients with applications has been investigated by Pham [91]. Infinite horizon optimal control problems for mean-field delay system with semi-Markov modulated jump-diffusion processes have been investigated by Deepa and Muthukumar [34]. First-order necessary conditions for optimal singular control problem for general mean-field SDEs, under convexity assumptions have been investigated by Hafayed et al. [41].

The maximum principle is one of the fundamental approaches for the study of optimal stochastic control problems. A pointwise second-order maximum principle for stochastic optimal controls was established by Zhang and Zhang [124] where both drift and diffusion terms may contain the control variable, and the control domain is assumed to be convex. The method was further developed in Zhang and Zhang [125] to derive a general pointwise second-order maximum principle, where the control domain is not assumed to be convex. First and second-order necessary conditions for stochastic optimal controls have been studied by [93] and [15]. A second-order maximum principle for singular optimal control for SDEs with uncontrolled diffusion coefficient has been obtained by Tang [92]. Second-order maximum principle for optimal control with recursive utilities has been obtained by Dong and Meng [33]. A second-order necessary conditions for singular optimal controls with recursive utilities of stochastic delay systems have been proved by Huo and Meng [63]. Singular optimal control problems with recursive utilities of mean-field type, where the second-order adjoint system was not a single mean-field backward stochastic differential equation, but a matrix-valued system have been studied in Hao and Meng [64].

Motivated by the recent works above, in this work we established a pointwise second-order necessary conditions for general mean-field optimal control problem. The first and second-order derivatives with respect to measure in *Wasserstein space* and the associated Itô formula with some appropriate estimates are applied to derive our result. The mean-field systems (4.15) occur naturally in the probabilistic analysis of financial optimization problems. Our problem is strongly motivated by the recent study of the mean-field games and the related mean-field stochastic control problem. This work extends the results obtained in Zhang and Zhang [124] to the general mean-field case.

The rest of this chapter is organized as follows. The formulation of the first and second-order derivatives with respect to probability measure, and basic notations are given in Section 2 . The formulation of the optimal control problem is given in Section 3 . In Sections 4 and 5 , we prove our mean-field type pointwise second-order maximum principle. The final section concludes the work and outlines some of the possible future developments.

4.2 First and second-order derivatives with respect to measure

We now recall briefly an important notion in mean-field control problems : the differentiability with respect to probability measures, in *Wasserstein space* which was introduced by P.Lions [88].

Let $\Gamma_2(\mathbb{R}^n)$ be Wasserstein space of probability measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ with finite second-moment, i.e ; $\int_{\mathbb{R}^n} |x|^2 \mu(\mathrm{d}x) < \infty$, endowed with the following 2$-$*Wasserstein metric* : for $\mu_1, \mu_2 \in \Gamma_2(\mathbb{R}^n)$,

$$\mathbb{T}(\mu_1, \mu_2) = \inf_{\rho(\cdot,\cdot)\in\Gamma_2(\mathbb{R}^{2n})} \left\{ \left[\int_{\mathbb{R}^{2n}} |x-y|^2 \rho(\mathrm{d}x, \mathrm{d}y) \right]^{\frac{1}{2}} \right\}, \tag{4.1}$$

where $\rho(\cdot, \mathbb{R}^n) = \mu_1$, and $\rho(\mathbb{R}^n, \cdot) = \mu_2$. Moreover, it has been shown that $(\Gamma_2(\mathbb{R}^n), \mathbb{T}(\cdot,\cdot))$ is a complete metric space.

The main idea is to identify a distribution $\mu \in \Gamma_2(\mathbb{R}^n)$ with a random variable $x \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ so that $\mu = P_x$ is the law of x. We assume that probability space $(\Omega, \mathcal{F}, P)$ is *rich-enough* in the sense that for every $\mu \in \Gamma_2(\mathbb{R}^n)$, there is a random variable $x \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ such that $\mu = P_x$. We suppose that there is a sub-$\sigma-$field $\mathcal{F}_0 \subset \mathcal{F}$ such that $\mathcal{F}_0$ is *rich-enough i.e,*

$$\Gamma_2(\mathbb{R}^n) := \left\{ P_x : x \in \mathbb{L}^2(\mathcal{F}_0, \mathbb{R}^n) \right\}. \tag{4.2}$$

By $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$, we denote the filtration generated by $W(\cdot)$, completed and augmented by $\mathcal{F}_0$. Next, for any function $g : \Gamma_2(\mathbb{R}^n) \to \mathbb{R}$ we define a function $\widetilde{g} : \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n) \to \mathbb{R}$

such that

$$\widetilde{g}\left(x\right)=g\left(P_{x}\right),\ x\in\mathbb{L}^{2}\left(\mathcal{F},\mathbb{R}^{n}\right). \tag{4.3}$$

Clearly, the function $\widetilde{g}$, called the *lift* of g, depends only on the law of $x\in\mathbb{L}^{2}(\mathcal{F},\mathbb{R}^{n})$ and is independent of the choice of the representative x, (see [19])

Let $g:\Gamma_{2}\left(\mathbb{R}^{n}\right)\to\mathbb{R}$. The function g is differentiable at a distribution $\mu_{0}\in\Gamma_{2}\left(\mathbb{R}^{n}\right)$ if there exists $x_{0}\in\mathbb{L}^{2}(\mathcal{F},\mathbb{R}^{n})$, with $\mu_{0}=P_{x_{0}}$ such that its lift $\widetilde{g}$ is *Fréchet-differentiable* at x_{0}. More precisely, there exists a continuous linear functional $\mathcal{D}\widetilde{g}(x_{0}):\mathbb{L}^{2}(\mathcal{F},\mathbb{R}^{n})\to\mathbb{R}$ such that

$$\widetilde{g}\left(x_{0}+\zeta\right)-\widetilde{g}\left(x_{0}\right)=\langle\mathcal{D}\widetilde{g}(x_{0})\cdot\zeta\rangle+o\left(\|\zeta\|_{2}\right)=\mathcal{D}_{\zeta}g(\mu_{0})+o\left(\|\zeta\|_{2}\right), \tag{4.4}$$

where $\langle.\cdot.\rangle$ is the dual product on $\mathbb{L}^{2}(\mathcal{F},\mathbb{R}^{n})$. We called $\mathcal{D}_{\zeta}g(\mu_{0})$ the *Fréchet-derivative* of g at μ_{0} in the direction ζ. In this case we have

$$\mathcal{D}_{\zeta}g(\mu_{0})=\langle\mathcal{D}\widetilde{g}(x_{0})\cdot\zeta\rangle=\left.\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{g}\left(x_{0}+t\zeta\right)\right|_{t=0},\ \text{with } \mu_{0}=P_{x_{0}}. \tag{4.5}$$

By applying *Riesz representation theorem*, there is a unique random variable $\Theta_{0}\in\mathbb{L}^{2}(\mathcal{F},\mathbb{R}^{n})$ such that $\langle\mathcal{D}\widetilde{g}(x_{0})\cdot\zeta\rangle=\left(\Theta_{0}\cdot\zeta\right)_{2}=E\left[\left(\Theta_{0}\cdot\zeta\right)_{2}\right]$ where $\zeta\in\mathbb{L}^{2}(\mathcal{F},\mathbb{R}^{n})$. It was shown, (see [19]) that there exists a Borel function $\Phi\left[\mu_{0}\right](\cdot):\mathbb{R}^{n}\to\mathbb{R}^{n}$, depending only on the law $\mu_{0}=P_{x_{0}}$ but not on the particular choice of the representative x_{0} such that

$$\Theta_{0}=\Phi\left[\mu_{0}\right]\left(x_{0}\right). \tag{4.6}$$

Thus we can write

$$g\left(P_{x}\right)-g\left(P_{x_{0}}\right)=\left(\Phi\left[\mu_{0}\right]\left(x_{0}\right)\cdot x-x_{0}\right)_{2}+o\left(\|x-x_{0}\|_{2}\right),\ \forall x\in\mathbb{L}^{2}\left(\mathcal{F},\mathbb{R}^{n}\right).$$

We denote

$$\partial_\mu g\left(P_{x_0}, x\right) = \Phi\left[\mu_0\right](x),\ x \in \mathbb{R}^n.$$

Moreover, we have the following identities

$$\mathcal{D}\widetilde{g}(x_0) = \Theta_0 = \Phi\left[\mu_0\right](x_0) = \partial_\mu g\left(P_{x_0}, x_0\right), \tag{4.7}$$

and

$$\mathrm{D}_\zeta g(P_{x_0}) = \left\langle \partial_\mu g\left(P_{x_0}, x_0\right) \cdot \zeta \right\rangle, \tag{4.8}$$

where $\zeta = x - x_0$.For each $\mu \in \Gamma_2\left(\mathbb{R}^n\right)$, $\partial_\mu g\left(P_x, \cdot\right) = \Phi\left[P_x\right](\cdot)$ is only defined in $P_x(\mathrm{d}x)-$ *a.e.* sense where $\mu = P_x$.

Among the different notions of differentiability of a function g defined over $\Gamma_2\left(\mathbb{R}^n\right)$, we apply the differentiability with respect to probability measures. We shall follow the approach introduced in P. Lions [88] and later detailed in Cardaliaguet [27]. We refer the reader to Buckdahn et al., [19] for more discussions.

We say that the function $g \in \mathbb{C}_b^{1,1}(\Gamma_2(\mathbb{R}^n))$ if for all $x \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ there exists a P_x-modification of $\partial_\mu g\left(P_x, \cdot\right)$ (denoted by $\partial_\mu g$) such that $\partial_\mu g : \Gamma_2\left(\mathbb{R}^n\right) \times \mathbb{R}^n \to \mathbb{R}^n$ is bounded and Lipschitz continuous. That is for some $C > 0$, it holds that

(1) $|\partial_\mu g(\mu, x)| \leq C$, $\forall \mu \in \Gamma_2(\mathbb{R}^n)$, $\forall x \in \mathbb{R}^n$.

(2) $|\partial_\mu g(\mu, x) - \partial_\mu g(\mu', x')| \leq C\left[\mathbb{T}\left(\mu, \mu'\right) + |x - x'|\right]$, $\forall \mu, \mu' \in \Gamma_2(\mathbb{R}^n)$, $\forall x, x' \in \mathbb{R}^n$.

We should note that if $g \in \mathbb{C}_b^{1,1}(\Gamma_2(\mathbb{R}^n))$, the version of $\partial_\mu g\left(P_x, \cdot\right)$, $x \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$, is unique (see [19, Remark 2.2], and [27]). We shall denote by $\partial_\mu g\left(t, x, \mu_0\right)$ the derivative with respect to μ computed at μ_0 whenever all the other variables (t, x) are held fixed, $\partial_\mu g\left(t, x, \mu_0\right) = \partial_\mu g\left(t, x, \mu\right)\big|_{\mu=\mu_0}$.

Second-order derivatives with respect to probability law : We present a second order derivatives with respect to measure of probability.

Let $g \in \mathbb{C}_b^{1,1}(\Gamma_2(\mathbb{R}^n))$ and consider the mapping $\left(\partial_\mu g\left(\cdot,\cdot\right)_1, \partial_\mu g\left(\cdot,\cdot\right)_2, ..., \partial_\mu g\left(\cdot,\cdot\right)_n\right)^\top$:

$\Gamma_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$.

We say that the function $g \in \mathbb{C}_b^{2,1}(\Gamma_2(\mathbb{R}^n))$ if $g \in \mathbb{C}_b^{1,1}(\Gamma_2(\mathbb{R}^n))$ such that $\partial_\mu g(\cdot, x) : \Gamma_2(\mathbb{R}^n) \to \mathbb{R}^n$

(1) $\partial_\mu g(\cdot, y)_i \in \mathbb{C}_b^{1,1}(\Gamma_2(\mathbb{R}^n))$, $\forall y \in \mathbb{R}^n$ and $i \in \{1, 2, ..., n\}$.

(2) $\partial_\mu g(\mu, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable, for every $\mu \in \Gamma_2(\mathbb{R}^n)$.

(3) The maps $\partial_x \partial_\mu g(\cdot, \cdot) : \Gamma_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ and $\partial_\mu^2 g(P_{x_0}, y, z) : \Gamma_2(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ are bounded and Lipschitz continuous, where

$$\partial_\mu^2 g(P_{x_0}, y, z) = \partial_\mu \left[\partial_\mu g(\cdot, y)\right](P_{x_0}, z).$$

Similarly, we define $\partial_u \partial_\mu g(\cdot, \cdot) : \Gamma_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ by

$$\partial_u \partial_\mu g(P_{x_0}, y, u, z) = \partial_u \left[\partial_\mu g(\cdot, y, u)\right](P_{x_0}, z).$$

Second-order Taylor expansion : Now, we give a second-order Taylor expansion that plays an essential role to establish our maximum principle. Let $g \in \mathbb{C}_b^{2,1}(\Gamma_2(\mathbb{R}^n))$, for $j \in \{1, 2, ..., n\}$.

$$\begin{aligned} \mathcal{D}\widetilde{g}_j(x_0 + \xi) - \mathcal{D}\widetilde{g}_j(x_0) &= \left[\partial_\mu g\right]_j (P_{x_0+\xi}, x_0 + \xi)) - \left[\partial_\mu g\right]_j (P_{x_0}, x_0)) \\ &= \left[\left[\partial_\mu g\right]_j (P_{x_0+\xi}, z)) - \left[\partial_\mu g\right]_j (P_{x_0}, z))\right]\Big|_{z=x_0+\xi} \\ &+ \left[\partial_\mu g\right]_j (P_{x_0}, z)\Big|_{z=x_0+\xi} - \left[\partial_\mu g\right]_j (P_{x_0}, z)\Big|_{z=x_0} \\ &= \int_0^1 \left\langle \mathcal{D}\widetilde{[\partial_\mu g]}_j (x_0 + \theta\xi, z) \cdot \xi \right\rangle d\theta \Big|_{z=x_0} \\ &+ \left(\partial_z \left[\partial_\mu g\right]_j (P_{x_0}, x_0), \xi\right) + o\left(\|\xi\|_2\right). \end{aligned} \tag{4.9}$$

Then, we obtain

$$\begin{aligned}\mathcal{D}\widetilde{[\partial_\mu g]}_j(x_0,y) &= \partial_\mu\left[[\partial_\mu g]_j(\cdot,y)\right](P_{x_0},x_0)\\ &= \left[\partial_\mu^2 g\right]_j(P_{x_0},y,z)\Big|_{z=x_0}.\end{aligned}$$

Second-order derivatives of f at a measure μ_0. Let $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P})$ be a copy of the probability space $(\Omega,\mathcal{F},P)$. For any pair of random variable $(Z,\xi)\in\mathbb{L}^2(\mathcal{F},\mathbb{R}^d)\times\mathbb{L}^2(\mathcal{F},\mathbb{R}^d)$, we let $(\widehat{Z},\widehat{\xi})$ be an independent copy of (Z,ξ) defined on $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P})$. On the product probability space $(\Omega\times\widehat{\Omega},\mathcal{F}\otimes\widehat{\mathcal{F}},P\otimes\widehat{P})$, we define $(\widehat{Z},\widehat{\xi})(w,\widehat{w})=(Z(\widehat{w}),\xi(\widehat{w}))$ for any $(w,\widehat{w})\in\Omega\times\widehat{\Omega}$. Let $(\widehat{u^*}(t),\widehat{x^*}(t))$ be an independent copy of $(u^*(t),x^*(t))$, so that $P_{x^*(t)}=\widehat{P}_{\widehat{x^*}(t)}$. We denote by $\widehat{E}$ the expectation under probability measure $\widehat{P}$, where $\widehat{E}(X)=\int_{\widehat{\Omega}}X(\widehat{w})\mathrm{d}\widehat{P}(\widehat{w})$.

Now, for any $\mu_0\in\Gamma_2(\mathbb{R}^n)$, in the direction ξ, we define the second-order derivatives of a function g at μ_0 with $\mu_0=P_{x_0}$

$$\begin{aligned}\mathcal{D}_\xi^2 g(\mu_0) &= \left\langle\left\langle\mathcal{D}\widetilde{[\partial_\mu g]}_j(\cdot,y)(P_{x_0},z)\mid_{z=\widehat{x_0}}\cdot\widehat{\xi}\right\rangle\mid_{y=\widehat{x_0}},\xi\right\rangle\\ &+\langle(\partial_y\partial_\mu g)(P_{x_0},x_0)\,\xi\cdot\xi\rangle\\ &= E\left[\widehat{E}\left[tr\left(\partial_\mu^2 g(P_{x_0},x_0,\widehat{x_0})\widehat{\xi}\otimes\xi\right)\right]\right]\\ &+E\left[tr\left(\partial_y\partial_\mu g(P_{x_0},x_0)\xi\otimes\xi\right)\right],\end{aligned}\tag{4.10}$$

where

$$\begin{aligned}&\widehat{E}\left[tr\left(\partial_\mu^2 g(P_{x_0},x_0,\widehat{x_0})\widehat{\xi}\otimes\xi\right)\right]\\ &=\int_{\widehat{\Omega}}tr\left[\partial_\mu^2 g(P_{x_0},x_0(w),\widehat{x_0}(\widehat{w}))\widehat{\xi}\otimes\xi(w,\widehat{w})\right]\mathrm{d}\widehat{P}(\widehat{w}).\end{aligned}\tag{4.11}$$

Furthermore, we have

$$E\left[\widehat{E}\left[tr\left[\partial_\mu^2 g(P_{x_0},x_0,\widehat{x_0})\widehat{\xi}\otimes\xi\right]\right]\right]$$
$$=\int_\Omega\int_{\widehat{\Omega}} tr\left[\partial_\mu^2 g(P_{x_0},x_0\left(w\right),\widehat{x_0}(\widehat{w}))\widehat{\xi}\otimes\xi(w,\widehat{w})\right]\mathrm{d}(P\otimes\widehat{P})(w,\widehat{w}). \tag{4.12}$$

For convenience, we will use the following notations throughout the work, for $\psi=f,\sigma,\ell,h$:

$$\begin{aligned}
\delta\psi(t)&=\psi(t,x^*(t),P_{x^*(t)},u^*(t))-\psi(t,x^\varepsilon(t),P_{x^\varepsilon(t)},u^\varepsilon(t)),\\
\psi_x(t)&=\frac{\partial\psi}{\partial x}(t,x^*(t),P_{x^*(t)},u^*(t)),\\
\psi_u(t)&=\frac{\partial\psi}{\partial u}(t,x^*(t),P_{x^*(t)},u^*(t)),\\
\widehat{\psi}_\mu(t)&=\partial_\mu\psi(t,x^*(t),P_{x^*(t)},u^*(t);\widehat{x^*}(t)),\\
\widehat{\psi}_\mu^*(t)&=\partial_\mu\psi(t,\widehat{x^*}(t),P_{x^*(t)},\widehat{u^*}(t);x^*(t)).
\end{aligned}\tag{4.13}$$

Furthermore, we denote

$$\begin{aligned}
\psi_{xx}(t)&=\frac{\partial^2\psi}{\partial x^2}(t,x^*(t),P_{x^*(t)},u^*(t)),\\
\psi_{uu}(t)&=\frac{\partial^2\psi}{\partial u^2}(t,x^*(t),P_{x^*(t)},u^*(t)),\\
\widehat{\psi}_{\mu\mu}(t)&=\partial_\mu^2\psi(t,x^*(t),P_{x^*(t)},u^*(t);x^*(t),\widehat{x^*}(t)),\\
\psi_{x\mu}(t)&=\partial_x\partial_\mu\psi(t,x^*(t),P_{x^*(t)},u^*(t);x^*(t)),\\
\widehat{\psi}_{x\mu}^*(t)&=\partial_x\partial_\mu\psi(t,\widehat{x^*}(t),P_{x^*(t)},\widehat{u^*}(t);\widehat{x^*}(t)).
\end{aligned}\tag{4.14}$$

4.3 Formulation of the control problem

Let us formulate the optimal mean-field type control problem. Let T be a fixed positive real number and $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in[0,T]},P)$ be a fixed filtered probability space satisfying the usual conditions in which *one*−dimensional Brownian motion $W(t)=\{W(t):0\le t\le T\}$ and $W(0)=0$ is defined. We study optimal solutions of general stochastic control problem driven by mean-field stochastic differential equation of the form :

$$\begin{cases} dx^u(t) = f\left(t, x^u(t), P_{x^u(t)}, u(t)\right) dt + \sigma\left(t, x^u(t), P_{x^u(t)}, u(t)\right) \mathrm{d}W(t), \\ x^u(0) = x_0. \end{cases} \tag{4.15}$$

The goal of our optimal control problem is to minimize the following cost functional

$$J\left(u(\cdot)\right) = E\left[h(x^u(T), P_{x^u(t)}) + \int_0^T \ell(t, x^u(t), P_{x^u(t)}, u(t))\mathrm{d}t\right]. \tag{4.16}$$

An admissible control $u(t)$ is an $\mathcal{F}_t-$predictable process with values in some non-empty convex subset $\mathbb{U}$ of $\mathbb{R}^k$ such that $E\int_0^T |u(t)|^2 \mathrm{d}t < +\infty$. We called $\mathbb{U}$ the control domain. We denote $\mathcal{U}\left([0,T]\right)$ the set of all admissible controls. That is,

$$\mathcal{U}\left([0,T]\right) = \left\{u\left(t\right)_{t\in[0,T]} : \text{is an} \mathcal{F}_t - \text{predictable process, and } E\int_0^T |u(t)|^2 \mathrm{d}t < \infty\right\}.$$

We suppose that an optimal control exists. Any admissible control $u^*(\cdot) \in \mathcal{U}\left([0,T]\right)$ satisfying

$$J\left(u^*(\cdot)\right) = \inf_{u(\cdot)\in\mathcal{U}([0,T])} J\left(u(\cdot)\right), \tag{4.17}$$

is called an optimal control. The maps

$$\begin{aligned} &f : [0,T] \times \mathbb{R}^n \times \Gamma_2\left(\mathbb{R}^n\right) \times \mathbb{U} \to \mathbb{R}^n, \\ &\sigma : [0,T] \times \mathbb{R}^n \times \Gamma_2\left(\mathbb{R}^n\right) \times \mathbb{U} \to \mathbb{R}^n, \\ &\ell : [0,T] \times \mathbb{R}^n \times \Gamma_2\left(\mathbb{R}^n\right) \times \mathbb{U} \to \mathbb{R}, \\ &h : \mathbb{R}^n \times \Gamma_2\left(\mathbb{R}^n\right) \to \mathbb{R}, \end{aligned}$$

are given deterministic functions.

To avoid excessive complexity in the notation of this work, we will make the simplifying assumption that all processes are *one*-dimensional (i.e., $n = m = 1$) in the subsequent sections.

We define a metric $d(\cdot,\cdot)$ on the space of admissible controls $\mathcal{U}([0,T])$ such that $(\mathcal{U}([0,T]), d)$ becomes a complete metric space. For any $u(\cdot)$ and $v(\cdot) \in \mathcal{U}([0,T])$ we set

$$d(u(\cdot), v(\cdot)) = \left[E \int_0^T |u(t) - v(t)|^2 \, dt \right]^{\frac{1}{2}}. \tag{4.18}$$

Moreover, it has been shown that $(\mathcal{U}([0,T]), d)$ is a complete metric space.

Assumptions. The following assumptions will be in force throughout this work, where x denotes the state variable, and u the control variable.

- **Assumption (H 4.1)** For fixed $\mu \in \Gamma_2(\mathbb{R})$, for any $(x,u) \in \mathbb{R}^d \times \mathbb{U}$, the coefficients f, σ, ℓ are measurable in all variables and continuously differentiable up to order-2 with respect to x, u; and al their partial derivatives are uniformly bounded. The function h is continuously differentiable up to order-2 with respect to x and u. Moreover the second-order derivatives $\psi_{xx}, \psi_{uu}, \psi_{xu}$, for $\psi = f, \sigma, \ell$ are bounded and Lipschitz in (x,u). The derivative h_{xx} is bounded and Lipschitz in x.

$$\begin{aligned} |\ell(t,x,\mu,u)| &\leq C(1 + |x|^2 + |u|^2), \\ |h(x,u)| &\leq C(1 + |x|^2), \\ |\ell_x(t,x,\mu,u)| + |\ell_u(t,x,\mu,u)| &\leq C(1 + |x| + |u|), \\ |h_x(x,u)| &\leq C(1 + |x|), \end{aligned}$$

where $C > 0$ is a generic positive constant, which may vary from line to line.

- **Assumption (H 4.2)** (1) For fixed $x \in \mathbb{R}$, for all $u(t) \in \mathbb{U} : f, \sigma, \ell \in \mathbb{C}_b^{1,1}(\Gamma_2(\mathbb{R}^d); \mathbb{R})$ and $h \in \mathbb{C}_b^{1,1}(\Gamma_2(\mathbb{R}); \mathbb{R})$.

  (2) All the derivatives with respect to measure f_μ, σ_μ, ℓ_μ, h_μ are bounded and Lipschitz continuous, with Lipschitz constants independent of u.**Assumption (H 4.3)** (1) For all $u(t) \in \mathbb{U}$, $f, \sigma, \ell \in \mathbb{C}_b^{2,1}(\Gamma_2(\mathbb{R}); \mathbb{R})$, and $h \in \mathbb{C}_b^{2,1}(\Gamma_2(\mathbb{R}); \mathbb{R})$. (2)

All the second-order derivatives of $\psi_{\mu\mu}$, $\psi_{x\mu}$ $\psi_{u\mu}$ for $\psi = f, \sigma, \ell$ are bounded and Lipschitz continuous in (x, μ, u) with Lipschitz constants independent of u. (3) The second-order derivative $h_{\mu\mu}$, $h_{x\mu}$ is bounded and Lipschitz in x and μ.

From assumption (H 4.3), Item 3, since the second-order derivatives are Lipschitz continuous, we have

$$\begin{cases} \forall \mu, \mu' \in \Gamma_2(\mathbb{R}^n), \forall x, x' \in \mathbb{R}^n, \forall u, u' \in \mathbb{U} : \\ \left|\left(\psi_{\mu\mu}, \psi_{x\mu}, \psi_{u\mu}\right)(t, x, \mu, u) - \left(\psi_{\mu\mu}, \psi_{x\mu}, \psi_{u\mu}\right)(t, x', \mu', u')\right| \\ \leq C\left[\mathbb{T}(\mu, \mu') + |x - x'| + |u - u'|\right]. \end{cases} \tag{4.19}$$

Similarly for Item 4, we deduce $\forall \mu, \mu' \in \Gamma_2(\mathbb{R}^n)$, and $\forall x, x' \in \mathbb{R}^n$:

$$\left|\left(h_{\mu\mu}, h_{x\mu}\right)(x, \mu) - \left(h_{\mu\mu}, h_{x\mu}\right)(x', \mu')\right| \leq C\left[\mathbb{T}(\mu, \mu') + |x - x'|\right]. \tag{4.20}$$

Under the assumptions (H1) and (H2), for each $u(\cdot) \in \mathcal{U}([0, T])$, Eq-(4.15) has a unique strong solution $x^u(\cdot)$ given by

$$x^u(t) = x_0 + \int_0^t f(r, x^u(r), P_{x^u(r)}, u(r))\mathrm{d}r + \int_0^t \sigma(r, x^u(r), P_{x^u(r)}, u(r))\mathrm{d}W(r),$$

such that $E\left[\sup_{t\in[0,T]} |x^u(t)|^2\right] < +\infty$, and the functional $J(\cdot)$ is well defined.

Let $u^*(\cdot) \in \mathcal{U}([0, T])$ be an optimal control for the problem (4.15)-(4.16), the corresponding state process $x^*(\cdot) = x^{u^*}(\cdot)$, solution of mean-field dynamic (4.15).

Finally, from assumption (H3) we define for $t \in [0, T]$:

$$\begin{aligned} \mathcal{L}_{xx}(t, \varphi, z) &= \frac{1}{2}\partial_{xx}\varphi(t, x^*(t), P_{x^*(t)}, u^*(t))z^2, \\ \mathcal{L}_{\mu y}(t, \widehat{\varphi}, z) &= \frac{1}{2}\partial_y\partial_\mu\varphi(t, x^*(t), P_{x^*(t)}, u^*(t); \widehat{x^*})z^2. \end{aligned} \tag{4.21}$$

The Hamiltonian. Let us define the Hamiltonian associated to our control problem.

For any $(t, x, \mu, u, p_1, q_1) \in [0,T] \times \mathbb{R} \times \Gamma_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$$H(t,x,\mu,u,p_1,q_1) = f(t,x,\mu,u)p_1 + \sigma(t,x,\mu,u)q_1 - \ell(t,x,\mu,u), \tag{4.22}$$

where $(p_1(\cdot), q_1(\cdot))$ is a pair of adapted processes, solution of the first-order adjoint equation (4.26).

We denote

$$H(t) = H(t, x^*, P_{x^*}, u^*, p_1, q_1). \tag{4.23}$$

We define

$$\begin{aligned}
\delta H(t) &= \delta f(t)p_1(t) + \delta\sigma(t)q_1(t) - \delta\ell(t), \\
H_x(t) &= f_x(t)p_1(t) + \sigma_x(t)q_1(t) - \ell_x(t), \\
H_u(t) &= f_u(t)p_1(t) + \sigma_u(t)q_1(t) - \ell_u(t), \\
H_\mu(t) &= f_\mu(t)p_1(t) + \sigma_\mu(t)q_1(t) - \ell_\mu(t), \\
H_{xx}(t) &= f_{xx}(t)p_1(t) + \sigma_{xx}(t) \otimes q_1(t) - \ell_{xx}(t), \\
H_{uu}(t) &= f_{uu}(t)p_1(t) + \sigma_{uu}(t) \otimes q_1(t) - \ell_{uu}(t), \\
H_{x\mu}(t) &= f_{x\mu}(t)p_1(t) + \sigma_{x\mu}(t) \otimes q_1(t) - \ell_{x\mu}(t).
\end{aligned} \tag{4.24}$$

To establish our integral-type second-order necessary condition for stochastic optimal control, we introduce the following notion.

Singular control in the classical sense : We call an admissible control $\overline{u}(\cdot)$ a singular control in the classical sense if $\overline{u}(\cdot)$ satisfies :

$$\begin{cases}
H_u(t, \overline{x}(t), P_{\overline{x}(t)}, \overline{u}(t), \overline{p}_1(t), \overline{q}_1(t)) = 0, \quad a.s.\ a.e.\ t \in [0,T], \\
H_{uu}(t, \overline{x}(t), P_{\overline{x}(t)}, \overline{u}(t), \overline{p}_1(t), \overline{q}_1(t)) + \overline{p}_2(t)\sigma_u\left(t, \overline{x}(t), P_{\overline{x}(t)}, \overline{u}(t)\right)^2] = 0, \\
\qquad a.s.\ a.e.\ t \in [0,T].
\end{cases} \tag{4.25}$$

We introduce the adjoint equations involved in the stochastic maximum principle for our control problem.

First-order adjoint equation. We consider the first-order adjoint equation, which is the following mean-field linear BSDE :

$$\begin{cases} -\mathrm{d}p_1(t) = \Big[f_x(t)p_1(t) + \widehat{E}(\widehat{f}^*_\mu(t)\widehat{p_1}(t)) + \sigma_x(t)q_1(t) + \widehat{E}(\widehat{\sigma}^*_\mu(t)\widehat{q_1}(t)) \\ \qquad\qquad - \ell_x(t) - \widehat{E}(\widehat{\ell}^*_\mu(t))\Big]\,\mathrm{d}t - q_1(t)\mathrm{d}W(t), \\ p_1(T) = -h_x(T) - \widehat{E}[\widehat{h}^*_\mu(T)]. \end{cases} \tag{4.26}$$

Here, from (4.13), $t \in [0,T]$, for $\varphi = f, \sigma, \ell$

$$\begin{aligned} \widehat{E}\left[\partial_\mu \widehat{\varphi}^*(t)\right] &= \widehat{E}\left[\partial_\mu \varphi(t, \widehat{x}^*(t), P_{x^*(t)}, \widehat{u}^*(t); z)\right]\Big|_{z=x^*(t)} \\ &= \int_{\widehat{\Omega}} \partial_\mu \varphi(t, \widehat{x}^*(t,\widehat{w}), P_{x^*(t,w)}, \widehat{u}^*(t,\widehat{w}); x^*(t,w))\mathrm{d}\widehat{P}(\widehat{w}), \end{aligned} \tag{4.27}$$

and the same argument allows to show that

$$\begin{aligned} \widehat{E}\left[\partial_\mu \widehat{h}^*(T)\right] &= \widehat{E}\left[\partial_\mu h(\widehat{x}^*(T), P_{x^*(T)}; z)\right]\Big|_{z=x^*(t)} \\ &= \int_{\widehat{\Omega}} \partial_\mu h(\widehat{x}^*(T,\widehat{w}), P_{x(T,w)}; x^*(T,w))\mathrm{d}\widehat{P}(\widehat{w}). \end{aligned} \tag{4.28}$$

Second-order adjoint equation. Consider the following linear BSDE :

$$\begin{cases} \mathrm{d}p_2(t) = -\Big\{2(f_x(t) + \widehat{E}[\widehat{f}^*_\mu(t)])p_2(t) + [\sigma_x(t) + \widehat{E}(\widehat{\sigma}^*_\mu(t))]^2 p_2(t) \\ \qquad + 2(\sigma_x(t) + \widehat{E}[\widehat{\sigma}_\mu(t)])q_2(t) + (H_{xx}(t) + \widehat{E}[\widehat{H}^*_{x\mu}(t)] + \widehat{E}[\widehat{H}^*_{\mu\mu}(t)])\Big\}\,\mathrm{d}t + q_2(t)\mathrm{d}W(t), \\ p_2(T) = -(h_{xx}(T) + \widehat{E}[\widehat{h}^*_{x\mu}(T)]). \end{cases} \tag{4.29}$$

Similar to (5.13) and (5.15), we have

$$\begin{aligned}\widehat{E}[\widehat{H}^*_{\mu y}(t)]) &= \widehat{E}\left[\partial_\mu \partial_y H(t, \widehat{x}^*(t), P_{x^*(t)}, \widehat{u}^*(t), \widehat{p_1}(t), \widehat{q_1}(t); y)\right]\Big|_{y=x^*(t)} \\ &= \int_{\widehat{\Omega}} \partial_\mu \partial_y H(t, \widehat{x}^*(t, \widehat{w}), P_{x^*(t)}, \widehat{u}^*(t, \widehat{w}), \widehat{p_1}(t), \widehat{q_1}(t); x^*(t)) \mathrm{d}\widehat{P}(\widehat{w}).\end{aligned}$$

Since the derivatives $f_x, f_\mu, \sigma_x, \sigma_\mu, \ell_x, \ell_\mu, h_x, h_\mu$ are bounded, (from assumptions (H 4.1) and (H 4.2)), the mean-field BSDE (4.26) admits a unique $\mathcal{F}_t$-adapted strong solution $(p_1(\cdot), q_1(\cdot))$ such that

$$\begin{aligned}p_1(t) = -(h_x(T) + \widehat{E}[\widehat{h}^*_\mu(T)]) &+ \int_t^T \Big[f_x(s) p_1(s) + \widehat{E}\left[\widehat{f}^*_\mu(s) \widehat{p_1}(s)\right] \\ &+ \sigma_x(s) q_1(s) + \widehat{E}(\widehat{\sigma}^*_\mu(s) \widehat{q_1}(s)) - \ell_x(s) - \widehat{E}(\widehat{\ell}^*_\mu(s)) \Big] \mathrm{d}s \\ &- \int_t^T q_1(s) \mathrm{d}W(s),\end{aligned}$$

which satisfies the following estimate

$$E\left[\sup_{t\in[0,T]} |p_1(t)|^2 + \int_0^T |q_1(t)|^2 \mathrm{d}t\right] < \infty. \tag{4.30}$$

Also, from the boundedness of the first and second-order derivatives of the coefficients f, σ, ℓ, h with respect to (x, μ), (assumptions (H3)), Eq-(4.29) has a unique $\mathcal{F}_t-$adapted strong solution $(p_2(\cdot), q_2(\cdot))$ such that

$$\begin{aligned}p_2(t) &= -(h_{xx}(T) + \widehat{E}[\widehat{h}^*_{x\mu}(T)]) \\ &+ \int_t^T \Big\{ 2(f_x(s) + \widehat{E}[\widehat{f}^*_\mu(s)] p_2(s) + [\sigma_x(s) + \widehat{E}(\widehat{\sigma}^*_\mu(s))]^2 p_2(s) \\ &+ 2(\sigma_x(s) + \widehat{E}[\widehat{\sigma}^*_\mu(s)]) q_2(s) + (H_{xx}(s) + \widehat{E}[\widehat{H}^*_{x\mu}(s)] + \widehat{E}[\widehat{H}^*_{\mu\mu}(s)]) \Big\} \mathrm{d}s \\ &- \int_t^T q_2(s) \mathrm{d}W(s),\end{aligned}$$

which satisfies the following estimate

$$E\left[\sup_{t\in[0,T]}|p_2(t)|^2+\int_0^T|q_2(t)|^2\,\mathrm{d}t\right]<\infty. \tag{4.31}$$

If the coefficients f,σ,ℓ,h do not explicitly depend on law of the solution, the mean-field BSDE-(4.26) and (4.29) reduce to a standard BSDE (see Zhang and Zhang [124]. Peng [90, Equation 19, page 974]), or Buckdahn et al., ([19]).

4.4 Mean-field second-order stochastic maximum principle in integral form

The purpose of the stochastic maximum principle is to establish a set of necessary conditions for optimality satisfied by an optimal control. In our work, the goal is to derive a set of second-order necessary conditions for the optimal control, where the system evolves according to controlled mean-field SDEs. To derive our main result, the approach that we use is based on the convex perturbation of the optimal control. This perturbation is described as follows : Let $u^*(\cdot)$ is an optimal control and $u(\cdot)$ is an arbitrary element of $\mathcal{F}_t-$measurable random variable with values in $\mathbb{U}$ which we consider as fixed from now on. We define a perturbed control $u^\varepsilon(\cdot)$ as follows. Let

$$u^\varepsilon(t)=u^*(t)+\varepsilon\left(u(t)-u^*(t)\right), \tag{4.32}$$

where $\varepsilon>0$ is sufficiently small. Since $\mathbb{U}$ is convex, $u^\varepsilon(\cdot)\in\mathcal{U}\left([0,T]\right)$. We denote by $x^\varepsilon(\cdot)$ the solution of Eq-(4.15) associated with $u^\varepsilon(\cdot)$.

Under assumptions (H 4.1), (H 4.2) and (H 4.3), we introduce the following new variational equations for our control problem.

First-order variational equation : let $t \in [0, T]$

$$\begin{cases} \mathrm{d}y_1(t) = \Big[f_x(t)y_1(t) + \widehat{E}[\widehat{f}_\mu(t)\widehat{y}_1(t)] + f_u(t)v(t)\Big]\,\mathrm{d}t \\ \qquad\qquad + \Big[\sigma_x(t)y_1(t) + \widehat{E}[\widehat{\sigma}_\mu(t)\widehat{y}_1(t)] + \sigma_u(t)v(t)\Big]\,\mathrm{d}W(t) \\ y_1(0) = 0. \end{cases} \tag{4.33}$$

Here the process $y_1(\cdot)$ is called the *first-order variational process*, associated to $u(\cdot)$. Since the coefficients $f_x, f_\mu, f_u, \sigma_x, \sigma_\mu, \sigma_u$ in (4.33) are bounded, it follows that there exists a unique solution $y_1(\cdot)$ such that

$$E\left[\sup_{t\in[0,T]} |y_1(t)|^k\right] < C_k, \quad \text{for} \quad k \geq 2. \tag{4.34}$$

We note that unless specified, for each $k \in \mathbb{R}_+$, we denote by $C_k > 0$ a generic positive constant depending only on k, which may vary from line to line.

Second-order variational equation :

$$\begin{cases} \mathrm{d}y_2(t) = \Big[f_x(t)y_2(t) + \widehat{E}[\widehat{f}_\mu(t)\widehat{y}_2(t)] + f_{xx}(t)y_1^2(t) + \widehat{E}[\widehat{f}_{x\mu}(t)\widehat{y}_1(t)]y_1(t) \\ \qquad + 2f_{xu}(t)y_1(t)v(t) + 2\widehat{E}[\widehat{f}_{u\mu}(t)\widehat{y}_1(t)]v(t) + f_{uu}(t)v^2(t)\Big]\,\mathrm{d}t \\ \qquad + \Big[\sigma_x(t)y_2(t) + \widehat{E}[\widehat{\sigma}_\mu(t)\widehat{y}_2(t)] + \sigma_{xx}(t)y_1^2(t) + \widehat{E}[\widehat{\sigma}_{x\mu}(t)\widehat{y}_1(t)]y_1(t) \\ \qquad + 2\sigma_{xu}(t)y_1(t)v(t) + 2\widehat{E}[\widehat{\sigma}_{u\mu}(t)\widehat{y}_1(t)]v(t) + \sigma_{uu}(t)v^2(t)\Big]\,\mathrm{d}W(t), \\ y_2(0) = 0. \end{cases} \tag{4.35}$$

Here the process $y_2(\cdot)$ is called the *second-order variational process.* Moreover, under assumptions (H 4.1), (H 4.2) and (H 4.3), equation (4.35) admits a unique $\mathcal{F}$-adapted strong solution such that : for any $k \geq 1$ we have

$$E(\sup_{t\in[0,T]} |y_2(t)|^k) \leq C_k. \tag{4.36}$$

We shall establish some fundamental estimates that will play the crucial roles for the proof of our stochastic maximum principle.

Proposition 4.4.1. Let $x^{\varepsilon}(\cdot)$ and $x^{*}(\cdot)$ be the states of (4.37) associated to $u^{\varepsilon}(\cdot)$ and $u^{*}(\cdot)$ respectively. Let $y_1(\cdot)$ be the solution of (4.33). Then the following estimates hold :

$$E\left[\sup_{t\in[0,T]}\left|x^{\varepsilon}(t)-x^{*}(t)\right|^{2k}\right]\leq C_k\varepsilon^{k}, \tag{4.37}$$

$$\lim_{\varepsilon\to 0}E\left[\sup_{0\leq t\leq T}\left|\frac{x^{\varepsilon}(t)-x^{*}(t)}{\varepsilon}-y_1(t)\right|^{2}\right]=0. \tag{4.38}$$

Proof. The proof of estimate (4.37) follows immediately from [19, Proposition 4.2, estimate (4.8)].

Let us turn to estimate (4.38). We put

$$\gamma^{\varepsilon}(t)=\frac{x^{\varepsilon}(t)-x^{*}(t)}{\varepsilon}-y_1(t),\ t\in[0,T]. \tag{4.39}$$

Since $D_{\xi}f(P_{Z_0})=\left\langle D\widetilde{f}(Z_0)\cdot\xi\right\rangle=\frac{d}{dt}\widetilde{f}\left(Z_0+t\xi\right)\Big|_{t=0}$, we have the following simple form of the *Taylor expansion*

$$f(P_{Z_0+\xi})-f(P_{Z_0})=D_{\xi}f(P_{Z_0})+\mathcal{R}(\xi),$$

where $\mathcal{R}(\xi)$ is of order $O\left(\|\xi\|_2\right)$ with $O\left(\|\xi\|_2\right) \to 0$ for $\xi \in \mathbb{L}^2\left(\mathcal{F}, \mathbb{R}^d\right)$.

$$
\begin{aligned}
\gamma^{\varepsilon}(t) = & \frac{1}{\varepsilon} \int_0^t \left[f\left(s, x^{\varepsilon}(s), P_{x^{\varepsilon}(s)}, u^{\varepsilon}(s)\right) - f\left(s, x^*(s), P_{x^*(s)}, u^*(s)\right)\right] \mathrm{d}s \\
& + \frac{1}{\varepsilon} \int_0^t \left[\sigma\left(s, x^{\varepsilon}(s), P_{x^{\varepsilon}(s)}, u^{\varepsilon}(s)\right) - \sigma\left(s, x^*(s), P_{x^*(s)}, u^*(s)\right)\right] \mathrm{d}W(s) \\
& - \int_0^t \Big\{f_x\left(s, x^*(s), P_{x^*(s)}, u^*(s)\right) y_1(s) + \widehat{E}\left[f_\mu(s, x^*(s), P_{x^*(s)}, u^*(s); \widehat{x}^*(s))\widehat{y}_1(s)\right] \\
& + f_u(s, x^*(s), P_{x^*(s)}, u^*(s))v(s)\Big\} \mathrm{d}s \\
& - \int_0^t \Big\{\sigma_x\left(s, x^*(s), P_{x^*(s)}, u^*(s)\right) y_1(s) + \widehat{E}\left[\sigma_\mu(s, x^*(s), P_{x^*(s)}, u^*(s); \widehat{x}^*(s))\widehat{y}_1(s)\right] \\
& + \sigma_u(s, x^*(s), P_{x^*(s)}, u^*(s))v(s)\Big\} \mathrm{d}W(s).
\end{aligned}
$$

We decompose the integral $\frac{1}{\varepsilon}\int_0^t [f(s, x^{\varepsilon}(s), P_{x^{\varepsilon}(s)}, u^{\varepsilon}(s)) - f(s, x^*(s), P_{x^*(s)}, u^*(s))]\mathrm{d}s$ into the following parts

$$
\begin{aligned}
& \frac{1}{\varepsilon} \int_0^t (f(s, x^{\varepsilon}(s), P_{x^{\varepsilon}(s)}, u^{\varepsilon}(s)) - f(s, x^*(s), P_{x^*(s)}, u^*(s)))\mathrm{d}s \\
& = \frac{1}{\varepsilon} \int_0^t (f(s, x^{\varepsilon}(s), P_{x^{\varepsilon}(s)}, u^{\varepsilon}(s)) - f(s, x^*(s), P_{x^{\varepsilon}(s)}, u^{\varepsilon}(s)))\mathrm{d}s \\
& + \frac{1}{\varepsilon} \int_0^t (f(s, x^*(s), P_{x^{\varepsilon}(s)}, u^{\varepsilon}(s)) - f(s, x^*(s), P_{x^*(s)}, u^{\varepsilon}(s)))\mathrm{d}s \\
& + \frac{1}{\varepsilon} \int_0^t (f(s, x^*(s), P_{x^*(s)}, u^{\varepsilon}(s)) - f(s, x^*(s), P_{x^*(s)}, u^*(s)))\mathrm{d}s.
\end{aligned}
$$

We notice that

$$
\begin{aligned}
& \frac{1}{\varepsilon} \int_0^t (f(s, x^{\varepsilon}(s), P_{x^{\varepsilon}(s)}, u^{\varepsilon}(s)) - f(s, x^*(s), P_{x^{\varepsilon}(s)}, u^{\varepsilon}(s)))\mathrm{d}s \\
& = \int_0^t \int_0^1 \left[f_x\left(s, x^*(s) + \lambda\varepsilon(\gamma^{\varepsilon}(s) + y_1(s)), P_{x^{\varepsilon}(s)}, u^{\varepsilon}(s)\right)\left(\gamma^{\varepsilon}(s) + y_1(s)\right)\right] \mathrm{d}\lambda \mathrm{d}s,
\end{aligned}
$$

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t (f(s,x^\varepsilon(s),P_{x^\varepsilon(s)},u^\varepsilon(s)) - f(s,x^\varepsilon(s),P_{x^*(s)},u^\varepsilon(s)))\mathrm{d}s \\
&= \int_0^t\int_0^1 \widehat{E}\left[\partial_\mu f(s,x^\varepsilon(s),P_{x^*(s)+\lambda\varepsilon(\gamma(s)+y_1(s))},u^\varepsilon(s);\widehat{x}^*(s))(\widehat{\gamma}(s)+\widehat{y}_1(s))\right]\mathrm{d}\lambda\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t (f(s,x^*(s),P_{x^*(s)},u^\varepsilon(s)) - f(s,x^*(s),P_{x^*(s)},u^*(s)))\mathrm{d}s \\
&= \int_0^t\int_0^1 \left[f_u\left(s,x^*(s),P_{x^*(s)},u^*(s)+\lambda\varepsilon(v(s)-u^*(s)\right)v(s)\right]\mathrm{d}\lambda\mathrm{d}s.
\end{aligned}$$

The analogue relations hold for σ. Therefore, we get

$$\begin{aligned}
&E\left[\sup_{s\in[0,t]}|\gamma^\varepsilon(s)|^2\right] \\
&\leq C(t)\Bigg[E\int_0^t\int_0^1 \left|f_x\left(s,x^*(s)+\lambda\varepsilon(\gamma(s)+y_1(s)),P_{x^*(s)},u^\varepsilon(s)\right)\gamma^\varepsilon(s)\right|^2 d\lambda ds \\
&+E\int_0^t\int_0^1 \widehat{E}\left|f_\mu(s,x^\varepsilon(s),P_{x^*(s)+\lambda\varepsilon(\widehat{\gamma}(s)+\widehat{y}_1(s))},u^\varepsilon(s);\widehat{x}^*(s))\widehat{\gamma}^\varepsilon(s)\right|^2 d\lambda ds \\
&+E\int_0^t\int_0^1 \left|\sigma_x\left(s,x^*(s)+\lambda\varepsilon(\gamma(s)+y_1(s)),P_{x^\varepsilon(s)},u^\varepsilon(s)\right)\gamma^\varepsilon(s)\right|^2 d\lambda ds \\
&+E\int_0^t\int_0^1 \widehat{E}\left|\sigma_\mu(s,x^\varepsilon(s),P_{x^*(s)+\lambda\varepsilon(\widehat{\gamma}(s)+\widehat{y}_1(s))},u^\varepsilon(s);\widehat{x}^*(s))\widehat{\gamma}^\varepsilon(s)\right|^2 d\lambda ds \\
&+E\left[\sup_{s\in[0,t]}|\beta^\varepsilon(s)|^2\right]\Bigg],
\end{aligned}$$

where

$$\begin{aligned}
\beta^\varepsilon(t) = & \int_0^t \int_0^1 [f_x\left(s, x^*(s) + \lambda\varepsilon(\gamma^\varepsilon(s) + y_1(s)), P_{x^*(s)}, u^\varepsilon(s)\right) \\
& - f_x\left(s, x^*(s), P_{x^*(s)}, u^*(s)\right)] y_1(s) d\lambda ds \\
& + \int_0^t \int_0^1 \widehat{E}\left[f_\mu(s, x^\varepsilon(s), P_{x^*(s)+\lambda\varepsilon(\widehat{\gamma}^\varepsilon(s)+\widehat{y}_1(s))}, u^\varepsilon(s); \widehat{x}^*(s))\right. \\
& - f_\mu(s, x^*(s), P_{x^*(s)}, u^*(s); \widehat{x}^*(s))] \widehat{y}_1(s) d\lambda ds \\
& + \int_0^t \int_0^1 [f_u\left(s, x^*(s), P_{x^*(s)}, u^*(s) + \lambda\varepsilon v(t)\right) \\
& - f_u(s, x^*(s), P_{x^*(s)}, u^*(s)] v(t) d\lambda ds
\end{aligned}$$

$$\begin{aligned}
& + \int_0^t \int_0^1 [\sigma_x\left(s, x^*(s) + \lambda\varepsilon(\gamma^\varepsilon(s) + y_1(s)), P_{x^*(s)}, u^\varepsilon(s)\right) \\
& - \sigma_x\left(s, x^*(s), P_{x^*(s)}, u^*(s)\right)] y_1(s) \mathrm{d}\lambda \mathrm{d}W(s) \\
& + \int_0^t \int_0^1 \widehat{E}\left[\sigma_\mu(s, x^\varepsilon(s), P_{x^*(s)+\lambda\varepsilon(\widehat{\gamma}^\varepsilon(s)+\widehat{y}_1(s))}, u^\varepsilon(s); \widehat{x}^*(s))\right. \\
& - \sigma_\mu(s, x^*(s), P_{x^*(s)}, u^*(s); \widehat{x}^*(s))] \widehat{y}_1(s) \mathrm{d}\lambda \mathrm{d}W(s) \\
& + \int_0^t \int_0^1 \sigma_u\left(s, x^*(s), P_{x^*(s)}, u^*(s) + \lambda\varepsilon v(t)\right) \\
& - \sigma_u(s, x^*(s), P_{x^*(s)}, u^*(s))] v(t) \mathrm{d}\lambda \mathrm{d}W(s).
\end{aligned}$$

Now, since the derivatives of f and σ with respect to x, μ, u are Lipschitz continuous in (x, μ, u) , we get

$$\lim_{\varepsilon \to 0} E\left[\sup_{s\in[0,T]} |\beta^\varepsilon(s)|^2\right] = 0.$$

Since the derivatives of f and σ with respect to variables $x, \mu,$ and u are bounded, we obtain $\forall t \in [0, T]$:

$$E\left[\sup_{s\in[0,t]} |\gamma^\varepsilon(s)|^2\right] \le C(t)\left\{E\int_0^t |\gamma^\varepsilon(s)|^2 ds + E\left[\sup_{s\in[0,t]} |\beta^\varepsilon(s)|^2\right]\right\}.$$

From *Gronwall's Lemma*, we have : for any $t \in [0,T]$,

$$E\left[\sup_{s\in[0,t]} |\gamma^\varepsilon(s)|^2\right] \leq C(t)E\left[\sup_{s\in[0,t]} |\beta^\varepsilon(s)|^2\right] \exp\left\{\int_0^t C(s)ds\right\}.$$

Finally, by putting $t = T$ and letting ε go to zero, the proof of *Propositions 4.1* is complete.

□

Proposition 4.4.2 Let $y_1(\cdot)$ and $y_2(\cdot)$ be the solutions of (4.33), (4.35), respectively. Let assumptions (H 4.1), (H 4.2) and (H 4.3) hold. Then, for any $k \geq 1$, and $\varepsilon > 0$, we have

$$E\left[\sup_{t\in[0,T]} \left|x^\varepsilon(t) - x^*(t) - \varepsilon y_1(t) - \frac{\varepsilon^2}{2} y_2(t)\right|^{2k}\right] \leq C_k \varepsilon^{6k}. \tag{4.40}$$

Proof. The proof is based on the first and second order expansions. We put

$$\lambda^\varepsilon(t) = x^\varepsilon(t) - x^*(t) - \varepsilon y_1(t) - \frac{\varepsilon^2}{2} y_2(t).$$

From (4.15), (4.33) and (4.35), we obtain

$$\begin{aligned}
\lambda^\varepsilon(t) = &\int_0^t \Big\{f\left(s, x^\varepsilon(s), P_{x^\varepsilon(s)}, u^\varepsilon(s)\right) - f\left(s, x^*(s), P_{x^*(s)}, u^*(s)\right) \\
&- \varepsilon\left[f_x(s)y_1(s) + \widehat{E}[\widehat{f}_\mu(t)\widehat{y}_1(t)] + f_u(s)v(s)\right] \\
&- \frac{\varepsilon^2}{2}\Big[f_x(s)y_2(s) + \widehat{E}[\widehat{f}_\mu(s)\widehat{y}_2(s)] + f_{xx}(s)y_1^2(s) + 2f_{xu}(s)y_1(s)v(s) \\
&+ \widehat{E}[\widehat{f}_{x\mu}(t)\widehat{y}_1(t)]y_1(t) + 2\widehat{E}[\widehat{f}_{u\mu}(s)\widehat{y}_1(s)]v(s) + f_{uu}(s)v^2(s)\Big]\Big\} \mathrm{d}s \\
&+ \int_0^t \Big\{\sigma\left(s, x^\varepsilon(s), P_{x^\varepsilon(s)}, u^\varepsilon(s)\right) - \sigma\left(s, x^*(s), P_{x^*(s)}, u^*(s)\right) \\
&- \varepsilon\left[\sigma_x(s)y_1(s) + \widehat{E}[\widehat{\sigma}_\mu(t)\widehat{y}_1(t)] + \sigma_u(s)v(s)\right] \\
&- \frac{\varepsilon^2}{2}\Big[\sigma_x(s)y_2(s) + \widehat{E}[\widehat{\sigma}_\mu(s)\widehat{y}_2(s)] + \sigma_{xx}(s)y_1^2(s) + 2\sigma_{xu}(s)y_1(s)v(s) \\
&+ \widehat{E}[\widehat{\sigma}_{x\mu}(t)\widehat{y}_1(t)]y_1(t) + 2\widehat{E}[\widehat{\sigma}_{u\mu}(s)\widehat{y}_1(s)]v(s) + \sigma_{uu}(s)v^2(s)\Big]\Big\} \mathrm{d}W(s).
\end{aligned} \tag{4.41}$$

We decompose the integral $\int_0^t[f(s, x^\varepsilon(s), P_{x^\varepsilon(s)}, u^\varepsilon(s)) - f(s, x^*(s), P_{x^*(s)}, x^*(s))]\mathrm{d}s$ into the following parts : for $\Psi = f, \sigma$

$$\begin{aligned}
&\int_0^t (\Psi(s, x^\varepsilon(s), P_{x^\varepsilon(s)}, u^\varepsilon(s)) - \Psi(s, x^*(s), P_{x^*(s)}, x^*(s)))\mathrm{d}s \\
&= \int_0^t (\Psi(s, x^\varepsilon(s), P_{x^\varepsilon(s)}, u^\varepsilon(s)) - \Psi(s, x^*(s), P_{x^\varepsilon(s)}, u^\varepsilon(s)))\mathrm{d}s \\
&+ \int_0^t (\Psi(s, x^*(s), P_{x^\varepsilon(s)}, u^\varepsilon(s)) - \Psi(s, x^*(s), P_{x^*(s)}, u^\varepsilon(s)))\mathrm{d}s \qquad (4.42)\\
&+ \int_0^t (\Psi(s, x^*(s), P_{x^*(s)}, u^\varepsilon(s)) - \Psi(s, x^*(s), P_{x^*(s)}, u^*(s)))\mathrm{d}s.
\end{aligned}$$

Let us denote $\Delta x^\varepsilon(t) = (x^\varepsilon(t) - x^*(t))$. We have for $\Psi = f, \sigma$:

$$\begin{aligned}
&\int_0^t (\Psi(s, x^\varepsilon(s), P_{x^\varepsilon(s)}, u^\varepsilon(s)) - \Psi(s, x^*(s), P_{x^\varepsilon(s)}, u^\varepsilon(s)))\mathrm{d}s \qquad (4.43)\\
&= \int_0^t \int_0^1 \left[\Psi_x\left(s, x^*(s) + \rho\Delta x^\varepsilon), P_{x^\varepsilon(s)}, u^\varepsilon(s)\right) \Delta x^\varepsilon(s)\right] \mathrm{d}\rho\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
&\int_0^t (\Psi(s, x^\varepsilon(s), P_{x^\varepsilon(s)}, u^\varepsilon(s)) - \Psi(s, x^\varepsilon(s), P_{x^*(s)}, u^\varepsilon(s)))\mathrm{d}s \qquad (4.44)\\
&= \int_0^t \int_0^1 \widehat{E}\left[\partial_\mu \Psi(s, x^\varepsilon(s), P_{x^*(s)+\rho\Delta x^\varepsilon}, u^\varepsilon(s); \widehat{x}^*(s))\Delta\widehat{x}^\varepsilon(s)\right] \mathrm{d}\rho\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t (\Psi(s, x^*(s), P_{x^*(s)}, u^\varepsilon(s)) - \Psi(s, x^*(s), P_{x^*(s)}, u^*(s)))\mathrm{d}s \qquad (4.45)\\
&= \int_0^t \int_0^1 \left[\Psi_u(s, x^*(s), P_{x^*(s)}, u^*(s) + \rho\varepsilon v(s))\varepsilon v(s)\right] \mathrm{d}\rho\mathrm{d}s.
\end{aligned}$$

By substituting (4.43), (4.44), and (4.45) into (4.42), we get

$$\begin{aligned}
&\int_0^t (\Psi\left(t, x^\varepsilon(s), P_{x^\varepsilon(s)}, u^\varepsilon(s)\right) - \Psi\left(s, x^*(s), P_{x^*(s)}, u^*(s)\right))\mathrm{d}s \\
&= \int_0^t \int_0^1 \left[\Psi_x\left(s, x^*(s) + \rho\Delta x^\varepsilon, P_{x^*(s)+\rho\Delta x^\varepsilon}, u^*(t)\right)\Delta x^\varepsilon\right. \\
&+ \widehat{E}\left[\widehat{\Psi}_\mu\left(s, \widehat{x}^*(s) + \rho\Delta\widehat{x}^\varepsilon(s), P_{\widehat{x}^*(s)+\rho\Delta\widehat{x}^\varepsilon(s)}, u^*(s); \widehat{x}^*(s) + \rho\lambda\Delta\widehat{x}^\varepsilon(s)\right)\Delta\widehat{x}^\varepsilon(s)\right] \\
&+ \left.\left[\Psi_u\left(s, x^*(s), P_{x^*(s)}, u^*(s) + \rho\varepsilon(v(s) - u^*(s)\right)\varepsilon v(s)\right]\right]\mathrm{d}\rho\mathrm{d}s.
\end{aligned} \tag{4.46}$$

By first-order expansion for $\Psi_x\left(t, x^\varepsilon(s), P_{x^\varepsilon(s)}, u^\varepsilon(s)\right) - \Psi_x\left(s, x^*(s), P_{x^*(s)}, u^*(s)\right)$, we have

$$\begin{aligned}
&\int_0^t (\Psi_x\left(s, x^\varepsilon(s), P_{x^\varepsilon(s)}, u^\varepsilon(s)\right) - \Psi_x\left(s, x^*(s), P_{x^*(s)}, u^*(s)\right))\mathrm{d}s \\
&= \rho\int_0^t \int_0^1 \left[\Psi_{xx}\left(s, x^*(s) + \rho\lambda\Delta x^\varepsilon, P_{x^*(s)+\rho\lambda\Delta x^\varepsilon}, u^*(s) + \rho\Delta u^\varepsilon\right)\Delta x^\varepsilon\right. \\
&+ \widehat{E}\left[\widehat{\Psi}_{x\mu}\left(s, \widehat{x}^*(s) + \rho\lambda\Delta\widehat{x}^\varepsilon(s), P_{\widehat{x}^*(s)+\rho\lambda\Delta\widehat{x}^\varepsilon(s)}, u^*(s) + \rho\Delta u^\varepsilon, \widehat{x}^*(s) + \rho\lambda\Delta\widehat{x}^\varepsilon(s)\right)\Delta\widehat{x}^\varepsilon(s)\right]\mathrm{d}\lambda\mathrm{d}s.
\end{aligned} \tag{4.47}$$

By applying similar method to $\widehat{E}\left(\Psi_\mu\left(t, x^\varepsilon(t), P_{x^\varepsilon(t)}, u^\varepsilon(t)\right) - \Psi_\mu\left(t, x^*(t), P_{x^*(t)}, u^*(t)\right)\right)$, we get

$$\begin{aligned}
&\int_0^t \widehat{E}(\Psi_\mu\left(s, x^\varepsilon(s), P_{x^\varepsilon(s)}, u^\varepsilon(s)\right) - \Psi_\mu\left(s, x^*(s), P_{x^*(s)}, u^*(s); \widehat{x}^*(s) + \rho\lambda\Delta\widehat{x}^\varepsilon(s)\right))\mathrm{d}s \\
&= \rho\int_0^t \int_0^1 \widehat{E}\left[\Psi_{x\mu}\left(s, x^*(s) + \rho\lambda\Delta x^\varepsilon, P_{x^*(s)+\rho\lambda\Delta x^\varepsilon}, u^*(s) + \rho\Delta u^\varepsilon; \widehat{x}^*(s) + \rho\lambda\Delta\widehat{x}^\varepsilon(s)\right)\Delta x^\varepsilon\right. \\
&+ \widehat{E}\left[\widehat{\Psi}_{\mu\mu}\left(s, \widehat{x}^*(s) + \rho\lambda\Delta\widehat{x}^\varepsilon(s), P_{\widehat{x}^*(s)+\rho\lambda\Delta\widehat{x}^\varepsilon(s)}, u^*(s) + \rho\Delta u^\varepsilon; \widehat{x}^*(s) + \rho\lambda\Delta\widehat{x}^\varepsilon(s)\right)\Delta\widehat{x}^\varepsilon(s)\right]\mathrm{d}\lambda\mathrm{d}s.
\end{aligned} \tag{4.48}$$

Now, since $f, \sigma \in \mathbb{C}_b^{2,1}(\Gamma_2(\mathbb{R}^n))$, then, by applying second-order expansion and the fact that $\Delta \mathrm{u}^{\varepsilon}(s) = u^{\varepsilon}(s) - u^{*}(s) = \varepsilon(u(s) - u^{*}(s)) = \varepsilon v(s)$, we have for each $t \in [0,T]$,

$$\begin{aligned}
&\int_0^t \left[\Psi\left(s, x^{\varepsilon}(s), P_{x^{\varepsilon}(s)}, u^{\varepsilon}(s)\right) - \Psi\left(s, x^{*}(s), P_{x^{\varepsilon}(s)}, u^{*}(s)\right)\right] \mathrm{d}s \\
&= \int_0^t \int_0^1 E\left[\partial_\mu \Psi\left(s, x^{*}(s) + \rho\Delta x^{\varepsilon}, P_{x^{*}(s)+\rho\Delta x^{\varepsilon}}, u^{*}(s)\right) \Delta x^{\varepsilon}(s)\right] d\rho \mathrm{d}s \\
&+ \frac{1}{2}\int_0^t \int_0^1 E\left[\widehat{E}\left[\partial_\mu^2 \Psi(s, x^{*}(s) + \rho\Delta x^{\varepsilon}, P_{x^{*}(s)+\rho\Delta x^{\varepsilon}}, u^{*}(s), \widehat{x}^{*}(s) + \rho\Delta x^{\varepsilon})\Delta\widehat{x}^{\varepsilon}(s)\Delta x^{\varepsilon}(s)\right]\right] d\rho \mathrm{d}s \\
&+ \frac{1}{2}\int_0^t \int_0^1 E\left[\partial_y \partial_\mu \Psi(s, x^{*}(s) + \rho\Delta x^{\varepsilon}, P_{x^{*}(s)+\rho\Delta x^{\varepsilon}}, u^{*}(s)) \left(\Delta x^{\varepsilon}(s)\right)^2\right] d\rho \mathrm{d}s \\
&+ \int_0^t \int_0^1 \Psi_x(s, x^{*}(s) + \rho\Delta x^{\varepsilon}, P_{x^{*}(s)+\rho\Delta x^{\varepsilon}}, u^{*}(s))\Delta \mathrm{x}^{\varepsilon}(t)\, d\rho \mathrm{d}s \qquad (4.49) \\
&+ \frac{1}{2}\int_0^t \int_0^1 \Psi_{xx}(s, x^{*}(s) + \rho\Delta x^{\varepsilon}, P_{x^{*}(s)+\rho\Delta x^{\varepsilon}}, u^{*}(s)) \left(\Delta \mathrm{x}^{\varepsilon}(s)\right)^2 d\rho \mathrm{d}s \\
&+ \int_0^t \int_0^1 \Psi_u(s, x^{*}(s), P_{x^{*}(s)}, u^{*}(s) + \rho\varepsilon v(s))\varepsilon v(s) d\rho \mathrm{d}s \\
&+ \frac{1}{2}\int_0^t \int_0^1 \Psi_{uu}(s, x^{*}(s), P_{x^{*}(s)}, u^{*}(s) + \rho\varepsilon v(s)) \left(\varepsilon v(s)\right)^2 d\rho \mathrm{d}s \\
&+ \int_0^t \int_0^1 \Psi_{xu}(s, x^{*}(s) + \rho\Delta x^{\varepsilon}, P_{x^{*}(s)+\rho\Delta x^{\varepsilon}}, u^{*}(s) + \rho\varepsilon v(s))\Delta \mathrm{x}^{\varepsilon}(s)\, \varepsilon v(s) d\rho \mathrm{d}s \\
&+ \int_0^t \int_0^1 \widehat{E}\left[\Psi_{\mu u}(s, x^{*}(s) + \rho\Delta x^{\varepsilon}, P_{x^{*}(s)+\rho\Delta x^{\varepsilon}}, u^{*}(s) + \rho\varepsilon v(s))\Delta\widehat{x}^{\varepsilon}(\mathrm{s})\varepsilon v(s)\right] d\rho \mathrm{d}s \\
&+ \int_0^t \int_0^1 \widehat{E}\left[\Psi_{x\mu}(s, x^{*}(s) + \rho\Delta x^{\varepsilon}, P_{x^{*}(s)+\rho\Delta x^{\varepsilon}}, u^{*}(s))\Delta\widehat{x}^{\varepsilon}(\mathrm{s})\Delta \mathrm{x}^{\varepsilon}(\mathrm{s})\right] d\rho \mathrm{d}s
\end{aligned}$$

From (5.16), $\Delta \mathrm{u}^{\varepsilon}(s) = u^{\varepsilon}(s) - u^{*}(s) = \varepsilon(u(s) - u^{*}(s)) = \varepsilon v(s)$.

Substituting (5.25), (4.47), (4.48) and (4.49) into (4.41), we get

$$\begin{aligned}
&\lambda^{\varepsilon}(t)\\
&=\int_0^t \Big[f_x(s)y_2(s)+\frac{1}{2}f_{xx}(s)(\Delta x^{\varepsilon})^2-\frac{1}{2}f_{xx}(s)(y_1(s))^2+f_{xu}(s)\Delta \mathrm{x}^{\varepsilon}(\mathrm{s})v(s)\\
&-f_{xu}(s)y_1(s)v(s)+\widehat{E}\left(f_{\mu}(s)\widehat{y}_2(s)\right)+\frac{1}{2}\widehat{E}\left(f_{\mu\mu}(s)(\Delta\widehat{x}^{\varepsilon})^2\right)-\frac{1}{2}\widehat{E}\left(f_{\mu\mu}(s)(\widehat{y}_1(s))^2\right)\\
&+\widehat{E}\left(f_{\mu u}(s)\Delta\widehat{\mathrm{x}}^{\varepsilon}(\mathrm{s})\right)v(s)-\widehat{E}\left(f_{\mu u}(s)\widehat{y}_1(s)\right)v(s)+\varphi_f(s)\Big]\mathrm{ds}\\
&+\int_0^t \Big[\sigma_x(s)y_2(s)+\frac{1}{2}\sigma_{xx}(s)(\Delta x^{\varepsilon})^2-\frac{1}{2}\sigma_{xx}(s)(y_1(s))^2+\sigma_{xu}(s)\Delta \mathrm{x}^{\varepsilon}(\mathrm{s})v(s)\\
&-\sigma_{xu}(s)y_1(s)v(s)+\widehat{E}\left(\sigma_{\mu}(s)\widehat{y}_2(s)\right)+\frac{1}{2}\widehat{E}\left(\sigma_{\mu\mu}(s)(\Delta\widehat{x}^{\varepsilon})^2\right)-\frac{1}{2}\widehat{E}\left(\sigma_{\mu\mu}(s)(\widehat{y}_1(s))^2\right)\\
&+\widehat{E}\left(\sigma_{\mu u}(s)\Delta\widehat{\mathrm{x}}^{\varepsilon}(\mathrm{s})\right)v(s)-\widehat{E}\left(\sigma_{\mu u}(s)\widehat{y}_1(s)\right)v(s)+\varphi_{\sigma}(s)\Big]\mathrm{d}W(s),
\end{aligned}$$

where

$$\begin{aligned}
&\varphi_f(s)\\
&=\int_0^1(1-\rho)\left[f_{xx}\left(s,x^*(s)+\rho\Delta x^{\varepsilon},P_{x^*(s)+\rho\Delta x^{\varepsilon}},u^*(s)+\rho\varepsilon v(s)\right)-f_{xx}\left(s,x^*(s),P_{x^*(s)},u^*(s)\right)\right.\\
&+\widehat{E}\left[f_{x\mu}\left(s,x^*(s)+\rho\lambda\Delta x^{\varepsilon},P_{x^*(s)+\rho\lambda\Delta x^{\varepsilon}},u^*(s)+\rho\varepsilon v(s)\right)-f_{x\mu}\left(s,x^*(s),P_{x^*(s)},u^*(s)\right)\right.\\
&\left.+\widehat{E}(f_{\mu\mu}\left(s,\widehat{x}^*(s)+\rho\Delta\widehat{x}^{\varepsilon}(s),P_{\widehat{x}^*(s)+\rho\Delta\widehat{x}^{\varepsilon}(s)},u^*(s)+\rho\varepsilon v(s)\right))-\widehat{E}(f_{\mu\mu}\left(s,x^*(s),P_{x^*(s)},u^*(s)\right))\right]\mathrm{d}\rho,
\end{aligned}$$

and

$$\begin{aligned}
&\varphi_{\sigma}(s)\\
&=\int_0^1(1-\rho)\left[\sigma_{xx}\left(s,x^*(s)+\rho\Delta x^{\varepsilon},P_{x^*(s)+\rho\Delta x^{\varepsilon}},u^*(s)+\rho\varepsilon v(s)\right)-\sigma_{xx}\left(s,x^*(s),P_{x^*(s)},u^*(s)\right)\right.\\
&+\widehat{E}\left[\sigma_{x\mu}\left(s,x^*(s)+\rho\lambda\Delta x^{\varepsilon},P_{x^*(s)+\rho\lambda\Delta x^{\varepsilon}},u^*(s)+\rho\varepsilon v(s)\right)-\sigma_{x\mu}\left(s,x^*(s),P_{x^*(s)},u^*(s)\right)\right.\\
&\left.+\widehat{E}(\sigma_{\mu\mu}\left(s,\widehat{x}^*(s)+\rho\Delta\widehat{x}^{\varepsilon}(s),P_{\widehat{x}^*(s)+\rho\Delta\widehat{x}^{\varepsilon}(s)},u^*(s)+\rho\varepsilon v(s)\right))-\widehat{E}(\sigma_{\mu\mu}\left(s,x^*(s),P_{x^*(s)},u^*(s)\right))\right]\mathrm{d}\rho.
\end{aligned}$$

Finally, applying similar arguments proved in Lemma 3.11 in Bonnans and Silva [15, Annex, Proof of (3.19)], we get

$$E\left[\sup_{t\in[0,T]}|\lambda^{\varepsilon}(t)|^{2k}\right]\leq C_k\varepsilon^{6k}, \tag{4.50}$$

then the desired result (4.40) is fulfilled, which completes the proof of Proposition 4.4.2□

As expected, the adjoint processes $(p_1(\cdot),q_1(\cdot))$, $(p_2(\cdot),q_2(\cdot))$ and the variational processes $(y_1(\cdot), y_2(\cdot))$ are related by the following duality relationship, which is essential for the proof of our main result.

Lemma 4.4.1 Let $(p_1(\cdot),q_1(\cdot))$ and $(p_2(\cdot),q_2(\cdot))$ be the solution to the adjoint equation (4.26) and (4.29) respectively. Let $y_1(\cdot)$ and $y_2(\cdot)$ be the solutions to the first and second-order variational equations (4.33) and (4.35), respectively associated to $u^*(\cdot)$. Then the following duality relations hold

$$\begin{aligned}&E\left[h_x(x^*(T)\,y_1(T)+\widehat{E}[\widehat{h}^*_\mu(T)\widehat{y}_1(T)]\right]\\&=-E\int_0^T\left[p_1(t)\,f_u(t)\,v(t)+q_1(t)\sigma_u(t)\,v(t)+y_1(t)\,(\ell_x(t)+\widehat{E}[\widehat{\ell}^*_\mu(t)])\right]\mathrm{d}t,\end{aligned} \tag{4.51}$$

$$\begin{aligned}&E\left[h_x(x^*T)y_2(T)+\widehat{E}[\widehat{h}^*_\mu(T)]y_2(T)\right]\\&=-E\int_0^T\Big\{p_1(t)\left[[f_{xx}(t)y_1(t)+\widehat{E}[\widehat{f}^*_{x\mu}(t)\widehat{y}_1(t)]+2f_{xu}(t)v(t)]y_1(t)\right.\\&\left.+\widehat{E}[\widehat{f}^*_{u\mu}(t)\widehat{y}_1(t)]v(t)+f_{uu}(t)v^2(t)\right]\\&+q_1(t)\left[\sigma_{xx}(t)y_1^2(t)+\widehat{E}[\widehat{\sigma}^*_{x\mu}(t)\widehat{y}_1(t)]y_1(t)\right.\\&\left.+2\sigma_{xu}(t)y_1(t)v(t)+\widehat{E}[\widehat{\sigma}^*_{u\mu}(t)\widehat{y}_1(t)]v(t)+\sigma_{uu}(t)v^2(t)\right]\\&+\ell_x(t)y_2(t)+\widehat{E}(\widehat{\ell}^*_\mu(t))y_2(t)\Big\}\,\mathrm{d}t,\end{aligned} \tag{4.52}$$

and

$$\begin{aligned}
&E\left[h_{xx}\left(x^{*}\left(T\right)\right)y_{1}^{2}\left(T\right)+\widehat{E}[\widehat{h}_{\mu y}^{*}\left(T\right)\widehat{y}_{1}^{2}\left(T\right)]\right] \\
&=-E\int_{0}^{T}\{[2y_{1}\left(t\right)\left[p_{2}\left(t\right)\left(f_{u}\left(t\right)+\sigma_{x}\left(t\right)\sigma_{u}\left(t\right)\right)+q_{2}\left(t\right)\sigma_{u}\left(t\right)\right] \\
&+p_{2}\left(t\right)\left[2\sigma_{u}\left(t\right)\widehat{E}[\widehat{\sigma}_{\mu}^{*}(t)\widehat{y}_{1}(t)]+\sigma_{u}^{2}\left(t\right)v(t)\right]\Big]\,v(t) \\
&-y_{1}^{2}(t)(H_{xx}\left(t\right)+\widehat{E}[\widehat{H}_{x\mu}^{*}(t)]+\widehat{E}[\widehat{H}_{\mu\mu}^{*}(t)])\Big\}\,\mathrm{d}t.
\end{aligned} \tag{4.53}$$

Proof.

Proof of (4.51). Applying Itô's formula to $p_1(t)\,y_1(t)$, we have

$$\begin{aligned}
&E\left[p_{1}\left(T\right)y_{1}\left(T\right)\right]-E\left[p_{1}\left(0\right)y_{1}\left(0\right)\right] \\
&=E\int_{0}^{T}p_{1}(t)\mathrm{d}y_{1}(t)+E\int_{0}^{T}y_{1}(t)\mathrm{d}p_{1}(t) \\
&+E\int_{0}^{T}q_{1}(t)[\sigma_{x}(t)y_{1}(t)+\widehat{E}[\widehat{\sigma}_{\mu}^{*}(t)\widehat{y}_{1}(t)]+\sigma_{u}(t)v(t)]\mathrm{d}t \\
&=\mathcal{I}_{1}(T)+\mathcal{I}_{2}(T)+\mathcal{I}_{3}(T).
\end{aligned} \tag{4.54}$$

From (5.17), we get

$$\begin{aligned}
\mathcal{I}_{1}(T)&=E\int_{0}^{T}p_{1}(t)\mathrm{d}y_{1}(t) \\
&=E\int_{0}^{T}p_{1}(t)\left[f_{x}(t)y_{1}(t)+\widehat{E}[\widehat{f}_{\mu}^{*}(t)\widehat{y}_{1}(t)]+f_{u}(t)v(t)\right]\mathrm{d}t,
\end{aligned} \tag{4.55}$$

and from (4.26), we get

$$\begin{aligned}
\mathcal{I}_{2}(T)&=E\int_{0}^{T}y_{1}(t)\mathrm{d}p_{1}(t), \\
&=-E\int_{0}^{T}y_{1}(t)\left[p_{1}(t)f_{x}(t)+\widehat{E}[\widehat{f}_{\mu}^{*}(t)\widehat{p}_{1}(t)]+q_{1}(t)\sigma_{x}(t)\right. \\
&\left.+\widehat{E}\left[\widehat{\sigma}_{\mu}^{*}(t)\widehat{q}_{1}(t)\right]-\ell_{x}(t)-\widehat{E}(\widehat{\ell}_{\mu}^{*}(t))\right]\mathrm{d}t.
\end{aligned} \tag{4.56}$$

Similarly, we can further write

$$\mathcal{I}_3(T) = E\int_0^T q_1(t)\left[\sigma_x(t)y_1(t) + \widehat{E}[\widehat{\sigma}_\mu^*(t)\widehat{y}_1(t)] + \sigma_u(t)v(t)\right]\mathrm{d}t. \tag{4.57}$$

Substituting (4.55), (4.56), and (4.57) into (4.54), with the fact that $y_1(0) = 0$, we get

$$\begin{aligned}
&E\left[p_1(T)\,y_1(T)\right]\\
&= E\int_0^T \left[p_1(t)\,f_u(t)\,v(t) + q_1(t)\sigma_u(t)\,v(t) + y_1(t)\left(\ell_x(t) + \widehat{E}[\widehat{\ell}_\mu^*(t)]\right)\right]\mathrm{d}t.
\end{aligned}$$

Since $p_1(T) = -h_x(T) - \widehat{E}[\widehat{h}_\mu^*(T)]$, we obtain

$$\begin{aligned}
&E\left[h_x(T)y_1(T) + \widehat{E}[\widehat{h}_\mu^*(T)]y_1(T)\right]\\
&= -E\left[p_1(T)\,y_1(T)\right]\\
&= -E\int_0^T \left[p_1(t)\,f_u(t)\,v(t) + q_1(t)\sigma_u(t)\,v(t) + y_1(t)\left(\ell_x(t) + \widehat{E}[\widehat{\ell}_\mu^*(t)]\right)\right]\mathrm{d}t,
\end{aligned}$$

then the desired result (4.51) is fulfilled.

Proof of (4.52). Applying Itô's formula to $p_1(T)\,y_2(T)$, we have

$$\begin{aligned}
&E\left[p_1(T)\,y_2(T)\right] - E\left[p_1(0)\,y_2(0)\right]\\
&= E\int_0^T p_1(t)\mathrm{d}y_2(t) + E\int_0^T y_2(t)\mathrm{d}p_1(t)\\
&+ E\int_0^T q_1(t)\left[\sigma_x(t)y_2(t) + \widehat{E}[\widehat{\sigma}_\mu^*(t)\widehat{y}_2(t)] + \sigma_{xx}(t)y_1^2(t) + \widehat{E}[\widehat{\sigma}_{x\mu}^*(t)\widehat{y}_1(t)]y_1(t)\right.\\
&\left.+ 2\sigma_{xu}(t)y_1(t)v(t) + 2\widehat{E}[\widehat{\sigma}_{u\mu}^*(t)\widehat{y}_1(t)]v(t) + \sigma_{uu}(t)v^2(t)\right]\mathrm{d}t\\
&= \mathcal{J}_1(T) + \mathcal{J}_2(T) + \mathcal{J}_3(T).
\end{aligned} \tag{4.58}$$

From (4.35), we have

$$\begin{aligned}\mathcal{J}_1(T) &= E\int_0^T p_1(t)\mathrm{d}y_2(t)\\ &= E\int_0^T p_1(t)\Big[f_x(t)y_2(t)+\widehat{E}[\widehat{f}_\mu(t)\widehat{y}_2(t)]+f_{xx}(t)y_1^2(t)+\widehat{E}[\widehat{f}^*_{x\mu}(t)\widehat{y}_1(t)]y_1(t)\\ &+2f_{xu}(t)y_1(t)v(t)+2\widehat{E}[\widehat{f}_{u\mu}(t)\widehat{y}_1(t)]v(t)+f_{uu}(t)v^2(t)\Big]\,\mathrm{d}t.\end{aligned} \tag{4.59}$$

From (4.26), it is easy to show that

$$\begin{aligned}\mathcal{J}_2(T) &= E\int_0^T y_2(t)\mathrm{d}p_1(t)\\ &= -E\int_0^T y_2(t)\Big[f_x(t)p_1(t)+\widehat{E}\left[\widehat{f}_\mu(t)\widehat{p}_1(t)\right]+\sigma_x(t)q_1(t)+\widehat{E}\left[\widehat{\sigma}^*_\mu(t)\widehat{q}_1(t)\right]\\ &-\ell_x(t)-\widehat{E}\left(\widehat{\ell}_\mu(t)\right)\Big]\,\mathrm{d}t,\end{aligned} \tag{4.60}$$

and similarly, we get

$$\begin{aligned}\mathcal{J}_3(T) = E\int_0^T q_1(t)\Big[&\sigma_x(t)y_2(t)+\widehat{E}[\widehat{\sigma}^*_\mu(t)\widehat{y}_2(t)]+\sigma_{xx}(t)y_1^2(t)+\widehat{E}[\widehat{\sigma}^*_{x\mu}(t)\widehat{y}_1(t)]y_1(t)\\ &+2\sigma_{xu}(t)y_1(t)v(t)+2\widehat{E}[\widehat{\sigma}^*_{u\mu}(t)\widehat{y}_1(t)]v(t)+\sigma_{uu}(t)v^2(t)\Big]\,\mathrm{d}t.\end{aligned} \tag{4.61}$$

Combining (4.59), (4.60), and (4.61) into (4.58), with the fact that $y_2(0) = 0$, we get

$$
\begin{aligned}
& E\left[p_1(T)\, y_2(T)\right] \\
& = E\int_0^T \Big\{ p_1(t)\Big[f_{xx}(t)y_1^2(t) + \widehat{E}[\widehat{f}^*_{x\mu}(t)\widehat{y}_1(t)]y_1(t) + 2f_{xu}(t)v(t)y_1(t) \\
& +2\widehat{E}[\widehat{f}^*_{u\mu}(t)\widehat{y}_1(t)]v(t) + f_{uu}(t)v^2(t)\Big] \\
& + q_1(t)\Big[\sigma_{xx}(t)y_1^2(t) + \widehat{E}[\widehat{\sigma}^*_{x\mu}(t)\widehat{y}_1(t)]y_1(t) \\
& + 2\sigma_{xu}(t)y_1(t)v(t) + 2\widehat{E}[\widehat{\sigma}^*_{u\mu}(t)\widehat{y}_1(t)]v(t) + \sigma_{uu}(t)v^2(t)\Big] \\
& +\ell_x(t)y_2(t) + \widehat{E}(\widehat{\ell}^*_\mu(t))y_2(t)\Big\}\, \mathrm{d}t.
\end{aligned}
$$

Since $p_1(T) = -h_x(T) - \widehat{E}[\widehat{h}^*_\mu(T)]$, we obtain

$$
\begin{aligned}
& E\Big[h_x(T)y_2(T) + \widehat{E}[\widehat{h}^*_\mu(T)]y_2(T)\Big] \\
& = -E\int_0^T \Big\{ p_1(t)\Big[f_{xx}(t)y_1^2(t) + \widehat{E}[\widehat{f}^*_{x\mu}(t)\widehat{y}_1(t)]y_1(t) + 2f_{xu}(t)v(t)y_1(t) \\
& +2\widehat{E}[\widehat{f}^*_{u\mu}(t)\widehat{y}_1(t)]v(t) + f_{uu}(t)v^2(t)\Big] \\
& + q_1(t)\Big[\sigma_{xx}(t)y_1^2(t) + \widehat{E}[\widehat{\sigma}^*_{x\mu}(t)\widehat{y}_1(t)]y_1(t) \\
& + 2\sigma_{xu}(t)y_1(t)v(t) + 2\widehat{E}[\widehat{\sigma}^*_{u\mu}(t)\widehat{y}_1(t)]v(t) + \sigma_{uu}(t)v^2(t)\Big] \\
& +\ell_x(t)y_2(t) + \widehat{E}(\widehat{\ell}^*_\mu(t))y_2(t)\Big\}\, \mathrm{d}t,
\end{aligned}
$$

then the desired result (4.52) is fulfilled.

Proof of (4.53). Applying Itô's formula to $p_2(t)\, y_1^2(t)$, we have

$$\begin{aligned} & E\left[p_2(T)\, y_1^2(T)\right] - E\left[p_2(0)\, y_1^2(0)\right] \\ & = E\int_0^T [p_2(t)y_1(t)]\,\mathrm{d}y_1(t) + E\int_0^T y_1(t)\mathrm{d}(p_2(t)y_1(t)) \\ & + E\int_0^T \Big[p_2(t)\Big(\sigma_x(t)y_1(t) + \widehat{E}[\widehat{\sigma}_\mu^*(t)\widehat{y}_1(t)] + \sigma_u(t)v(t)\Big) + y_1(t)q_2(t)\Big] \\ & \times \Big[\sigma_x(t)y_1(t) + \widehat{E}[\widehat{\sigma}_\mu^*(t)\widehat{y}_1(t)] + \sigma_u(t)v(t)\Big]\,\mathrm{d}t \\ & = \mathcal{K}_1(T) + \mathcal{K}_2(T) + \mathcal{K}_3(T), \end{aligned} \tag{4.62}$$

where $\mathcal{K}_1(T)$ is given by

$$\begin{aligned} \mathcal{K}_1(T) &= E\int_0^T [p_2(t)y_1(t)]\,\mathrm{d}y_1(t) \\ &= E\int_0^T [p_2(t)y_1(t)]\Big[f_x(t)y_1(t) + \widehat{E}[\widehat{f}_\mu^*(t)\widehat{y}_1(t)] + f_u(t)v(t)\Big]\,\mathrm{d}t. \end{aligned} \tag{4.63}$$

We again applying *Itô's formula*, to $p_2(t)y_1(t)$, we obtain

$$\begin{aligned} \mathcal{K}_2(T) &= E\int_0^T y_1(t)\mathrm{d}\,[p_2(t)y_1(t)] \\ &= E\int_0^T y_1(t)p_2(t)\mathrm{d}y_1(t) + E\int_0^T y_1^2(t)\mathrm{d}p_2(t) \\ &+ E\int_0^T y_1(t)q_2(t)(\sigma_x(t)y_1(t) + \widehat{E}[\widehat{\sigma}_\mu^*(t)\widehat{y}_1(t)] + \sigma_u(t)v(t))\mathrm{d}t, \end{aligned}$$

which implies that

$$\begin{aligned} \mathcal{K}_2(T) &= E\int_0^T y_1(t)p_2(t)\Big(f_x(t)y_1(t) + \widehat{E}[\widehat{f}_\mu^*(t)\widehat{y}_1(t)] + f_u(t)v(t)\Big)\,\mathrm{d}t \\ &- E\int_0^T y_1^2(t)\Big\{2(f_x(t) + \widehat{E}[\widehat{f}_\mu^*(t)])p_2(t) + [\sigma_x(t) + \widehat{E}(\widehat{\sigma}_\mu^*(t))]^2 p_2(t) \\ &+ 2(\sigma_x(t) + \widehat{E}[\widehat{\sigma}_\mu^*(t)])q_2(t) + (H_{xx}(t) + \widehat{E}[\widehat{H}_{x\mu}(t)]) + \widehat{E}[\widehat{H}_{\mu\mu}(t)])\Big\}\,\mathrm{d}t \\ &+ E\int_0^T y_1(t)q_2(t)(\sigma_x(t)y_1(t) + \widehat{E}[\widehat{\sigma}_\mu^*(t)\widehat{y}_1(t)] + \sigma_u(t)v(t))\mathrm{d}t. \end{aligned}$$

Analogously, by simple computations, we get

$$\begin{aligned}\mathcal{K}_2(T) &= E\int_0^T y_1(t)\left(p_2(t)f_u(t)+q_2(t)\sigma_u(t)\right)v(t)\mathrm{d}t\\ &- E\int_0^T y_1^2(t)\Big\{(f_x(t)+\widehat{E}[\widehat{f}_\mu^*(t)])p_2(t)+[\sigma_x(t)+\widehat{E}(\widehat{\sigma}_\mu^*(t))]^2p_2(t)\\ &+(\sigma_x(t)+\widehat{E}[\widehat{\sigma}_\mu^*(t)])q_2(t)+(H_{xx}(t)+\widehat{E}[\widehat{H}_{x\mu}^*(t)]+\widehat{E}[\widehat{H}_{\mu\mu}(t)])\Big\}\,\mathrm{d}t.\end{aligned} \tag{4.64}$$

The same argument allows to show that

$$\begin{aligned}\mathcal{K}_3(T) = E\int_0^T &\Big[p_2(t)\Big(\sigma_x(t)y_1(t)+\widehat{E}[\widehat{\sigma}_\mu^*(t)\widehat{y}_1(t)]+\sigma_u(t)v(t)\Big)+y_1(t)q_2(t)\Big]\\ &\times\Big[\sigma_x(t)y_1(t)+\widehat{E}[\widehat{\sigma}_\mu^*(t)\widehat{y}_1(t)]+\sigma_u(t)v(t)\Big]\,\mathrm{d}t.\end{aligned} \tag{4.65}$$

By substituting (4.63), (4.64), and (4.65) into (4.62), with the fact that $y_1(0)=0$, we conclude that

$$\begin{aligned}&E\left[p_2(T)\,y_1^2(T)\right]\\ &=E\int_0^T\{[2y_1(t)\,[p_2(t)\,(f_u(t)+\sigma_x(t)\,\sigma_u(t))+q_2(t)\,\sigma_u(t)]\\ &+p_2(t)\Big(2\sigma_u(t)\,\widehat{E}[\widehat{\sigma}_\mu^*(t)\widehat{y}_1(t)]+\sigma_u^2(t)\,v(t)\Big)\Big]\,v(t)\\ &-y_1^2(t)(H_{xx}(t)+\widehat{E}[\widehat{H}_{x\mu}^*(t)]+\widehat{E}[\widehat{H}_{\mu\mu}^*(t)])\Big\}\,\mathrm{d}t.\end{aligned} \tag{4.66}$$

Finally, since $p_2(T)=-h_{xx}(x(T))-\widehat{E}[\widehat{h}_{x\mu}^*(x(T))]$, then the desired result (4.53) follows, which completes the proof of Lemma 4.4.1 □

Proposition 4.4.3 Let assumptions (H 4.1), (H 4.2) and (H 4.3) hold. Then the following variational equality holds. For any control $u(\cdot) \in \mathcal{U}([0,T])$,

$$\begin{aligned}
&J(u^{\varepsilon}(\cdot)) - J(u^{*}(\cdot)) \\
&= -E\int_0^T \left[\varepsilon H_u(t)v(t) + \frac{\varepsilon^2}{2}\left(H_{uu}(t) + p_2(t)\sigma_u^2(t)\right)v^2(t)\right. \\
&\left.+\varepsilon^2\mathcal{H}(t)y_1(t)\,v(t)\right] \mathrm{d}t + o\left(\varepsilon^2\right) \quad \left(\varepsilon \longrightarrow 0^{+}\right).
\end{aligned} \tag{4.67}$$

where $v(t) = u(t) - u^{*}(t)$ and

$$\begin{aligned}
\mathcal{H}(t) &= \mathcal{H}(t, x, u, \mu, p_1, q_1, p_2, q_2) \\
&= H_{xu}(t) + \widehat{E}[\widehat{H}_{\mu u}(t)] + f_u(t, x, \mu, u)p_2(t) + \sigma_u(t, x, \mu, u)q_2(t) \\
&+ p_2(t)\sigma_u(t, x, \mu, u)(\sigma_x(t, x, \mu, u) + \widehat{E}[\widehat{\sigma}_{\mu}(t, x, \mu, u)]).
\end{aligned} \tag{4.68}$$

Proof. From (4.16), we have

$$\begin{aligned}
&J(u^{\varepsilon}(\cdot)) - J(u^{*}(\cdot)) \\
&= E\int_0^T \delta\ell(t)\,\mathrm{d}t + E[h(x^{\varepsilon}(T), P_{x^{\varepsilon}(T)}) - h(x^{*}(T), P_{x^{*}(T)})].
\end{aligned}$$

Applying *Taylor-Young's formula*, we get

$$
\begin{aligned}
& J\left(u^{\varepsilon}(\cdot)\right)-J\left(u^{*}(\cdot)\right) \\
& =E\left[\int_{0}^{T}\left\{\ell_{x}(t)\left(x^{\varepsilon}(t)-x^{*}(t)\right)+\widehat{E}\left[\widehat{\ell}_{\mu}^{*}(t)\left(\widehat{x}^{\varepsilon}(t)-\widehat{x}^{*}(t)\right)\right]+\ell_{u}(t)\left(u^{\varepsilon}(t)-u^{*}(t)\right)\right.\right. \\
& +\frac{1}{2} \ell_{xx}(t)\left(x^{\varepsilon}(t)-x^{*}(t)\right)^{2}+\frac{1}{2} \widehat{E}[\widehat{\ell}_{\mu\mu}^{*}(t)\left(\widehat{x}^{\varepsilon}(t)-\widehat{x}^{*}(t)\right)^{2}]+\frac{1}{2} \ell_{uu}(t)\left(u^{\varepsilon}(t)-u^{*}(t)\right)^{2} \\
& +\ell_{xu}(t)\left(x^{\varepsilon}(t)-x^{*}(t)\right)\left(u^{\varepsilon}(t)-u^{*}(t)\right)+\widehat{E}[\widehat{\ell}_{\mu u}^{*}(t)\left(\widehat{x}^{\varepsilon}(t)-\widehat{x}^{*}(t)\right)\left(u^{\varepsilon}(t)-u^{*}(t)\right)] \\
& \left.+\widehat{E}[\widehat{\ell}_{x\mu}^{*}(t)\left(\widehat{x}^{\varepsilon}(t)-\widehat{x}^{*}(t)\right)]\left(x^{\varepsilon}(t)-x^{*}(t)\right)\right\} \mathrm{d}t \\
& +E\left[h_{x}(T)\left(x^{\varepsilon}(T)-x^{*}(T)\right)+\frac{1}{2} h_{xx}(T)\left(x^{\varepsilon}(T)-x^{*}(T)\right)^{2}\right] . \\
& +E\left[\widehat{E}[h_{\mu}^{*}(T)\left(\widehat{x}^{\varepsilon}(T)-\widehat{x}^{*}(T)\right)]+\frac{1}{2} \widehat{E}[\widehat{h}_{\mu\mu}^{*}(T)\left(\widehat{x}^{\varepsilon}(T)-\widehat{x}^{*}(T)\right)^{2}]\right. \\
& \left.+\widehat{E}[\widehat{h}_{x\mu}^{*}(T)\left(x^{\varepsilon}(T)-x^{*}(T)\right)\left(\widehat{x}^{\varepsilon}(T)-\widehat{x}^{*}(T)\right)]\right]+o\left(\varepsilon^{2}\right) .
\end{aligned}
$$

Applying Proposition 4.4.1 and Proposition 4.4.2and the fact that $u^{\varepsilon}(t)-u^{*}(t)=\varepsilon\left(u(t)-u^{*}(t)\right)=\varepsilon v$, we obtain

$$
\begin{aligned}
& J\left(u^{\varepsilon}(\cdot)\right)-J\left(u^{*}(\cdot)\right) \\
& =E\left[\int_{0}^{T}\left\{\varepsilon \ell_{x}(t) y_{1}(t)+\varepsilon \widehat{E}\left[\widehat{\ell}_{\mu}(t) \widehat{y}_{1}(t)\right]+\frac{\varepsilon^{2}}{2} \ell_{x}(t) y_{2}(t)+\frac{\varepsilon^{2}}{2} \widehat{E}\left[\widehat{\ell}_{\mu}^{*}(t) \widehat{y}_{2}(t)\right]\right.\right. \\
& +\varepsilon \ell_{u}(t) v(t)+\frac{\varepsilon^{2}}{2}\left(\ell_{xx}(t) y_{1}(t)^{2}+\widehat{E}(\widehat{\ell}_{\mu\mu}^{*}(t) \widehat{y}_{1}(t)^{2})+\ell_{uu}(t) v(t)^{2}\right. \\
& \left.\left.+2 \ell_{xu}(t) y_{1}(t) v(t)+2 \widehat{E}(\widehat{\ell}_{\mu u}^{*}(t) \widehat{y}_{1}(t)) v(t)+2 \widehat{E}(\widehat{\ell}_{x\mu}^{*}(t) \widehat{y}_{1}(t)) y_{1}(t)\right)\right\} \mathrm{d}t \\
& +E\left[\varepsilon[h_{x}(T)) y_{1}(T)+\widehat{E}(\widehat{h}_{\mu}^{*}(T) \widehat{y}_{1}(T))]+\frac{\varepsilon^{2}}{2}[h_{x}(T) y_{2}(T)+\widehat{E}(\widehat{h}_{\mu}^{*}(T) \widehat{y}_{2}(T))]\right. \qquad (4.69)\\
& \left.+\frac{\varepsilon^{2}}{2}\left[h_{xx}(T) y_{1}^{2}(T)+2 \widehat{E}[\widehat{h}_{x\mu}(T) \widehat{y}_{1}(T) y_{1}(T)]+\widehat{E}[\widehat{h}_{\mu\mu}(T) \widehat{y}_{1}^{2}(T)]\right]+o\left(\varepsilon^{2}\right)\right], \\
& \left(\varepsilon \longrightarrow 0^{+}\right) .
\end{aligned}
$$

Further, from Lemma 4.4.1 , we have

$$
\begin{aligned}
& J\left(u^{\varepsilon}(\cdot)\right)-J\left(u^{*}(\cdot)\right) \\
& =E \int_{0}^{T}\left\{\varepsilon \ell_{x}(t) y_{1}(t)+\varepsilon \widehat{E}(\widehat{\ell}_{\mu}^{*}(t) \widehat{y}_{1}(t))+\frac{\varepsilon^{2}}{2} \ell_{x}(t) y_{2}(t)+\frac{\varepsilon^{2}}{2} \widehat{E}(\widehat{\ell}_{\mu}^{*}(t) \widehat{y}_{2}(t))+\varepsilon \ell_{u}(t) v(t)\right. \\
& +\frac{\varepsilon^{2}}{2}\left(\ell_{xx}(t) y_{1}(t)^{2}+\widehat{E}(\widehat{\ell}_{\mu\mu}^{*}(t) \widehat{y}_{1}(t)^{2})+\ell_{uu}(t) v(t)^{2}\right. \\
& \left.\left.+\ 2 \ell_{xu}(t) y_{1}(t) v(t)+2 \widehat{E}(\widehat{\ell}_{\mu u}^{*}(t) \widehat{y}_{1}(t)) v(t)+2 \widehat{E}(\widehat{\ell}_{x\mu}^{*}(t) \widehat{y}_{1}(t)) y_{1}(t)\right)\right\} \mathrm{d} t \\
& -\varepsilon E \int_{0}^{T}\left[p_{1}(t) f_{u}(t) v(t)+q_{1}(t) \sigma_{u}(t) v(t)+y_{1}(t)(\ell_{x}(t)+\widehat{E}(\widehat{\ell}_{\mu}^{*}(t)))\right] \mathrm{d} t \\
& -\frac{\varepsilon^{2}}{2} E \int_{0}^{T}\left\{p_{1}(t)\left[f_{xx}(t) y_{1}^{2}(t)+2 \widehat{E}[\widehat{f}_{x\mu}^{*}(t) \widehat{y}_{1}(t)] y_{1}(t)+\widehat{E}[\widehat{f}_{\mu\mu}^{*}(t) \widehat{y}_{1}(t)^{2}]+2 f_{xu}(t) v(t) y_{1}(t)\right.\right. \\
& \left.+2 \widehat{E}[\widehat{f}_{u\mu}^{*}(t) \widehat{y}_{1}(t)] v(t)+f_{uu}(t) v^{2}(t)\right] \\
& +q_{1}(t)\left[\sigma_{xx}(t) y_{1}^{2}(t)+\widehat{E}[\widehat{\sigma}_{x\mu}^{*}(t) \widehat{y}_{1}(t)] y_{1}(t)+2 \sigma_{xu}(t) y_{1}(t) v(t)\right. \\
& \left.\left.+\ 2 \widehat{E}[\widehat{\sigma}_{u\mu}^{*}(t) \widehat{y}_{1}(t)] v(t)+\sigma_{uu}(t) v^{2}(t)\right]+\ell_{x}(t) y_{2}(t)+\widehat{E}(\widehat{\ell}_{\mu}^{*}(t)) y_{2}(t)\right\} \mathrm{d} t \\
& -\frac{\varepsilon^{2}}{2} E \int_{0}^{T}\{2 f_{u}(t) p_{2}(t) y_{1}(t) v(t)+2 \sigma_{x}(t) \sigma_{u}(t) p_{2}(t) y_{1}(t) v(t)+2 q_{2}(t) \sigma_{u}(t) y_{1}(t) v(t) \\
& +2 \sigma_{u}(t) \widehat{E}[\widehat{\sigma}_{\mu}(t) \widehat{y}_{1}(t)] p_{2}(t) v(t)+\sigma_{u}^{2}(t) p_{2}(t) v^{2}(t) \\
& \left.-\ y_{1}^{2}(t)(H_{xx}(t)+\widehat{E}[\widehat{H}_{x\mu}^{*}(t)]+\widehat{E}[\widehat{H}_{\mu\mu}^{*}(t)])\right\} \mathrm{d} t \\
& +o\left(\varepsilon^{2}\right), \quad\left(\varepsilon \longrightarrow 0^{+}\right) .
\end{aligned}
$$

Since $\widehat{E}[\widehat{H}_{x\mu}^{*}(t)]=\widehat{E}(\widehat{f}_{x\mu}^{*}(t) p_{1}(t))+\widehat{E}(\widehat{\sigma}_{x\mu}^{*}(t) q_{1}(t))-\widehat{E}(\widehat{\ell}_{x\mu}^{*}(t))$, then we get

$$
\begin{aligned}
& J\left(u^{\varepsilon}(\cdot)\right)-J\left(u^{*}(\cdot)\right) \\
& =-E \int_{0}^{T}\{\varepsilon\left[f_{u}(t) p_{1}(t)+\sigma_{u}(t) q_{1}(t)-\ell_{u}(t)\right] v(t) \\
& +\frac{\varepsilon^{2}}{2}\left[f_{uu}(t) p_{1}(t)+\sigma_{uu}(t) q_{1}(t)-\ell_{uu}(t)\right] v^{2}(t) \\
& \left.+\varepsilon^{2}\left[\mathcal{H}(t) y_{1}(t) v(t)\right]+\frac{\varepsilon^{2}}{2} p_{2}(t) \sigma_{u}^{2}(t) v^{2}(t)\right\} \mathrm{d} t+o\left(\varepsilon^{2}\right),
\end{aligned}
$$

where $\mathcal{H}(t)$ is given by (4.68). From (4.22) and (4.24), we get

$$\begin{aligned} & J\left(u^{\varepsilon}(\cdot)\right)-J\left(u^{*}(\cdot)\right) \\ & =-E \int_{0}^{T}\left[\varepsilon H_{u}(t) v(t)+\frac{\varepsilon^{2}}{2} H_{u u}(t) v^{2}(t)+\varepsilon^{2} \mathcal{H}(t) y_{1}(t) v(t)\right. \\ & \left.+\frac{\varepsilon^{2}}{2} p_{2}(t) \sigma_{u}^{2}(t) v^{2}(t)\right] \mathrm{d} t+o\left(\varepsilon^{2}\right) \quad\left(\varepsilon \longrightarrow 0^{+}\right) . \end{aligned} \tag{4.70}$$

This completes the proof of Proposition 4.4.3 □

Applying Proposition 4.4.3 , we can derive the following second order necessary condition in integral form for our stochastic optimal control (4.15)-(4.16).

Theorem 4.4.1 *(Maximum principle in integral form)* Let assumption (H1), (H2) and (H3) hold. If $u^{*}(\cdot)$ is a singular optimal control in the classical sense for the control problem (4.15)-(4.16). Then we have

$$E \int_{0}^{T} \mathcal{H}(t) y_{1}(t)(u(t)-u^{*}(t)) \mathrm{d} t \leq 0, \tag{4.71}$$

for any $u(\cdot) \in \mathcal{U}([0, T])$, where $\mathcal{H}(t)$ is defined by the formula (4.68) and $y_{1}(\cdot)$ is the solution of first-order variational equation given by

$$\begin{aligned} y_{1}(t)= & \int_{0}^{t}\left[f_{x}(s) y_{1}(s)+\widehat{E}[\widehat{f}_{\mu}^{*}(s) \widehat{y}_{1}(s)]+f_{u}(s) v(s)\right] \mathrm{d} s \\ & +\int_{0}^{t}\left[\sigma_{x}(s) y_{1}(s)+\widehat{E}[\widehat{\sigma}_{\mu}^{*}(s) \widehat{y}_{1}(s)]+\sigma_{u}(s) v(s)\right] \mathrm{d} W(s). \end{aligned}$$

Proof. From the optimality of $u^{*}(\cdot)$, and Proposition 4.4.3 , for $v(t)=u(t)-u^{*}(t)$

we have

$$
\begin{aligned}
0 &\leq \frac{J\left(u^{\varepsilon}(\cdot)\right)-J\left(u^{*}(\cdot)\right)}{\varepsilon^{2}} \\
&= -E \int_{0}^{T}\left[\frac{1}{\varepsilon} H_{u}(t) v(t)+\frac{1}{2} H_{u u}(t) v^{2}(t)\right. \\
&\left.+\mathcal{H}(t) y_{1}(t) v(t)+\frac{1}{2} p_{2}(t) \sigma_{u}^{2}(t) v^{2}(t)\right] \mathrm{d} t \\
&+\frac{1}{\varepsilon^{2}} o\left(\varepsilon^{2}\right) \quad\left(\varepsilon \longrightarrow 0^{+}\right),
\end{aligned}
$$

by applying (4.25), we deduce

$$
\frac{1}{\varepsilon} H_{u}(t) v(t)=0,
$$

and

$$
\frac{1}{2} H_{u u}(t) v^{2}(t)+\frac{1}{2} p_{2}(t) \sigma_{u}^{2}(t) v^{2}(t)=\frac{1}{2}\left[H_{u u}(t)+p_{2}(t) \sigma_{u}^{2}(t)\right] v^{2}(t)=0,
$$

the desired result (4.71) follows immediately. This completes the proof of Theorem 4.4.1

□

4.5 Pointwise mean-field second order maximum principle

In this section, by using the property of Itô's integrals and the martingale representation theorem, we establish the second order necessary condition for singular optimal controls, which is pointwise mean-field maximum principle in terms of the martingale with respect to the time variable t. The following lemma plays an important role to establish our result.

Lemma 4.5.1 Under assumptions (H 4.1), (H4.2) and (H 4.3),.the first variational equation (5.17) admits a unique strong solution $y_1(\cdot)$, which is represented by the following :

$$y_1(t) = \Phi(t)\left[\int_0^t \Psi(s)\left(f_u(s) - (\sigma_x(s) + \widehat{E}[\widehat{\sigma}_\mu^*(s)])\sigma_u(s)\right)v(s)\mathrm{d}s + \int_0^t \Psi(s)\sigma_u(s)v(s)\mathrm{d}W(s)\right], \tag{4.72}$$

where $\Phi(t)$ is a defined by the following linear stochastic differential equation :

$$\begin{cases} d\Phi(t) &= \left[f_x(t) + \widehat{E}(\widehat{f}_\mu^*(t))\right]\Phi(t)\mathrm{d}t + \left[\sigma_x(t) + \widehat{E}[\widehat{\sigma}_\mu^*(t)]\right]\Phi(t)\mathrm{d}W(t), \\ \Phi(0) &= 1, \end{cases} \tag{4.73}$$

and $\Psi(t)$ its inverse.

Proof. Equation (4.73) is linear with bounded coefficients, then it admits a unique strong solution. Moreover, this solution is invertible and its inverse $\Psi(t) = \Phi^{-1}(t)$ given by the following equation

$$\begin{cases} d\Psi(t) = \left[\left(\sigma_x(t) + \widehat{E}(\widehat{\sigma}_\mu^*(t))\right)^2\Psi(t) - f_x(t)\Psi(t) - \widehat{E}(\widehat{f}_\mu^*(t))\Psi(t)\right]\mathrm{d}t \\ \qquad - \left(\sigma_x(t) + \widehat{E}[\widehat{\sigma}_\mu^*(t)]\right)\Psi(t)\mathrm{d}W(t), \\ \Psi(0) = 1. \end{cases} \tag{4.74}$$

Applying Itô's formula to $\Psi(t)y_1(t)$, we deduce

$$\begin{aligned} &d[\Psi(t)y_1(t)] \\ &= y_1(t)\,\mathrm{d}\Psi(t) + \Psi(t)\mathrm{d}y_1(t) \\ &- \left[(\sigma_x(t) + \widehat{E}[\widehat{\sigma}_\mu^*(t)])\Psi(t)\right]\left[\sigma_x(t)y_1(t) + \widehat{E}[\widehat{\sigma}_\mu^*(t)\widehat{y}_1(t)] + \sigma_u(t)v(t)\right]\mathrm{d}t \end{aligned} \tag{4.75}$$

$$= I_1(t) + I_2(t) + I_3(t), \tag{4.76}$$

where

$$\begin{aligned} I_1(t) &= y_1(t)\,\mathrm{d}\Psi(t) \qquad (4.77)\\ &= \left[y_1(t)\left(\sigma_x(t)+\widehat{E}[\widehat{\sigma}^*_\mu(t)]\right)^2\Psi(t) - y_1(t)f_x(t)\Psi(t) - y_1(t)\widehat{E}(\widehat{f}^*_\mu(t))\Psi(t)\right]\mathrm{d}t\\ &- y_1(t)\left(\sigma_x(t)+\widehat{E}[\widehat{\sigma}^*_\mu(t)]\right)\Psi(t)\mathrm{d}W(t). \end{aligned}$$

By simple computations, we obtain

$$\begin{aligned} I_2(t) &= \Psi(t)\mathrm{d}y_1(t)\\ &= \left[\Psi(t)f_x(t)y_1(t) + \Psi(t)\widehat{E}[\widehat{f}^*_\mu(t)\widehat{y}_1(t)] + \Psi(t)f_u(t)v(t)\right]\mathrm{d}t \qquad (4.78)\\ &+ \left[\Psi(t)\sigma_x(t)y_1(t) + \Psi(t)\widehat{E}[\widehat{\sigma}^*_\mu(t)\widehat{y}_1(t)] + \Psi(t)\sigma_u(t)v(t)\right]\mathrm{d}W(t), \end{aligned}$$

and

$$I_3(t) = -\left[(\sigma_x(t)+\widehat{E}[\widehat{\sigma}^*_\mu(t)])\Psi(t)\right]\left[\sigma_x(t)y_1(t)+\widehat{E}[\widehat{\sigma}^*_\mu(t)\widehat{y}_1(t)]+\sigma_u(t)v(t)\right]\mathrm{d}t. \qquad (4.79)$$

Substituting (4.77), (4.78) and (4.79) into (4.76), we get

$$\begin{aligned} &\Psi(t)y_1(t) - \Psi(0)y_1(0)\\ &= \left[\int_0^t \Psi(s)\left[f_u(s) - \left(\sigma_x(s)+\widehat{E}[\widehat{\sigma}^*_\mu(s)]\right)\sigma_u(s)\right]v(s)\mathrm{d}s\right.\\ &+ \int_0^t \Psi(s)\sigma_u(s)v(s)\mathrm{d}W(s). \qquad (4.80) \end{aligned}$$

Since $y_1(0)=0$ and $\Psi^{-1}(t)=\Phi(t)$, then from (4.80) the desired result (4.73) is fulfilled. This completes the proof of Lemma 4.5.1 □

To prove the main theorem we need the following technical Lemma.

Lemma 4.5.2 Let assumptions (H 4.1), (H 4.2) and (H 4.3) hold. Then we have

(1) $\mathcal{H}(\cdot) \in \mathbb{L}^2_{\mathbb{F}}([0,T];\mathbb{R})$.

(2) For any $v \in U$, there exists a unique stochastic process $\xi_v(\cdot, t) \in \mathbb{L}_{\mathbb{F}}^2([0,T];\mathbb{R})$, with $E\left(\left[\int_0^T |\xi_v(s,t)|^2 ds\right]^2\right) < \infty$ such that

$$\mathcal{H}(t)(v - \bar{u}(t)) = E\left[\mathcal{H}(t)(v - \overline{u}(t))\right] + \int_0^t \xi_v(s,t)\, \mathrm{d}W(s) \tag{4.81}$$
$$a.e.\ t \in [0,T],\ \ P - a.s.$$

Proof. Since the derivatives f_{xu}, $f_{\mu u}$, σ_{xu}, $\sigma_{\mu u}$, ℓ_{xu}, $\ell_{\mu u}$, f_u, σ_u, σ_x, and σ_μ, are bounded, (see assumptions (H 4.2) and (H 4.3)), we have $E\left(\left[\int_0^T |\mathcal{H}(t)|^2 dt\right]^2\right) < \infty$, the desired result in (4.81) follows immediately. The second item follows by applying *Martingale Representation Theorem*, (see also Lemma 3.9 in [124].). □

Now, in order to derive a pointwise second order necessary condition from the integral form in (4.71), we need to choose the following *spike variation* (needle variation) for the optimal control $\overline{u}(\cdot)$:

$$u(t) = \begin{cases} v, & t \in \mathcal{G}_\epsilon, \\ \overline{u}(t), & t \in [0,T] \mid \mathcal{G}_\epsilon, \end{cases} \tag{4.82}$$

where $\tau \in [0,T)$, $v \in U$, $\mathcal{G}_\epsilon$ is a Borel subset, $\mathcal{G}_\epsilon = [\tau, \tau+\epsilon) \subset [0,T]$ so that $\epsilon > 0$ and $\tau + \epsilon \leq T$. Denote by $\mathbf{I}_{\mathcal{G}_\epsilon}(\cdot)$ the characteristic function of the set $\mathcal{G}_\epsilon$. Then we have

$$v(\cdot) = u(\cdot) - \overline{u}(\cdot) = (v - \overline{u}(\cdot))\, \mathbf{I}_{\mathcal{G}_\epsilon}.$$

The following theorem constitutes the main contribution of this work.

Theorem 4.5.1 Let assumptions (H 4.1), (H 4.2) and (H 4.3) hold. If $\overline{u}(\cdot)$ is a singular optimal control in the classical sense for the stochastic control (4.15)-(4.16), then for any $v \in U$, it holds that

$$E\left[\mathcal{H}(\tau) f_u(\tau)(v - \overline{u}(\tau))^2\right] + \partial_\tau^+\left(\mathcal{H}(\tau)\sigma_u(\tau)(v - \overline{u}(\tau))^2\right) \leq 0,\ \ a.e.\ \tau \in [0,T], \tag{4.83}$$

where $\mathcal{H}(\tau)$ given by (4.68) at τ

$$\mathcal{H}(\tau) = H_{xu}(\tau) + \widehat{E}[\widehat{H}^*_{\mu u}(\tau)] + p_2(\tau)\left[f_u(\tau) + \sigma_u(\tau)\left(\sigma_x(\tau) + \widehat{E}[\widehat{\sigma}^*_\mu(\tau)]\right)\right]$$
$$+ \sigma_u(\tau) q_2(\tau),$$

and

$$\partial^+_\tau \left(\mathcal{H}(\tau)\sigma_u(\tau)(v - \overline{u}(\tau))^2\right) \tag{4.84}$$
$$:= 2 \lim_{\epsilon\to 0^+} \sup \frac{1}{\epsilon^2} E \int_\tau^{\tau+\epsilon} \int_\tau^t \left[\xi_v(s,t)\Phi(\tau)\Psi(s)\sigma_u(s)(v - \overline{u}(s))\right] \mathrm{d}s\mathrm{d}t,$$

where $\xi_v(\cdot, t)$ is given by (4.81), and $\Psi(\cdot)$ is determined by (4.74).

Proof. From (4.82), we have $v(\cdot) = u(\cdot) - \overline{u}(\cdot) = (v - \overline{u}(\cdot))\mathbf{I}_{\mathcal{G}_\epsilon}(\cdot)$ and the corresponding solution $y_1(\cdot)$ to (4.72) is given by the following mean-field equation :

$$y_1(t) = \Phi(t) \int_0^t \Psi(s)\left(f_u(s) - (\sigma_x(s) + \widehat{E}[\widehat{\sigma}^*_\mu(s)])\sigma_u(s)\right)(v - \overline{u}(s))\mathbf{I}_{\mathcal{G}_\epsilon}(s)\mathrm{d}s \tag{4.85}$$
$$+ \Phi(t) \int_0^t \Psi(s)\sigma_u(s)(v - \overline{u}(s))\mathbf{I}_{\mathcal{G}_\epsilon}(s)\mathrm{d}W(s).$$

Substituting $v(\cdot) = (v - \overline{u}(\cdot))\mathbf{I}_{\mathcal{G}_\epsilon}(\cdot)$ and (4.85) into (4.71), we have

$$0 \geq \frac{1}{\epsilon^2} E \int_\tau^{\tau+\epsilon} \left[\mathcal{H}(t)\, y_1(t)\,(v - \overline{u}(t))\right] \mathrm{d}t$$
$$= \frac{1}{\epsilon^2} E \int_\tau^{\tau+\epsilon} \left[\mathcal{H}(t)\,\Phi(t) \int_\tau^t \Psi(s)\left(f_u(s) - (\sigma_x(s) + \widehat{E}[\widehat{\sigma}^*_\mu(s)])\sigma_u(s)\right)\right.$$
$$\left.\times\ (v - \overline{u}(s))\mathrm{d}s\,(v - \overline{u}(t))\right] \mathrm{d}t$$
$$+ \frac{1}{\epsilon^2} E \int_\tau^{\tau+\epsilon} \left[\mathcal{H}(t)\,\Phi(t) \int_\tau^t \Psi(s)\,\sigma_u(s)(v - \overline{u}(s))\mathrm{d}W(s)\,(v - \overline{u}(t))\right] \mathrm{d}t$$
$$= J^\epsilon_1(\tau) + J^\epsilon_2(\tau), \tag{4.86}$$

where

$$J_1^{\epsilon}(\tau) = \frac{1}{\epsilon^2} E \int_{\tau}^{\tau+\epsilon} \Big[\mathcal{H}(t) \, \Phi(t) \int_{\tau}^{t} \Psi(s) \Big(f_u(s) - (\sigma_x(s) + \widehat{E}[\widehat{\sigma}_{\mu}^*(s)])\sigma_u(s) \Big) \\ \times (v - \overline{u}(s)) \mathrm{d}s \, (v - \overline{u}(t)) \Big] \mathrm{d}t,$$

and

$$J_2^{\epsilon}(\tau) = \frac{1}{\epsilon^2} E \int_{\tau}^{\tau+\epsilon} \Big[\mathcal{H}(t) \, \Phi(t) \int_{\tau}^{t} \Psi(s) \, \sigma_u(s)(v - \overline{u}(s)) \mathrm{d}W(s) \, (v - \overline{u}(t)) \Big] \mathrm{d}t.$$

Applying similar arguments developed in [124], we obtain

$$\lim_{\epsilon \to 0^+} J_1^{\epsilon}(\tau) = \frac{1}{2} E \Big[\mathcal{H}(\tau) \Big(f_u(\tau) - (\sigma_x(\tau) + \widehat{E}[\widehat{\sigma}_{\mu}^*(\tau)])\sigma_u(\tau) \Big) (v - \overline{u}(\tau))^2 \Big]. \tag{4.87}$$

Let us turn to estimate the second term $J_2^{\epsilon}(\tau)$. From (4.73), and since

$$\Phi(t) = \Phi(\tau) + \int_{\tau}^{t} (f_x(s) + \widehat{E}(\widehat{f}_{\mu}^*(s)))\Phi(s) \mathrm{d}s \\ + \int_{\tau}^{t} (\sigma_x(s) + \widehat{E}[\widehat{\sigma}_{\mu}^*(s)])\Phi(s) \mathrm{d}W(s),$$

it follows that

$$\begin{aligned} J_2^{\epsilon}(\tau) &= \frac{1}{\epsilon^2} E \int_{\tau}^{\tau+\epsilon} \Big[\mathcal{H}(t) \, \Phi(\tau) \int_{\tau}^{t} \Psi(s) \, \sigma_u(s)(v - \overline{u}(s)) \mathrm{d}W(s) \, (v - \overline{u}(t)) \Big] \mathrm{d}t \\ &+ \frac{1}{\epsilon^2} E \int_{\tau}^{\tau+\epsilon} \Big\{ \mathcal{H}(t) \int_{\tau}^{t} (f_x(s) + \widehat{E}(\widehat{f}_{\mu}^*(s)))\Phi(s) \mathrm{d}s \\ &\times \int_{\tau}^{t} \Psi(s) \, \sigma_u(s)(v - \overline{u}(s)) \mathrm{d}W(s) \, (v - \overline{u}(t)) \Big\} \mathrm{d}t \\ &+ \frac{1}{\epsilon^2} E \int_{\tau}^{\tau+\epsilon} \Big\{ \mathcal{H}(t) \int_{\tau}^{t} (\sigma_x(s) + \widehat{E}[\widehat{\sigma}_{\mu}^*(s)])\Phi(s) \mathrm{d}W(s) \\ &\times \int_{\tau}^{t} \Psi(s) \, \sigma_u(s)(v - \overline{u}(s)) \mathrm{d}W(s) \, (v - \overline{u}(t)) \Big\} \mathrm{d}t \\ &= J_{2,1}^{\epsilon}(\tau) + J_{2,2}^{\epsilon}(\tau) + J_{2,3}^{\epsilon}(\tau). \end{aligned} \tag{4.88}$$

Now, we proceed to derive estimates for the terms $J_{2,1}^{\epsilon}(\tau)$, $J_{2,2}^{\epsilon}(\tau)$, and $J_{2,3}^{\epsilon}(\tau)$.

Arguing as in [124, Eq-(4.8)], with the help of Lemma 4.5.2 , we get

$$\begin{aligned}\lim_{\epsilon\to0^+}\sup J_{2,1}^{\epsilon}(\tau) &= \lim_{\epsilon\to0^+}\sup\frac{1}{\epsilon^2}E\int_{\tau}^{\tau+\epsilon}\left[\mathcal{H}(t)\,\Phi(\tau)\int_{\tau}^{t}\Psi(s)\,\sigma_u(s)(v-\overline{u}(s))\mathrm{d}W(s)\right.\\ &\quad\left.\times(v-\overline{u}(t))\right]\mathrm{d}t\\ &= \frac{1}{2}\partial_{\tau}^{+}\left(\mathcal{H}(\tau)\,(v-\overline{u}(\tau))^2\sigma_u(\tau)\right),\ \forall\tau\in[0,T].\end{aligned}\tag{4.89}$$

Let us turn to second term $J_{2,2}^{\epsilon}(\tau)$ in the right-hand side of (**??**). Since f_x, f_μ are bounded, then by applying similar arguments developed in [124, Eq-(4.9)], we have

$$\begin{aligned}\lim_{\epsilon\to0^+}\sup J_{2,2}^{\epsilon}(\tau) &= \lim_{\epsilon\to0^+}\sup\frac{1}{\epsilon^2}E\int_{\tau}^{\tau+\epsilon}\left\{\mathcal{H}(t)\int_{\tau}^{t}(f_x(s)+\widehat{E}(\widehat{f}_{\mu}^{*}(s)))\Phi(s)\mathrm{d}s\right.\\ &\quad\left.\times\int_{\tau}^{t}\Psi(s)\,\sigma_u(s)(v-\overline{u}(s))\mathrm{d}W(s)\,(v-\overline{u}(t))\right\}\mathrm{d}t\\ &= 0,\quad a.e.\ \tau\in[0,T].\end{aligned}\tag{4.90}$$

Let us turn to third term $J_{2,3}^{\epsilon}(\tau)$ in the right-hand side of (4.88). Since $E\left|\int_{\tau}^{t}|\sigma_x(s)\Phi(s)|^2\,ds\right|^2$ and $E\left|\int_{\tau}^{t}\left|\widehat{E}[\widehat{\sigma}_{\mu}^{*}(s)])\Phi(s)\right|^2ds\right|^2$ are bounded, then by applying similar arguments developed in [124, Eq-(4.10)], we have

$$\begin{aligned}&\lim_{\epsilon\to0^+}\sup J_{2,3}^{\epsilon}(\tau)\\ &= \lim_{\epsilon\to0^+}\sup\frac{1}{\epsilon^2}E\int_{\tau}^{\tau+\epsilon}\left\{\mathcal{H}(t)\int_{\tau}^{t}(\sigma_x(s)+\widehat{E}[\widehat{\sigma}_{\mu}^{*}(s)])\Phi(s)\mathrm{d}W(s)\right.\\ &\times\left.\int_{\tau}^{t}\Psi(s)\,\sigma_u(s)(v-\overline{u}(s))\mathrm{d}W(s)\,(v-\overline{u}(t))\right\}\mathrm{d}t\\ &= \frac{1}{2}E\left[\mathcal{H}(\tau)\,(\sigma_x(\tau)+\widehat{E}[\widehat{\sigma}_{\mu}^{*}(\tau)])\sigma_u(\tau)(v-\overline{u}(\tau))^2\right].\ a.e.\ \tau\in[0,T].\end{aligned}\tag{4.91}$$

Substituting (4.89), (4.90), (4.91) in (4.88), we obtain

$$\lim_{\epsilon\to 0^+}\sup J_2^{\epsilon}(\tau) = \frac{1}{2}E\left[\mathcal{H}(\tau)\left(\sigma_x(\tau)+\widehat{E}[\widehat{\sigma}_{\mu}^{*}(\tau)]\right)\sigma_u(\tau)(v-\overline{u}(\tau))^2\right] + \frac{1}{2}\partial_{\tau}^{+}\left(\mathcal{H}(\tau)(v-\overline{u}(\tau))^2\sigma_u(\tau)\right),\ a.e.\ \tau\in[0,T]. \tag{4.92}$$

Finally, by substituting (4.92), (4.87) in (**??**), we get

$$0 \geq \frac{1}{2}E\left[\mathcal{H}(\tau)\left(f_u(\tau)-(\sigma_x(\tau)+\widehat{E}[\widehat{\sigma}_{\mu}^{*}(\tau)])\sigma_u(\tau)\right)(v-\overline{u}(\tau))^2\right] + \frac{1}{2}E\left[\mathcal{H}(\tau)\left(\sigma_x(\tau)+\widehat{E}[\widehat{\sigma}_{\mu}^{*}(\tau)]\right)\sigma_u(\tau)(v-\overline{u}(\tau))^2\right] + \frac{1}{2}\partial_{\tau}^{+}\left(\mathcal{H}(\tau)\sigma_u(\tau)(v-\overline{u}(\tau))^2\right),\ a.e.\ \tau\in[0,T],$$

then the desired result (4.83) is fulfilled. This completes the proof of Theorem 4.5.1. □

Chapitre 5

Lions's partial derivatives with respect to probability measures for general mean-field stochastic control problem

5.1 Introduction

In this chapter,we establish a necessary stochastic maximum principle for stochastic model governed by mean-field nonlinear controlled Itô-stochastic differential equations. The coefficients of our model are nonlinear and depend explicitly on the control variable, the state process as well as of its probability distribution. The control region is assumed to be bounded and convex. Our main result is derived by applying the Lions's partial-derivatives with respect to random measures in Wasserstein space. The associated Itô-formula and convex-variation approach are applied to establish the optimal control.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\tau]}, P)$ be a fixed filtered probability space and τ be a fixed positive real number. In this chapter, we study the following mean-field-type stochastic optimal nonlinear control problem : We denote by :

Problem A. Minimize a mean-field cost functional

$$J(\alpha(\cdot)) = E\int_{\mathbb{R}^d} \Phi(\mathrm{y}_\alpha(\tau), \mu^{\mathrm{y}_\alpha(\tau)})\mu(d\mathrm{y}_\alpha),$$

subject to $\mathrm{y}_\alpha(\cdot)$ solution of the (MF-SDE) : $t \in [0, \tau]$

$$\begin{cases} d\mathrm{y}_\alpha(t) = \int_{\mathbb{R}^d} \varphi\left(t, \mathrm{y}_\alpha(t), \mu^{\mathrm{y}_\alpha(t)}, \alpha(t)\right)\mu(\mathrm{dy}_\alpha)dt + \int_{\mathbb{R}^d} \psi\left(t, \mathrm{y}_\alpha(t), \mu^{\mathrm{y}_\alpha(t)}, \alpha(t)\right)\mu(\mathrm{dy}_\alpha)\mathrm{d}W(t), \\ \mathrm{y}_\alpha(0) = \mathrm{y}_0. \end{cases}$$

In the above, $\alpha(\cdot)$ is the control variable valued in a convex bounded subset $\mathbb{U} \subset \mathbb{R}^k$, $\mathrm{y}_\alpha(\cdot)$ is the controlled state variable, $W(\cdot)$ is a standard Brownian motion, $\mu^{\mathrm{y}_\alpha(t)}$ is the distribution of $\mathrm{y}_\alpha(t)$ and Φ, φ and ψ are a given maps.

The mean-field control theory has found important applications and has become a powerful tool in many fields, such as mathematical finance, economics, and stochastic mean-field games, see Lasry and Lions [76], Buckdahn et al. [19]. Under partial information, necessary maximum principle of optimality for MF-SDEs has been proved in Wang et al. [87]. Stochastic optimal control of mean-field jump-diffusion systems with delay has been studied by Meng and Shen [84]. Under partial information, the necessary and sufficient conditions for optimal continous and singular controls for mean-field SDEs with Teugels martingales have been studied in Hafayed et al. [42, 61]. Necessary conditions for mean-field FBSDEs have been studied by Hafayed et al. [62]. The general maximum principle for MF-SDEs has been established in Buckdahn et al. [12]. Mean-field game has been studied by Lions [88]. The convex maximum principle for mean-field delay SDE have been investigated in Shen et al. [89]. General maximum principle for optimal stochastic control has been established in Peng [90]. A Peng's type maximum principle for SDEs of mean-field type was proved by Buckdahn et al., [19]. Forward-backward stochastic differential equations (FBSDs) and controlled McKean-Vlasov dynamics have been investigated in

Carmona and Delarue [26]. Linear quadratic optimal control problem for conditional mean-field equation with random coefficients with applications has been investigated by Pham [91]. Necessary maximum principle for optimal continuous-singular control problem for general MF-SDEs, under convexity assumptions have been investigated by Hafayed et al. [41]. Second-order necessary maximum principle for MF-SDEs has been proved in Boukaf et al. [17].

In this chapter, we apply the Lions's partial-derivatives with respect to probability measure to establish our maximum principle. This approach introduced by Lions [88] and later detailed in Buckdahn et al. [19], Cardaliaguet [27] and Guo et al. [39]. Motivated by the recent works above, in this chapter, we derive the necessary maximum principle for our mean-field optimal control problem (5.6) -(5.7) The Lions's partial-derivatives with respect to probability measure in *Wasserstein space* and the associated Itô-formula with some appropriate estimates are applied to prove our result. This approach of derivatives over Wasserstein space has turned out to be crucial in the study of our maximum principle. Our stochastic mean-field model occur naturally in the probabilistic models of financial optimization problems.

Our control problem is strongly motivated by the recent study of the McKean-Vlasov games and the related McKean-Vlasov control problem.

The rest of the chapter is organized as follows. The formulation of the partial derivatives with respect to probability measures, and basic notations are given in Sect. 2 . The formulation of the control problem is given in Sect. 3 . In Sect. 4 , we prove our main results. Finally, to illustrate our theoretical result, we give an example in the last section.

5.2 Lions's partial-derivatives with respect to probability measure

We now recall briefly an important notion in mean-field control problems : the Lions's partial derivatives with respect to probability measures, over *Wasserstein space* which was introduced by P.Lions [88], see also Cardaliaguet [27], and Guo et al. [39] and the recent references therein.

Throughout this chapter, we let $\mathbb{K}_2(\mathbb{R}^n)$ be *Wasserstein space of probability measures* on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ with finite second-moment, i.e; $\int_{\mathbb{R}^n} |\mathrm{y}|^2 \mu(\mathrm{dy}) < \infty$, endowed with the following *Wasserstein metric* : for $\mu_1, \mu_2 \in \mathbb{K}_2(\mathbb{R}^n)$,

$$\mathbb{T}(\mu_1, \mu_2) = \inf_{\rho(\cdot,\cdot) \in \mathbb{K}_2(\mathbb{R}^{2n})} \left[\int_{\mathbb{R}^{2n}} |\mathrm{x} - y|^2 \rho(\mathrm{dx}, \mathrm{d}y) \right]^{\frac{1}{2}}, \tag{5.1}$$

where $\rho(\cdot, \mathbb{R}^n) = \mu_1$, and $\rho(\mathbb{R}^n, \cdot) = \mu_2$.

The main idea in Lions's partial-derivatives is to identify a distribution (measure of probability) $\mu \in \mathbb{K}_2(\mathbb{R}^n)$ with a random variable $\mathrm{y}(\cdot) \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ so that $\mu = P_{\mathrm{y}}$ is the law of $\mathrm{y}(\cdot)$. We assume that probability space $(\Omega, \mathcal{F}, P)$ is *rich-enough* in the sense that for every $\mu \in \mathbb{K}_2(\mathbb{R}^n)$, there is a random variable $\mathrm{y}(\cdot) \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ such that $\mu = P_{\mathrm{y}}$. We suppose that there is a sub-$\sigma-$field $\mathcal{G}_0 \subset \mathcal{F}$ such that $\mathcal{G}_0$ is *rich-enough i.e,*

$$\mathbb{K}_2(\mathbb{R}^n) := \left\{ \mu^{\mathrm{y}} = P_{\mathrm{y}} : \mathrm{y}(\cdot) \in \mathbb{L}^2(\mathcal{G}_0, \mathbb{R}^n) \right\}. \tag{5.2}$$

By $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,\tau]}$, we denote the filtration generated by $W(\cdot)$, completed and augmented by $\mathcal{G}_0$. Next, for any function $f : \mathbb{K}_2(\mathbb{R}^n) \to \mathbb{R}$ we define a function $\widetilde{f} : \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n) \to \mathbb{R}$ such that

$$\widetilde{f}(\mathrm{y}) = f(\mu^{\mathrm{y}}) = f(P_{\mathrm{y}}), \ \mathrm{y}(\cdot) \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n). \tag{5.3}$$

Clearly, the function $\widetilde{f}$, called the *lift-function* of f, depends only on the law of $\mathrm{y} \in$

$\mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ and is independent of the choice of the representative y.

Let $g : \mathbb{K}_2(\mathbb{R}^n) \to \mathbb{R}$. The function g is differentiable at a distribution $\mu_0 \in \mathbb{K}_2(\mathbb{R}^n)$ if there exists $\mathrm{y}_0 \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$, with $\mu^{\mathrm{y}_0} = P_{\mathrm{y}_0}$ such that its lift $\widetilde{g}$ is *Fréchet-differentiable* at y_0. More precisely, there exists a continuous linear functional $\mathcal{D}\widetilde{g}(\mathrm{y}_0) : \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n) \to \mathbb{R}$ such that

$$\widetilde{g}(\mathrm{y}_0 + \zeta) - \widetilde{g}(\mathrm{y}_0) = \langle \mathcal{D}\widetilde{g}(\mathrm{y}_0) \cdot \zeta \rangle + o(\|\zeta\|_2) = \mathcal{D}_\zeta g(\mu^{\mathrm{y}_0}) + o(\|\zeta\|_2), \tag{5.4}$$

where $\langle . \cdot . \rangle$ is the dual product on $\mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$. We called $\mathcal{D}_\zeta g(\mu^{\mathrm{y}_0})$ the *Fréchet-derivative* of g at μ_0 in the direction ζ. In this case we have

$$\mathcal{D}_\zeta g(\mu^{\mathrm{y}_0}) = \langle \mathcal{D}\widetilde{g}(\mathrm{y}_0) \cdot \zeta \rangle = \frac{\mathrm{d}}{\mathrm{d}t} \widetilde{g}(\mathrm{y}_0 + t\zeta) \Big|_{t=0}, \text{ with } \mu^{\mathrm{y}_0} = P_{\mathrm{y}_0}. \tag{5.5}$$

Now, from *Riesz representation theorem*, there exists a unique random variable $\psi_0 \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ such that $\langle \mathcal{D}\widetilde{g}(\mathrm{y}_0) \cdot \zeta \rangle = (\psi_0 \psi_0 \cdot \zeta)_2 = E[(\psi_0 \cdot \zeta)_2]$ where $\zeta \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$. It was shown, (see [19]) that there exists a Borel function $\Psi[\mu^{\mathrm{y}_0}](\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, depending only on the law $\mu^{\mathrm{y}_0} = P_{\mathrm{y}_0}$ but not on the choice of the representative y_0 such that $\psi_0 = \Psi[\mu^{\mathrm{y}_0}](\mathrm{y}_0)$. Thus we can write (5.4) as :for any $\mathrm{y} \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$, we have

$$g(\mu^{\mathrm{y}}) - g(\mu^{\mathrm{y}_0}) = (\Psi[\mu^{\mathrm{y}_0}](\mathrm{y}_0) \cdot \mathrm{y} - \mathrm{y}_0)_2 + o(\|\mathrm{y} - \mathrm{y}_0\|_2).$$

We denote $\partial_\mu g(\mu^{\mathrm{y}_0}, \mathrm{y}) = \Psi[\mu^{\mathrm{y}_0}](\mathrm{y})$, $\mathrm{y} \in \mathbb{R}^n$. Moreover, we have the following identities

$$\mathcal{D}\widetilde{g}(\mathrm{y}_0) = \psi_0 = \Psi[\mu^{\mathrm{y}_0}](\mathrm{y}_0) = \partial_\mu g(\mu^{\mathrm{y}_0}, \mathrm{y}_0),$$

and $\mathrm{D}_\zeta g(\mu^{\mathrm{y}_0}) = \langle \partial_\mu g(\mu^{\mathrm{y}_0}, \mathrm{y}_0) \cdot \zeta \rangle$, where $\zeta = (\mathrm{y} - \mathrm{y}_0)$.

Remark 5.1.1 (1) For each $\mu \in \mathbb{K}_2(\mathbb{R}^n)$, the partial derivatives $\partial_\mu g(\mu^{\mathrm{y}}, \cdot) = \Psi[\mu^{\mathrm{y}}](\cdot)$ are only defined in $\mu(\mathrm{dy}) - a.e.$ sense.

(2) A function f is said to be differentiable at $\mu_0 \in \mathbb{K}_2(\mathbb{R}^n)$ if there exists a random variable y_0 with law μ_0 such that the lift function $\widetilde{f}$ is Fréchet differentiable at y_0.

Definition 5.1.2 We say that the function $g \in \mathbb{C}_b^{1,1}(\mathbb{K}_2(\mathbb{R}^n))$ if for all $\mathrm{y} \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ there exists a $P_{\mathrm{y}}-$modification of $\partial_\mu g\left(\mu^{\mathrm{y}}, \cdot\right)$ (denoted by $\partial_\mu g$) such that $\partial_\mu g : \mathbb{K}_2\left(\mathbb{R}^n\right) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is bounded and Lipschitz continuous. That is for some $C > 0$, it holds that

(1) $\left|\partial_\mu g(\mu, \mathrm{y})\right| \leq C$, $\forall \mu \in \mathbb{K}_2(\mathbb{R}^n)$, $\forall \mathrm{y} \in \mathbb{R}^n$.

(2) $\left|\partial_\mu g(\mu, \mathrm{y}) - \partial_\mu g(\mu', \mathrm{y}')\right| \leq C\left[\mathbb{T}\left(\mu, \mu'\right) + |\mathrm{y} - \mathrm{y}'|\right]$, $\forall \mu, \mu' \in \mathbb{K}_2(\mathbb{R}^n)$, $\forall \mathrm{y}, \mathrm{y}' \in \mathbb{R}^n$.

We should note that if the function $g \in \mathbb{C}_b^{1,1}(\mathbb{K}_2(\mathbb{R}^n))$, the version of $\partial_\mu g\left(\mu^{\mathrm{y}}, \cdot\right)$, $\mathrm{y} \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$, presented in *Definition 5.2* is unique (see [19, Remark 2.2], and [27]). We shall denote by $\partial_\mu g\left(t, \mathrm{y}, \mu_0\right)$ the derivative with respect to μ computed at μ_0 whenever all the other variables (t, y) are held fixed, $\partial_\mu g\left(t, \mathrm{y}, \mu_0\right) = \partial_\mu g\left(t, \mathrm{y}, \mu\right)\big|_{\mu=\mu_0}$ $\mu(\mathrm{dy}) - a.e.$.

Throughout this work, we will use the following notations, for $\psi = f, h$: $\psi_{\mathrm{y}}(t) = \frac{\partial \psi}{\partial \mathrm{y}}(t, \mathrm{y}^*(t), \mu^*, \alpha^*(t))$, $\psi_\alpha(t) = \frac{\partial \psi}{\partial \alpha}(t, \mathrm{y}^*(t), \mu^*, \alpha^*(t))$, and $\widehat{\psi}_\mu(t) = \partial_\mu \psi(t, \mathrm{y}(t), \mu, \alpha(t); \widehat{\mathrm{y}}(t))$, $\mu(\mathrm{dy}) - a.e.$.

5.3 Formulation of the mean-field control problem

Let $\tau > 0$ be a fixed positive real number and $\left(\Omega, \mathcal{F}, \left\{\mathcal{F}_t\right\}_{t \in [0,\tau]}, P\right)$ be a fixed filtered probability space satisfying the usual conditions in which *one*$-$dimensional Brownian motion $W(t) = \{W(t) : 0 \leq t \leq \tau\}$ and $W(0) = 0$ is defined. We study optimal solutions of stochastic control problem driven by controlled mean-field model :

$$\begin{cases} d\mathrm{y}(t) = \int_{\mathbb{R}^d} \varphi\left(t, \mathrm{y}(t), \mu^{\mathrm{y}(t)}, \alpha(t)\right) \mu(dy) dt + \int_{\mathbb{R}^d} \psi\left(t, \mathrm{y}(t), \mu^{\mathrm{y}(t)}, \alpha(t)\right) \mu(dy) \mathrm{d}W(t), \\ \mathrm{y}(0) = \mathrm{y}_0, \end{cases} \tag{5.6}$$

where $\mu^{y(t)} = P_{y(t)}$ is the probability distribution of $y(t)$. The goal of our mean-field optimal control problem is to minimize the following cost functional

$$J\left(\alpha(\cdot)\right) = E\int_{\mathbb{R}^d} \Phi(y(\tau), \mu^{y(\tau)})\mu(dy), \tag{5.7}$$

where

$$\varphi : [0, \tau] \times \mathbb{R}^n \times \mathbb{K}_2\left(\mathbb{R}^n\right) \times \mathbb{U} \rightarrow \mathbb{R}^n,$$

$$\psi : [0, \tau] \times \mathbb{R}^n \times \mathbb{K}_2\left(\mathbb{R}^n\right) \times \mathbb{U} \rightarrow \mathbb{R}^n,$$

$$\Phi : \mathbb{R}^n \times \mathbb{K}_2\left(\mathbb{R}^n\right) \rightarrow \mathbb{R},$$

are a given deterministic functions.

An admissible control $\alpha(\cdot)$ is an $\mathcal{F}_t$−predictable process with values in some non-empty convex subset $\mathbb{U}$ of $\mathbb{R}^k$ such that $E\int_0^\tau |\alpha(t)|^2 \, dt < \infty$. We called $\mathbb{U}$ the control domain. We denote by $\mathcal{U}\left([0, \tau]\right)$ the set of all admissible controls. We suppose that an optimal control exists. Any admissible control $\alpha^*(\cdot) \in \mathcal{U}\left([0, \tau]\right)$ satisfying

$$J\left(\alpha^*(\cdot)\right) = \inf_{\alpha(\cdot) \in \mathcal{U}([0,\tau])} J\left(\alpha(\cdot)\right), \tag{5.8}$$

is called an optimal control. The maps

$$\begin{aligned} f\left(t, \mu, \alpha\right) &= \int_{\mathbb{R}^d} \varphi\left(t, y(t), \mu^{y(t)}, \alpha(t)\right) \mu(dx), \\ \sigma\left(t, \mu, \alpha\right) &= \int_{\mathbb{R}^d} \psi\left(t, y(t), \mu^{y(t)}, \alpha(t)\right) \mu(dx), \\ h\left(\mu\right) &= \int_{\mathbb{R}^d} \Phi\left(y(\tau), \mu^{y(\tau)}\right) \mu(dx), \end{aligned}$$

are a given deterministic functions such that

$$f : [0, \tau] \times \mathbb{K}_2 (\mathbb{R}^n) \times \mathbb{U} \to \mathbb{R}^n,$$

$$\sigma : [0, \tau] \times \mathbb{K}_2 (\mathbb{R}^n) \times \mathbb{U} \to \mathbb{R}^{n\times d},$$

$$h : \mathbb{K}_2 (\mathbb{R}^n) \to \mathbb{R}.$$

To avoid excessive complexity in the notation, we will make the simplifying assumption that all processes are 1-dimensional (i.e., $n = m = 1$) in the subsequent sections.
We define a metric $d(\cdot, \cdot)$ on the space of admissible controls $\mathcal{U}([0, \tau])$ such that $(\mathcal{U}([0, \tau]), d)$ becomes a complete metric space. For any $\alpha(\cdot)$ and $\alpha'(\cdot) \in \mathcal{U}([0, \tau])$ we set

$$d(\alpha(\cdot), \alpha'(\cdot)) = \left[E \int_0^\tau |\alpha(t) - \alpha'(t)|^2 \, dt \right]^{\frac{1}{2}}. \tag{5.9}$$

Assumptions. The following assumptions will be in force throughout this work, where y denotes the state variable, and α the control variable.

- **Assumption (H 5.1)** The control region is assumed to be bounded and convex.
- **Assumption (H 5.2)** For fixed measure $\mu \in \mathbb{K}_2(\mathbb{R})$, for any $(\mathrm{y}, \alpha) \in \mathbb{R}^d \times \mathbb{U}$, the functions φ, ψ are measurable in all variables and continuously differentiable with respect to y, α; and al their partial derivatives are uniformly bounded.

  The function Φ is continuously differentiable with respect to y and Moreover $|\Phi(\mathrm{y})| \leq C(1 + |\mathrm{y}|^2)$, and $|\Phi_{\mathrm{y}}(\mathrm{y})| \leq C(1 + |\mathrm{y}|)$, where $C > 0$ is a generic positive constant, which may vary from line to line.
- **Assumption (H 5.3)** (1) For fixed $\mathrm{y} \in \mathbb{R}$, for all $\alpha(t) \in \mathbb{U} : \varphi, \psi \in \mathbb{C}_b^{1,1}(\mathbb{K}_2(\mathbb{R}^d); \mathbb{R})$ and $\Phi \in \mathbb{C}_b^{1,1}(\mathbb{K}_2(\mathbb{R}); \mathbb{R})$.

  (2) All the derivatives with respect to measure φ_μ, ψ_μ are bounded and Lipschitz continuous, with Lipschitz constants independent of α.

Under the assumptions (H 5.2) and (H 5.3), for each $\alpha(\cdot) \in \mathcal{U}([0, \tau])$, Eq-(5.6) has a unique strong solution $\mathrm{y}(\cdot)$ given by

$$\mathrm{y}(t) = \mathrm{y}_0 + \int_0^t \int_{\mathbb{R}^d} \varphi\left(s, \mathrm{y}(s), \mu^{\mathrm{y}(s)}, \alpha(s)\right) \mu(\mathrm{dy}) \mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \psi\left(s, \mathrm{y}(s), \mu^{\mathrm{y}(s)}, \alpha(s)\right) \mu(\mathrm{dy}) \mathrm{d}W(s),$$

such that $E\left[\sup_{t\in[0,\tau]} |\mathrm{y}(t)|^2\right] < \infty$, and the functional $J(\cdot)$ is well defined.

Let $\alpha^*(\cdot) \in \mathcal{U}([0,\tau])$ be an optimal control for the *problem* A, and $\mathrm{y}^*(\cdot) = \mathrm{y}^{\alpha^*}(\cdot)$ the corresponding optimal state process.

Hamiltonian. Let us define the Hamiltonian associated to our control problem. For any $(t, \mathrm{y}, \mu, \alpha, p, q) \in [0,\tau] \times \mathbb{R} \times \mathbb{K}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$$H(t, \mathrm{y}, \mu, \alpha, p(t), q(t)) = p(t) \int_{\mathbb{R}^d} \varphi(t, \mathrm{y}, \mu^{\mathrm{y}(t)}, \alpha) \mu(\mathrm{dy}) + q(t) \int_{\mathbb{R}^d} \psi(t, \mathrm{y}, \mu^{\mathrm{y}(t)}, \alpha) \mu(\mathrm{dy}), \tag{5.10}$$

where $(p(\cdot), q(\cdot))$ is a pair of adapted processes, solution of the adjoint equation (5.12). The derivatives of H with respect to control variable $\alpha(\cdot)$ has the form

$$\begin{aligned} &\frac{\partial H}{\partial \alpha}(t, \mathrm{y}^*(t), \mu^{\mathrm{y}(t)}, \alpha^*(t), p(t), q(t)) \\ &= \int_{\mathbb{R}^d} \varphi_\alpha\left(t, \mathrm{y}, \mu^{\mathrm{y}(t)}, \alpha\right) p(t) \mu(\mathrm{dy}) + \int_{\mathbb{R}^d} \psi_\alpha\left(t, \mathrm{y}, \mu^{\mathrm{y}(t)}, \alpha\right) q(t) \mu(\mathrm{dy}). \end{aligned} \tag{5.11}$$

Adjoint equation : we consider the new adjoint equation, which is the following MF-BSDE :

$$\begin{cases} \mathrm{d}p(t) = -\widehat{E}\left(\partial_{\mathrm{y}} \widehat{\varphi}(t, \mathrm{y}, \mu, \alpha) \widehat{p}(t) + \int_{\mathbb{R}^d} \partial_\mu \widehat{\varphi}(t, \mathrm{y}, \mu, \alpha) \widehat{p}(t) \mu(\mathrm{dy}) \right. \\ \qquad\qquad \left. + \partial_{\mathrm{y}} \widehat{\psi}(t, \mathrm{y}, \mu, \alpha) \widehat{q}(t) + \int_{\mathbb{R}^d} \partial_\mu \widehat{\psi}(t, \mathrm{y}, \mu, \alpha) \widehat{q}(t) \mu(\mathrm{dy}) \right) \mathrm{d}t \\ \qquad\qquad + q(t) \mathrm{d}W(t), \\ p(\tau) = -\widehat{E}\left[\partial_{\mathrm{y}} \widehat{\Phi}(\mathrm{y}, \mu, \alpha) + \int_{\mathbb{R}^d} \partial_\mu \widehat{\Phi}(\mathrm{y}, \mu, \alpha) \mu(\mathrm{dy})\right]. \end{cases} \tag{5.12}$$

Here, for $t \in [0, \tau]$, we have

$$\begin{aligned} \widehat{E}(\widehat{\varphi}_\mu(t)) &= \widehat{E}\left[\partial_\mu \widehat{\varphi}(t, \widehat{\mathrm{y}}^*(t), \mu^{\mathrm{y}^*(t)}, \widehat{\alpha}^*(t); z)\right]\Big|_{z=\mathrm{y}^*(t)} \\ &= \int_{\widehat{\Omega}} \partial_\mu \varphi(t, \widehat{\mathrm{y}}^*(t, \widehat{w}), P_{\mathrm{y}^*(t,w)}, \widehat{\alpha}^*(t, \widehat{w}); \mathrm{y}^*(t, w)) \mathrm{d}\widehat{P}(\widehat{w}), \end{aligned} \tag{5.13}$$

$$\begin{aligned} \widehat{E}(\widehat{\psi}_\mu(t)) = E_{\widehat{P}}(\widehat{\psi}_\mu(t)) &= E_{\widehat{P}}\left[\partial_\mu \widehat{\psi}(t, \widehat{\mathrm{y}}^*(t), \mu^{\mathrm{y}^*(t)}, \widehat{\alpha}^*(t); z)\right]\Big|_{z=\mathrm{y}^*(t)} \\ &= \int_{\widehat{\Omega}} \partial_\mu \psi(t, \widehat{\mathrm{y}}^*(t, \widehat{w}), P_{\mathrm{y}^*(t,w)}, \widehat{\alpha}^*(t, \widehat{w}); \mathrm{y}^*(t, w)) \mathrm{d}\widehat{P}(\widehat{w}). \end{aligned} \tag{5.14}$$

Similarly, we get

$$\begin{aligned} \widehat{E}(\widehat{\Phi}_\mu(\tau)) = E_{\widehat{P}}(\widehat{\Phi}_\mu(\tau)) &= E_{\widehat{P}}\left[\partial_\mu \Phi(\widehat{\mathrm{y}}^*(\tau), P_{\mathrm{y}^*(\tau)}; z)\right]\Big|_{z=\mathrm{y}^*(t)} \\ &= \int_{\widehat{\Omega}} \partial_\mu \Phi(\widehat{\mathrm{y}}^*(\tau, \widehat{w}), P_{\mathrm{y}(\tau,w)}; \mathrm{y}^*(\tau, w)) \mathrm{d}\widehat{P}(\widehat{w}). \end{aligned} \tag{5.15}$$

Under the assumptions (H5.2) and (H5.3), the mean-field BSDE (5.12) admits a unique $\mathcal{F}_t$-adapted strong solution $(p(\cdot), q(\cdot))$ such that $E(\sup_{t\in[0,\tau]} |p(t)|^2 + \int_0^\tau |q(t)|^2 \,\mathrm{d}t) < \infty$. See Guo et al. [39] for some examples and different models of derivatives with respect to probability measures.

5.4 Main results

5.4.1 Maximum principle

In this work, our purpose is to derive mean-field-type necessary maximum principle for the optimal control, where the dynamic driven by controlled mean-field model (5.6). To establish our necessary optimality conditions, we apply the convex perturbation method of the optimal control. This perturbation method is described as follows : Let $\alpha^*(\cdot)$ be an optimal control and $\alpha(\cdot)$ is an arbitrary element of $\mathcal{F}_t-$measurable random variable with values in convex bounded set $\mathbb{U}$ which we consider as fixed from now on. We define a perturbed control $\alpha^{\theta}(\cdot)$ as follows. Let

$$\alpha^{\theta}(t) = \alpha^{*}(t) + \theta\left(\alpha(t) - \alpha^{*}(t)\right), \tag{5.16}$$

where $\theta > 0$ is sufficiently small. Since the control region $\mathbb{U}$ is convex, then $\alpha^{\theta}(\cdot) \in \mathcal{U}\left([0,\tau]\right)$. We denote by $\mathrm{y}^{\theta}(\cdot)$ the solution of Eq-(5.6) associated with $\alpha^{\theta}(\cdot)$.

Under assumptions (H 5.1), (H 5.2) and (H 5.3), we introduce the following new variational equation for our control problem.

Variational equation : let $t \in [0,\tau]$, and $v(t) = \alpha(t) - \alpha^{*}(t)$.

$$\begin{cases} \mathrm{dZ}(t) = \Big[\widehat{E}\left(\partial_{\mathrm{y}}\widehat{\varphi}\left(t,\mathrm{y},\mu,\alpha\right)\widehat{\mathrm{Z}}(t) + \int_{\mathbb{R}^d}\partial_{\mu}\widehat{\varphi}\left(t,\mathrm{y},\mu,\alpha\right)\widehat{\mathrm{Z}}(t)\mu(\mathrm{dy})\right) \\ \qquad +\varphi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)v(t) + \int_{\mathbb{R}^d}\varphi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)v(t)\mu(\mathrm{dy})\Big]\,\mathrm{d}t \\ \qquad + \Big[\widehat{E}\left(\partial_{\mathrm{y}}\widehat{\varphi}\left(t,\mathrm{y},\mu,\alpha\right)\widehat{\mathrm{Z}}(t) + \int_{\mathbb{R}^d}\partial_{\mu}\widehat{\varphi}\left(t,\mathrm{y},\mu,\alpha\right)\widehat{\mathrm{Z}}(t)\mu(\mathrm{dy})\right) \\ \qquad +\psi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)v(t) + \int_{\mathbb{R}^d}\psi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)v(t)\mu(\mathrm{dy})\Big]\,\mathrm{d}W(t) \\ \mathrm{Z}(0) = 0. \end{cases} \tag{5.17}$$

Here the process $\mathrm{Z}(\cdot)$ is called the *first-order variational process*, associated to $\alpha(\cdot)$. Since the derivatives in (5.17) are bounded, it follows that there exists a unique solution $\mathrm{Z}(\cdot)$ such that

$$E\left[\sup_{t\in[0,\tau]}\left|\mathrm{Z}\left(t\right)\right|^{k}\right] < C_k, \quad \text{for} \quad k \geq 2. \tag{5.18}$$

We note that unless specified, for each $k \in \mathbb{R}_{+}$, we denote by $C_k > 0$ a generic positive constant depending only on k, which may vary from line to line.

We shall establish some fundamental estimates that will play the crucial roles for the proof of our stochastic maximum principle.

Our aim in this section is to establish a stochastic maximum principle for optimal stochastic control for systems driven by nonlinear controlled SDEs. Since the control domain is assumed to be convex, the proof of our result based on convex perturbation. Now, the main result of this chapter is stated in the following theorem.

Theorem 5.4.1. *(Maximum principle in integral form via Lions's derivative). Let assumptions (H 5.1), (H 5.2) and (H 5.3) hold. Then there exists a unique pair of $\mathcal{F}_t-$adapted processes $(p(\cdot), q(\cdot))$ solution of the mean-field BSDE (5.12) such that for all $\alpha \in \mathbb{U}$*

$$E\int_0^\tau \frac{\partial H}{\partial \alpha}(t, \mathrm{y}^*(t), \mu^{\mathrm{y}^*(t)}, \alpha^*(t), p(t), q(t))\,(\alpha(t) - \alpha^*(t))\,\mathrm{d}t \geq 0. \tag{5.19}$$

Corollary 5.4.1. *Under assumptions of Theorem 5.4.1, Then there exists a unique pair of $\mathcal{F}_t-$adapted processes $(p(\cdot), q(\cdot))$ solution of mean-field BSDE-(5.12) such that for all $\alpha \in \mathbb{U}$*

$$\frac{\partial H}{\partial \alpha}(t, \mathrm{y}^*(t), \mu^{\mathrm{y}(t)}, \alpha^*(t), p(t), q(t))\,(\alpha(t) - \alpha^*(t))\,\mathrm{d}t \geq 0.$$

$$P-a.s.,\ a.e.\ t \in [0, \tau].$$

To prove *Theorem 5.4.1* we need the following results

5.4.2 Proof of main result

Let $(\alpha^*(\cdot), \mathrm{y}^*(\cdot))$ be the optimal solution of the control problem (5.6)-(5.7). We derive the variational inequality from :

$$J\left(\alpha^\theta(\cdot)\right) \geq J\left(\alpha^*(\cdot)\right), \tag{5.20}$$

where $\alpha^\theta(\cdot)$ is the so called convex-perturbation of $\alpha^*(\cdot)$ defined as follows : $\forall s \in [0, \tau]$

$$\alpha^\theta(s) = \alpha^*(s) + \theta(\alpha(s) - \alpha^*(s)), \tag{5.21}$$

where $\theta > 0$ is sufficiently small and $\alpha(s) \in \mathbb{U}$ is an element of $\mathcal{U}([0, \tau])$.

Proposition 5.4.1. Let $\mathrm{y}^\theta(\cdot)$ and $\mathrm{y}^*(\cdot)$ be the states of (5.22) corresponding to $\alpha^\theta(\cdot)$

and $\alpha^*(\cdot)$ respectively. Let $\mathrm{Z}(\cdot)$ be the solution of (5.17). Then we have

$$\lim_{\theta\to 0} E\left[\sup_{s\in[0,\tau]}\left|\mathrm{y}^\theta(s)-\mathrm{y}^*(s)\right|^{2k}\right]=0, \tag{5.22}$$

$$\lim_{\theta\to 0} E\left[\sup_{s\leq\tau}\left|\theta^{-1}\left[\mathrm{y}^\theta(s)-\mathrm{y}^*(s)\right]-\mathrm{Z}(s)\right|^{2}\right]=0. \tag{5.23}$$

Proof. By using Proposition 5.4.2, estimate (4.8) in [19], we have

$$E\left[\sup_{s\in[0,\tau]}\left|\mathrm{y}^\theta(s)-\mathrm{y}^*(s)\right|^{2k}\right]\leq C_k\theta^k,$$

then the proof of estimate (5.22) follows immediately by letting $\theta\to 0$. Let us turn to prove estimate (5.23). We consider

$$\gamma^\theta(s)=\theta^{-1}\left[\mathrm{y}^\theta(s)-\mathrm{y}^*(s)\right]-\mathrm{Z}(s),\ \ s\in[0,\tau]. \tag{5.24}$$

Since $D_\xi f(\mu^{Z_0(t)})=\left\langle D\widetilde{f}(Z_0)\cdot\xi\right\rangle=\frac{d}{dt}\widetilde{f}\left(Z_0+t\xi\right)\Big|_{t=0}$, then we have the following simple form of the first order *Taylor expansion*

$$f(\mu^{Z_0(t)+\xi})-f(\mu^{Z_0(t)})=D_\xi f(\mu^{Z_0(t)})+\mathcal{E}(\xi),$$

where $\mathcal{E}(\xi)$ is of order $O\left(\|\xi\|_2\right)$ with $O\left(\|\xi\|_2\right)\to 0$ for $\xi\in\mathbb{L}^2\left(\mathcal{F},\mathbb{R}^d\right)$. From (5.24), we have

$$\begin{aligned}\gamma^\theta(t)=&\frac{1}{\theta}\int_0^t\int_{\mathbb{R}^d}\left[\varphi\left(s,\mathrm{y}^\theta(s),\mu^{\mathrm{y}^\theta(s)},\alpha^\theta(s)\right)-\varphi\left(s,\mathrm{y}^*(s),\mu^{\mathrm{y}^*(s)},\alpha^*(s)\right)\right]\mu(\mathrm{dy})\mathrm{d}s\\&+\frac{1}{\theta}\int_0^t\int_{\mathbb{R}^d}\left[\psi\left(s,\mathrm{y}^\theta(s),\mu^{\mathrm{y}^\theta(s)},\alpha^\theta(s)\right)-\psi\left(s,\mathrm{y}^*(s),\mu^{\mathrm{y}^*(s)},\alpha^*(s)\right)\right]\mu(\mathrm{dy})\mathrm{d}W(s)\end{aligned} \tag{5.25}$$

$$-\mathrm{Z}(t).$$

We put

$$f(t,\mu,\alpha) = \int_{\mathbb{R}^d} \varphi\left(t, \mathrm{y}(t), \mu^{\mathrm{y}(t)}, \alpha(t)\right)\mu(dy),$$
$$\sigma(t,\mu,\alpha) = \int_{\mathbb{R}^d} \psi\left(t, \mathrm{y}(t), \mu^{\mathrm{y}(t)}, \alpha(t)\right)\mu(dy), \tag{5.26}$$
$$h(\mu) = \int_{\mathbb{R}^d} \Phi\left(\mathrm{y}(\tau), \mu^{\mathrm{y}(\tau)}\right)\mu(dy).$$

By applying (5.25), we get

$$\begin{aligned}
\gamma^\theta(t) &= \frac{1}{\theta}\int_0^t \left[f\left(s, \mu^{\mathrm{y}^\theta(s)}, \alpha^\theta(s)\right) - f\left(s, \mu^{\mathrm{y}^*(s)}, \alpha^*(s)\right)\right]\mathrm{d}s \\
&+ \frac{1}{\theta}\int_0^t \left[\sigma\left(s, \mu^{\mathrm{y}^\theta(s)}, \alpha^\theta(s)\right) - \sigma\left(s, \mu^{\mathrm{y}^*(s)}, \alpha^*(s)\right)\right]\mathrm{d}W(s) \\
&- \int_0^t \left\{\widehat{E}\left[f_\mu(s, \mu^{\mathrm{y}^*(s)}, \alpha^*(s); \widehat{\mathrm{y}}^*(s))\widehat{\mathrm{Z}}(s)\right] + f_\alpha(s, \mu^{\mathrm{y}^*(s)}, \alpha^*(s))v(s)\right\}\mathrm{d}s \\
&- \int_0^t \left\{\widehat{E}\left[\sigma_\mu(s, \mu^{\mathrm{y}^*(s)}, \alpha^*(s); \widehat{\mathrm{y}}^*(s))\widehat{\mathrm{Z}}(s)\right] + \sigma_\alpha(s, \mu^{\mathrm{y}^*(s)}, \alpha^*(s))v(s)\right\}\mathrm{d}W(s).
\end{aligned}$$

By simple computations, we have

$$\begin{aligned}
&\int_0^t [f(s, \mu^{\mathrm{y}^\theta(s)}, \alpha^\theta(s)) - f(s, \mu^{\mathrm{y}^*(s)}, \alpha^*(s))]\mathrm{d}s \\
&= \int_0^t (f(s, \mu^{\mathrm{y}^\theta(s)}, \alpha^\theta(s)) - f(s, \mu^{\mathrm{y}^*(s)}, \alpha^\theta(s)))\mathrm{d}s \\
&+ \int_0^t (f(s, \mu^{\mathrm{y}^*(s)}, \alpha^\theta(s)) - f(s, \mu^{\mathrm{y}^*(s)}, \alpha^*(s)))\mathrm{d}s.
\end{aligned}$$

Applying first-order expansion, we get

$$\begin{aligned}
&\frac{1}{\theta}\int_0^t (f(s, \mu^{\mathrm{y}^\theta(s)}, \alpha^\theta(s)) - f(s, \mu^{\mathrm{y}^*(s)}, \alpha^\theta(s)))\mathrm{d}s \\
&= \int_0^t \int_0^1 \widehat{E}\left[\partial_\mu f(s, \mu^{\mathrm{y}^*(s)+\lambda\varepsilon(\gamma(s)+\mathrm{Z}(s))}, \alpha^\theta(s); \widehat{\mathrm{y}}^*(s))(\widehat{\gamma}(s) + \widehat{\mathrm{Z}}(s))\right]\mathrm{d}\lambda\mathrm{d}s.
\end{aligned}$$

Using similar arguments developed above, we can easily prove that

$$\begin{aligned}
&\frac{1}{\theta}\int_0^t (f(s,\mu^{\mathrm{y}^\theta(s)},\alpha^\theta(s)) - f(s,\mu^{\mathrm{y}^\theta(s)},\alpha^*(s)))\mathrm{d}s \\
&= \int_0^t \int_0^1 \left[f_\alpha \left(s, \mu^{\mathrm{y}^\theta(s)}, \alpha^*(s) + \lambda\varepsilon(\alpha(s)-\alpha^*(s)) \right) v(s) \right] \mathrm{d}\lambda \mathrm{d}s.
\end{aligned}$$

The analogue arguments hold for σ, then we get

$$\begin{aligned}
&\frac{1}{\theta}\int_0^t [\sigma(s,\mu^{\mathrm{y}^\theta(s)},\alpha^\theta(s)) - \sigma(s,\mu^{\mathrm{y}^*(s)},\alpha^*(s))]\mathrm{d}s \\
&= \int_0^t \int_0^1 \widehat{E}\left[\partial_\mu \sigma(s,\mu^{\mathrm{y}^*(s)+\lambda\varepsilon(\gamma(s)+\mathrm{Z}(s))},\alpha^\theta(s);\widehat{\mathrm{y}}^*(s))(\widehat{\gamma}(s)+\widehat{\mathrm{Z}}(s)) \right] \mathrm{d}\lambda \mathrm{d}s \\
&+ \int_0^t \int_0^1 \left[\sigma_\alpha \left(s, \mu^{\mathrm{y}^\theta(s)}, \alpha^*(s) + \lambda\varepsilon(\alpha(s)-\alpha^*(s) \right) v(s) \right] \mathrm{d}\lambda \mathrm{d}s.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&E\left[\sup_{s\in[0,t]} \left| \gamma^\theta(s) \right|^2 \right] \\
&\le C_t \left[E\int_0^t \int_0^1 \widehat{E}\left| f_\mu(s,\mu^{\mathrm{y}^*(s)+\lambda\varepsilon(\widehat{\gamma}(s)+\widehat{\mathrm{Z}}(s))},\alpha^\theta(s);\widehat{\mathrm{y}}^*(s))\widehat{\gamma}^\theta(s) \right|^2 d\lambda ds \right. \\
&+ E\int_0^t \int_0^1 \widehat{E}\left| \sigma_\mu(s,\mu^{\mathrm{y}^*(s)+\lambda\varepsilon(\widehat{\gamma}(s)+\widehat{\mathrm{Z}}(s))},\alpha^\theta(s);\widehat{\mathrm{y}}^*(s))\widehat{\gamma}^\theta(s) \right|^2 d\lambda ds \\
&\left. + E\left[\sup_{s\in[0,t]} \left| A^\theta(s) \right|^2 \right] \right],
\end{aligned}$$

where

$$
\begin{aligned}
A^{\theta}(t) = \int_0^t \int_0^1 \widehat{E}\Big[& f_{\mu}(s, \mu^{\mathrm{y}^*(s)+\lambda\varepsilon(\widehat{\gamma}(s)+\widehat{\mathrm{Z}}(s))}, \alpha^{\theta}(s); \widehat{\mathrm{y}}^*(s)) \\
& - f_{\mu}(s, \mu^{\mathrm{y}^*(s)}, \alpha^*(s); \widehat{\mathrm{y}}^*(s))]\widehat{Z}(s)\mathrm{d}\lambda\mathrm{d}s \\
& + \int_0^t \int_0^1 [f_{\alpha}\left(s, \mu^{\mathrm{y}^*(s)}, \alpha^*(s) + \lambda\varepsilon v(t)\right) \\
& - f_{\alpha}(s, \mu^{\mathrm{y}^*(s)}, \alpha^*(s)]v(t)\mathrm{d}\lambda\mathrm{d}s \\
& + \int_0^t \int_0^1 \widehat{E}\Big[\sigma_{\mu}(s, \mu^{\mathrm{y}^*(s)+\lambda\varepsilon(\widehat{\gamma}(s)+\widehat{\mathrm{Z}}(s))}, \alpha^{\theta}(s); \widehat{\mathrm{y}}^*(s)) \\
& - \sigma_{\mu}(s, \mu^{\mathrm{y}^*(s)}, \alpha^*(s); \widehat{\mathrm{y}}^*(s))]\widehat{\mathrm{Z}}(s)\mathrm{d}\lambda\mathrm{d}W(s) \\
& + \int_0^t \int_0^1 [\sigma_{\alpha}\left(s, \mu^{\mathrm{y}^*(s)}, \alpha^*(s) + \lambda\varepsilon v(t)\right) \\
& - \sigma_{\alpha}(s, \mu^{\mathrm{y}^*(s)}, \alpha^*(s)]v(t)\mathrm{d}\lambda\mathrm{d}W(s).
\end{aligned}
$$

Now, since the partial derivatives of f and σ with respect to μ, α are Lipschitz continuous in μ, α, then we get

$$
\lim_{\theta \to 0} E\left[\sup_{s\in[0,\tau]} \left|A^{\theta}(s)\right|^2\right] = 0.
$$

Moreover, since the partial-derivatives of f and σ with respect to variables μ, and α are bounded, we obtain $\forall t \in [0, \tau]$:

$$
E\left[\sup_{s\in[0,t]} \left|\gamma^{\theta}(s)\right|^2\right] \leq C(t)\left\{E\int_0^t \left|\gamma^{\theta}(s)\right|^2 ds + E\left[\sup_{s\in[0,t]} \left|A^{\theta}(s)\right|^2\right]\right\}.
$$

By using *Gronwall's theorem*, we get

$$
E\left[\sup_{s\in[0,t]} \left|\gamma^{\theta}(s)\right|^2\right] \leq C_s E\left[\sup_{s\in[0,t]} \left|A^{\theta}(s)\right|^2\right] \exp\left(\int_0^t C_s ds\right).
$$

Finally, putting $t = \tau$ the proof of *Proposition 5.4.1* is fulfilled by sending θ to zero. □

Proposition 5.4.2. For any $\alpha(\cdot) \in \mathcal{U}([0,\tau])$, we have

$$0 \leq E\left(\partial_{\mathrm{y}}\Phi\left(\mathrm{y}^*(\tau),\mu^{\mathrm{y}^*(\tau)}\right) + \int_{\mathbb{R}^d}\widehat{E}(\partial_\mu\Phi\left(\mathrm{y}^*(\tau),\mu^{\mathrm{y}^*(\tau)};\widehat{\mathrm{y}}^*(\tau)\right)\mu(\mathrm{dy})\right)\mathrm{Z}(\tau). \tag{5.27}$$

Proof. From (5.7) and (5.20), we have

$$\begin{aligned} 0 &\leq J\left(\alpha^\theta(\cdot)\right) - J\left(\alpha^*(\cdot)\right) \\ &= E\left[h(\mathrm{y}^\theta(\tau),\mu^{\mathrm{y}^\theta(\tau)}) - h(\mathrm{y}^*,\mu^{\mathrm{y}^*(\tau)})\right]. \end{aligned}$$

By applying first-order expansion, we get

$$\begin{aligned} &h(\mathrm{y}^\theta(\tau),\mu^{\mathrm{y}^\theta(\tau)}) - h(\mathrm{y}^*,\mu^{\mathrm{y}^*(\tau)}) \\ &= \int_0^1\left[h_{\mathrm{y}}\left(\mathrm{y}^*(\tau)+\rho\Delta x^\theta(\tau)),\mu^{\mathrm{y}^*(\tau)+\rho\Delta x^\theta}\right)\Delta x^\theta(\tau)\right]\mathrm{d}\rho \\ &+ \int_0^1\widehat{E}\left[h_\mu(\mathrm{y}^*(\tau)+\rho\Delta x^\theta(\tau),\mu^{\mathrm{y}^*(\tau)+\rho\Delta x^\theta(\tau)};\widehat{\mathrm{y}}^*(\tau))\Delta\widehat{\mathrm{y}}^\theta(\tau)\right]\mathrm{d}\rho \\ &= \int_0^1\left(\partial_{\mathrm{y}}\Phi\left(\mathrm{y}^*(\tau)+\rho\Delta x^\theta(\tau),\mu^{\mathrm{y}^*(\tau)+\rho\Delta x^\theta(\tau)};\widehat{\mathrm{y}}^*(\tau)\right)\right)\Delta\widehat{\mathrm{y}}^\theta(\tau)\mathrm{d}\rho \\ &+ \int_0^1\int_{\mathbb{R}^d}\widehat{E}(\partial_\mu\Phi\left(\mathrm{y}^*(\tau)+\rho\Delta x^\theta(\tau),\mu^{\mathrm{y}^*(\tau)+\rho\Delta x^\theta(\tau)};\widehat{\mathrm{y}}^*(\tau)\right)\mu(\mathrm{dy})\Delta\widehat{\mathrm{y}}^\theta(\tau)\mathrm{d}\rho, \end{aligned}$$

where $\Delta x^\theta(t) = \mathrm{y}^\theta(t) - \mathrm{y}^*(t)$. Finally, by using Proposition 5.4.1, the desired result (5.27) is fulfilled. This completes the proof of Propositionn 5.4.2. □

Proof of Theorem 5.4.1. Itô's formula is one of the most fundamental building blocks in stochastic calculus and maximum principle, see Guo et al. [39]. By applying Itô's formula to stochastic process $p(t)\mathrm{Z}(t)$ and take expectation, where $\mathrm{Z}(0) = 0$, then a simple computations shows that

$$\begin{aligned}
&E(p(\tau)\mathrm{Z}(\tau)) - E(p(0)\mathrm{Z}(0)) \\
&= E\int_0^\tau p(t)d\mathrm{Z}(t) + E\int_0^\tau \mathrm{Z}(t)dp(t) \\
&+ E\int_0^\tau q(t)\left[\widehat{E}\left(\partial_{\mathrm{y}}\widehat{\varphi}\left(t,\mathrm{y},\mu,\alpha\right)\widehat{\mathrm{Z}}(t) + \int_{\mathbb{R}^d}\partial_\mu\widehat{\varphi}\left(t,\mathrm{y},\mu,\alpha\right)\widehat{\mathrm{Z}}(t)\mu(\mathrm{dy})\right)\right. \\
&\left.+\psi_\alpha\left(t,\mathrm{y},\mu,\alpha\right)v(t) + \int_{\mathbb{R}^d}\psi_\alpha\left(t,\mathrm{y},\mu,\alpha\right)v(t)\mu(\mathrm{dy})\right]\mathrm{d}t \\
&= I_1 + I_2 + I_3,
\end{aligned} \tag{5.28}$$

where

$$\begin{aligned}
I_1 &= E\int_0^\tau p(t)\mathrm{dZ}(t) \\
&= E\int_0^\tau p(t)\left[\widehat{E}\left(\partial_{\mathrm{y}}\widehat{\varphi}\left(t,\mathrm{y},\mu,\alpha\right)\widehat{\mathrm{Z}}(t) + \textstyle\int_{\mathbb{R}^d}\partial_\mu\widehat{\varphi}\left(t,\mathrm{y},\mu,\alpha\right)\widehat{\mathrm{Z}}(t)\mu(\mathrm{dy})\right)\right]\mathrm{d}t \\
&\quad + E\int_0^\tau p(t)\left[\varphi_\alpha\left(t,\mathrm{y},\mu,\alpha\right)v(t) + \textstyle\int_{\mathbb{R}^d}\varphi_\alpha\left(t,\mathrm{y},\mu,\alpha\right)v(t)\mu(\mathrm{dy})\right]\mathrm{d}t.
\end{aligned} \tag{5.29}$$

Let us turn to estimate the second term I_2. From (5.12), we have

$$\begin{aligned}
I_2 &= E\int_0^\tau \mathrm{Z}(t)dp(t) \\
&= -E\int_0^\tau \mathrm{Z}(t)\widehat{E}\left(\partial_{\mathrm{y}}\widehat{\varphi}\left(t,\mathrm{y},\mu,\alpha\right)\widehat{p}(t) + \textstyle\int_{\mathbb{R}^d}\partial_\mu\widehat{\varphi}\left(t,\mathrm{y},\mu,\alpha\right)\widehat{p}(t)\mu(\mathrm{dy})\right)\mathrm{d}t \\
&- E\int_0^\tau \mathrm{Z}(t)\widehat{E}\left(\partial_{\mathrm{y}}\widehat{\psi}\left(t,\mathrm{y},\mu,\alpha\right)\widehat{q}(t) + \textstyle\int_{\mathbb{R}^d}\partial_\mu\widehat{\psi}\left(t,\mathrm{y},\mu,\alpha\right)\widehat{q}(t)\mu(\mathrm{dy})\right)\mathrm{d}t;
\end{aligned} \tag{5.30}$$

From (5.17), we have

$$\begin{aligned}
I_3 &= E\int_0^\tau q(t)\left[\widehat{E}\left(\partial_{\mathrm{y}}\widehat{\varphi}\left(t,\mathrm{y},\mu,\alpha\right)\widehat{\mathrm{Z}}(t) + \int_{\mathbb{R}^d}\partial_\mu\widehat{\varphi}\left(t,\mathrm{y},\mu,\alpha\right)\widehat{\mathrm{Z}}(t)\mu(\mathrm{dy})\right)\right. \\
&\left.+\psi_\alpha\left(t,\mathrm{y},\mu,\alpha\right)v(t) + \int_{\mathbb{R}^d}\psi_\alpha\left(t,\mathrm{y},\mu,\alpha\right)v(t)\mu(\mathrm{dy})\right]\mathrm{d}t.
\end{aligned} \tag{5.31}$$

Substituting (5.29), (5.30) and (5.31) into (5.28), with the fact that

$$p(\tau)=\widehat{E}\left[\partial_{\mathrm{y}}\widehat{\Phi}\left(\mathrm{y}(\tau),\mu^{\mathrm{y}(\tau)}\right)+\int_{\mathbb{R}^d}\partial_{\mu}\widehat{\Phi}\left(\mathrm{y}(\tau),\mu^{\mathrm{y}(\tau)}\right)\mu(\mathrm{dy})\right],$$

we get

$$\begin{aligned}
&E\left(\widehat{E}\left[\partial_{\mathrm{y}}\widehat{\Phi}\left(\mathrm{y}(\tau),\mu^{\mathrm{y}(\tau)}\right)+\int_{\mathbb{R}^d}\partial_{\mu}\widehat{\Phi}\left(\mathrm{y}(\tau),\mu^{\mathrm{y}(\tau)}\right)\mu(\mathrm{dy})\right]\mathrm{Z}(\tau)\right)\\
&=E\int_0^{\tau}p(t)\left[\varphi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\left(\alpha(t)-\alpha^*(t)\right)+\int_{\mathbb{R}^d}\varphi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\left(\alpha(t)-\alpha^*(t)\right)\mu(\mathrm{dy})\right]\mathrm{d}t\\
&+E\int_0^{\tau}q(t)\left[\psi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\left(\alpha(t)-\alpha^*(t)\right)+\int_{\mathbb{R}^d}\psi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\left(\alpha(t)-\alpha^*(t)\right)\mu(\mathrm{dy})\right]\mathrm{d}t.
\end{aligned}$$

Applying Proposition 5.4.1, *we* obtain

$$\begin{aligned}
0\le\, &E\int_0^{\tau}p(t)\left[\varphi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\left(\alpha(t)-\alpha^*(t)\right)+\int_{\mathbb{R}^d}\varphi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\left(\alpha(t)-\alpha^*(t)\right)\mu(\mathrm{dy})\right]\mathrm{d}t\\
&+E\int_0^{\tau}q(t)\left[\psi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\left(\alpha(t)-\alpha^*(t)\right)+\int_{\mathbb{R}^d}\psi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\left(\alpha(t)-\alpha^*(t)\right)\mu(\mathrm{dy})\right]\mathrm{d}t.
\end{aligned}$$

Finally, by simple computations, with the helps of (5.11), we get

$$\begin{aligned}
&E\int_0^{\tau}p(t)\left[\varphi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\left(\alpha(t)-\alpha^*(t)\right)+\int_{\mathbb{R}^d}\varphi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\left(\alpha(t)-\alpha^*(t)\right)\mu(\mathrm{dy})\right]\mathrm{d}t\\
&+E\int_0^{\tau}q(t)\left[\psi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\left(\alpha(t)-\alpha^*(t)\right)+\int_{\mathbb{R}^d}\psi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\left(\alpha(t)-\alpha^*(t)\right)\mu(\mathrm{dy})\right]\mathrm{d}t\\
&=E\int_0^{\tau}\left[p(t)\left(\varphi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\left(\alpha(t)-\alpha^*(t)\right)+\int_{\mathbb{R}^d}\varphi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\mu(\mathrm{dy})\right)\right.\\
&\left.+\,q(t)\left(\psi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\left(\alpha(t)-\alpha^*(t)\right)+\int_{\mathbb{R}^d}\psi_{\alpha}\left(t,\mathrm{y},\mu,\alpha\right)\mu(\mathrm{dy})\right)\right]\left(\alpha(t)-\alpha^*(t)\right)\mathrm{d}t\\
&=E\int_0^{\tau}\frac{\partial H}{\partial\alpha}(t,\mathrm{y}^*(t),\mu^{\mathrm{y}^*(t)},\alpha^*(t),p(t),q(t))\left(\alpha(t)-\alpha^*(t)\right)\mathrm{d}t,
\end{aligned}$$

then (5.19) is fulfilled. This completes the proof of *Theorem 4.1* □

5.5 Examples : Gamma process via Lévy measure

The Gamma process is a Lévy process (of bounded variation) $(G(t))_{t\geq 0}$, with Lévy measure given by

$$\mu(\mathrm{d}y) = \frac{e^{-y}}{y} I_{\{y>0\}} \mathrm{d}y. \tag{5.32}$$

It is called *Gamma process* because the probability law of $G(\cdot)$ is a Gamma distribution with mean t and scale-parameter equal to one.

5.5.1 Examples (Derivatives with respect to measure)

Let $(G(t))_{t\geq 0}$ be Gamma process with Lévy measure $\mu(\cdot)$ given by (5.32). We give some examples.

1) If $\Phi(\mu) = \int_{\mathbb{R}} \varphi(y)\mu(\mathrm{d}y)$, then the Lions's derivatives of $\Phi(\mu)$ with respect to measure at z is given by

$$\partial_\mu \Phi(\mu)(z) = \frac{\partial \varphi}{\partial y}(z).$$

2) If $\Phi(\mu) = \int_{\mathbb{R}} \varphi(y, \mu)\mu(\mathrm{d}y)$, then the Lions's derivatives of $\Phi(\mu)$ with respect to measure at z is given by

$$\begin{aligned}\partial_\mu \Phi(\mu)(z) &= \frac{\partial \varphi}{\partial y}(z, \mu) + \int_{\mathbb{R}} \frac{\partial \varphi}{\partial \mu}(y, \mu)(z)\, \mu(\mathrm{d}y) \\ &= \frac{\partial \varphi}{\partial y}(z, \mu) + \int_{\mathbb{R}} \frac{e^{-y}}{y} \frac{\partial \varphi}{\partial \mu}(y, \mu)(z)\, I_{\{y>0\}} \mathrm{d}y.\end{aligned}$$

5.5.2 Maximum principle

We consider $\varphi(t, \mathrm{y}(t), \mu, \alpha(t)) = \mathrm{y}(t)\alpha(t)$, $\psi(t, \mathrm{y}(t), \mu, \alpha(t)) = \mathrm{y}(t)\alpha(t)$. Our purpose is to minimize $Var(\mathrm{y}(\tau)) - \mu^{\mathrm{y}(\tau)}$.

From (5.26), then a simple computations shows that

$$f(t, \mu, \alpha) = \int_{\mathbb{R}} \varphi(t, \mathrm{y}(t), \mu^{\mathrm{y}(t)}, \alpha(t))\mu(dy) = \alpha(t), \tag{5.33}$$

$$\sigma(t, \mu, \alpha) = \int_{\mathbb{R}} \psi(t, \mathrm{y}(t), \mu^{\mathrm{y}(t)}, \alpha(t))\mu(dy) = \alpha(t). \tag{5.34}$$

From (5.10) we get

$$H(t, \mathrm{y}, \mu, \alpha, p(t), q(t)) = \alpha(t)p(t) + \alpha(t)q(t), \tag{5.35}$$

Since, the Hamiltonian H is linear in the control variable $\alpha(\cdot)$, then by considering the first-order condition for minimizing the Hamiltonian that yields

$$H_\alpha(t, \mathrm{y}, \mu, \alpha, p(t), q(t)) = p(t) + q(t) = 0, \tag{5.36}$$

From (5.12) and (5.32), with simple computations, we have

$$\begin{cases} \mathrm{d}p(t) = q(t)\mathrm{d}W(t) \\ p(\tau) = 2\left[\mathrm{y}(t) - \mu^{\mathrm{y}(\tau)}\right] - 1. \end{cases} \tag{5.37}$$

Conjecture of the adjoint process. Looking at the terminal condition $p(\tau)$, it is reasonable to try a solution of the form :

$$p(t) = U_1(t)\left[\mathrm{y}(t) - \mu^{\mathrm{y}(\tau)}\right] + U_2(t), \tag{5.38}$$

where $U_1(\cdot)$, and $U_2(\cdot)$ are deterministic differentiable functions, and $U_1(\tau) = 2$, and $U_2(\tau) = -1$.

On the other hand, by applying Itô's formula to $U_1(t)\left(\mathrm{y}(t) - \mu^{\mathrm{y}(t)}\right)$ in (5.38), we get

$$\begin{aligned} \mathrm{d}p(t) &= \mathrm{d}(U_1(t)(\mathrm{y}(t) - \mu^{\mathrm{y}(t)})) + \mathrm{d}U_2(t) \\ &= U_1(t)\,\mathrm{d}(\mathrm{y}(t) - \mu) + (\mathrm{y}(t) - \mu)\,U_1'(t)\,\mathrm{d}t + U_2'(t)\,\mathrm{d}t \\ &= U_1(t)\,\alpha(t)\mathrm{d}t - U_1(t)\,\mathrm{d}\mu + (\mathrm{y}(t) - \mu)\,U_1'(t)\,\mathrm{d}t + U_2'(t)\,\mathrm{d}t \\ &+ U_1(t)\,\alpha(t)\mathrm{d}W(t). \end{aligned} \tag{5.39}$$

From (5.39) and (5.37), we conclude

$$(\mathrm{y}(t)-\mu)\,U_1'\,(t)+U_1\,(t)\,\alpha(t)+U_1\,(t)\,\mu+U_2'\,(t)=0. \tag{5.40}$$

and

$$q(t)=U_1\,(t)\,\alpha(t). \tag{5.41}$$

Substituting (5.41) into (5.36), we obtain a candidate optimal control in feedback form

$$\begin{aligned}\alpha(t)&=\frac{q(t)}{U_1\,(t)}=\frac{-p(t)}{U_1\,(t)}=\frac{-U_1(t)(\mathrm{y}(t)-\mu)+U_2(t)}{U_1\,(t)}\\&=-\mathrm{y}(t)+\mu-\frac{U_2(t)}{U_1\,(t)},\end{aligned} \tag{5.42}$$

By comparing the coefficient of $y(t)$ and μ, in (5.40), we obtain

$$U_1\,(t)-U_1'\,(t)=0,\;\;U_1(\tau)=2, \tag{5.43}$$

and

$$U_2'\,(t)=0,\;\;U_2(\tau)=-1. \tag{5.44}$$

By solving the ordinary differential equations (5.43)-(5.44), we obtain for $t\in[0,\tau]$

$$\begin{aligned}U_1\,(t)&=2\exp\left[t-\tau\right],\\U_2(t)&=-1.\end{aligned} \tag{5.45}$$

Finally, by substituting (5.42) into (5.45), the optimal control is given in the feedback form by

$$\alpha^*(t,\mathrm{y}^*(t),\mu^{\mathrm{y}^*(t)})=-\mathrm{y}^*(t)+\mu^{\mathrm{y}^*(t)}+\frac{1}{2}\exp\left[\tau-t\right]. \tag{5.46}$$

Conclusion, perspectives and future Developments

In this thesis, we establish a set of necessary conditions of optimal stochastic for different stochastic models. More precisely, in the second chapter, we have developed a necessary conditions for partially observed singular stochastic optimal control problem, where the controlled state dynamics is influenced by unobserved uncertainties. The system is governed by general McKean-Vlasov differential equations. By transforming the partial observation problem to a related problem with full information, a stochastic maximum principle for optimal singular control has been established via the derivative with respect to probability measure in P.Lions' sense. The main feature of these results is to explicitly solve some new mathematical finance problems such as general conditional mean-variance portfolio selection problem in incomplete market.

Apparently, there are many problems left unsolved :

1. One possible problem is to establish some optimality conditions (or near-optimality) for partially observed singular stochastic optimal control for systems governed forward-backward stochastic differential equations of general McKean-Vlasov type with some recent applications.
2. The partially observed singular control in the case when the control domain is not necessarily convex.
3. It would be quite interesting to derive a general maximum principle for partially observed optimal control for fully coupled forward-backward stochastic differential equations FBSEDs following Yong's maximum principle.

In the fourth chapter, pointwise second-order necessary conditions, in the form of Pontryagin maximum principal for optimal stochastic singular control have been established. The control dynamic system was governed by nonlinear controlled stochastic differential equation. In our class of control problem, we have studied two types of singularity, the predictable ones which come from the singular control part and the second ones which come from the irregularity in some senses.

We note that if the coefficients $G(t) = M(t) = 0$ our results coincides with second-order maximum principle developed in [124, Theorem 3.5]. Apparently, there are many problems left unsolved such as :

1. The case when the control domain is not assumed to be convex (general action space).
2. One possible problem is to study the second-order maximum principle for optimal singular control for McKean-Vlasov stochastic differential equations.
3. Another challenging problem left unsolved is to derive a various second-order maximum principles in the case where the coefficients G and M depend on the state of the solution process $x^{u,\xi}(\cdot)$.
4. It would be quite interesting to establish second order maximum principle for systems governed by forward-backward stochastic differential equations with some applications.

We plane to study these interesting problems in forthcoming works.

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ملخص الاطروحة

يندرج موضوع هذه الاطروحة في مجال الاحتمالات وتطبيقاتها. وقد تطرقنا فيها الى موضوع الامثلية التوافقية لمختلف للمعادلات التفاضلية العشوائية من الصنف الحقل المتوسط.

ولقد قدمنا في هذه الاطروحة خمسة فصول على النحو الاتي

احتوى الفصل الاول على مجموعة من التعاريف و المفاهيم العامة و طرق الحلول الممكنة في مسائل المراقبة الاحتمالية . كما احتوي الفصل الثاني على عرض خاص للطريقة الاشتقاق الجديدة بالنسبة للقياسات والتى تم استخدامها في اعمالنا البحثية للقياسات الاحتمالية. وقد ساعدنا في ذلك- نظرية رايز- . اما الفصول الثالث و الرابع و الخامس فكانت عرضا دقيقا لنتائجنا ونظريات مبدا الاعظمية لمختلف الانظمة التفاضلية و لاعمالنا البحثية الجديدة مع بعض التطبيقات في المالية.